On a class of nonlinear elliptic equations in Hilbert spaces

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Abstract. We consider elliptic nonlinear equations in a separable Hilbert space and their solutions in spaces of Sobolev type.

1. Introduction. We study equations of the form

\[ P(D)u = F(x, (\partial^\alpha u)), \quad (D = -i\partial), \]

with a strongly elliptic polynomial \( P \) of \( n \) variables, defined for \( u : \mathbb{R}^n \to H \), where \( H \) is a separable Hilbert space. The equations are understood in a weak sense (see Definition 2). We make assumptions giving an a priori bound for solutions in a space of Sobolev type. As an example, we consider assumptions of Bernstein type. Assumptions of this kind appear in the papers [1], [5] concerning equations on a bounded interval, and in [8], [3] and [4] concerning equations on the half-line, on the line and in \( \mathbb{R}^n \), respectively.

2. Spaces of Sobolev type

Definition 1. We denote by \( H^s = H^s(\mathbb{R}^n) \), for \( s \in \mathbb{R} \), the Sobolev space of real tempered distributions \( u \) such that

\[
\|u\|_s^2 := (2\pi)^{-n} \int |\mathcal{F}u(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty
\]

where \( \mathcal{F} \) stands for the Fourier transform.

Remark 1. Definition 1 may be used for both real and complex Sobolev spaces, depending on whether we consider real or complex functions and distributions.

\( H^s \) is a Hilbert space with the scalar product

\[
\langle u, w \rangle_s := (2\pi)^{-n} \int (\mathcal{F}u(\xi))(\mathcal{F}w(\xi))(1 + |\xi|^2)^s d\xi
\]

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in the complex case or
\[ (u, w)_s := \text{Re} \left( 2\pi^{-n} \int (\mathcal{F}u)(\xi)(\mathcal{F}w)(\overline{\xi})(1 + |\xi|^2)^s \, d\xi \right) \]
in the real case.

We denote the local Sobolev space by \( H^s_{\text{loc}} = H^s_{\text{loc}}(\mathbb{R}^n) \) and treat it as a Fréchet space in the standard way (see for example [6]).

Note two important lemmas.

**Lemma 1.** The embedding \( H^s_{\text{loc}} \hookrightarrow H^{s'}_{\text{loc}} \), for \( s > s' \), is completely continuous.

The proof is in [6], Theorem 10.1.27.

**Lemma 2.** If \( u \in H^s \) then any \( \partial^\alpha u \), for \( |\alpha| < s - n/2 \), is a continuous bounded function and there exists a constant \( C \) such that
\[ (2) \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| < s - n/2} |\partial^\alpha u(x)| \leq C \|u\|_s. \]

**Proof.** See [6], Corollary 7.9.4. One can obtain inequality (2) by a standard calculation.

Assume that \( (H, \langle \cdot, \cdot \rangle_H) \) is a complex Hilbert space. Let \( L^2(\mathbb{R}^n, H) \) be the Hilbert space of measurable functions \( u : \mathbb{R}^n \to H \) for which
\[ \|u\|_{L^2(\mathbb{R}^n, H)}^2 := \int \|u(x)\|_H^2 \, dx < \infty. \]
The scalar product in \( L^2(\mathbb{R}^n, H) \) is defined by
\[ \langle u, w \rangle_{L^2(\mathbb{R}^n, H)} := \int \langle u(x), w(x) \rangle_H \, dx. \]

Let \( (e_\gamma)_{\gamma \in \Gamma} \) be a complete orthonormal system in \( H \). For \( u \in L^2(\mathbb{R}^n, H) \), let
\[ u_\gamma(x) := \langle u(x), e_\gamma \rangle_H. \]

By the Bessel inequality, for any finite set \( \Gamma' \subseteq \Gamma \), we have
\[ \|u\|_{L^2(\mathbb{R}^n, H)}^2 = \int \|u(x)\|_H^2 \, dx \geq \sum_{\gamma \in \Gamma'} |u_\gamma(x)|^2 \, dx \]
\[ = \sum_{\gamma \in \Gamma'} \int |u_\gamma(x)|^2 \, dx = \sum_{\gamma \in \Gamma'} \|u_\gamma\|_0^2. \]

Hence at most countable many \( u_\gamma \) are nonzero outside a set of measure zero. From the Lebesgue theorem and the Parseval equality, we have
\[ (3) \|u\|_{L^2(\mathbb{R}^n, H)}^2 = \sum_{\gamma \in \Gamma} \|u_\gamma\|_0^2. \]
We define the Fourier transform for $L^2(\mathbb{R}^n, H)$ by means of the Fourier transform in $L^2(\mathbb{R}^n, C)$:

(4) \[ F u := \sum_{\gamma \in \Gamma} (F u_\gamma)e_\gamma. \]

One can verify that this definition is independent of the choice of a complete orthonormal system $(e_\gamma)_{\gamma \in \Gamma}$ and that $F$ is an isomorphism of $L^2(\mathbb{R}^n, H)$ onto itself and, by (3),

(5) \[ \|u\|_{L^2(\mathbb{R}^n, H)}^2 = (2\pi)^{-n}\|F u\|_{L^2(\mathbb{R}^n, H)}^2 \]

for any $u \in L^2(\mathbb{R}^n, H)$.

For $s \geq 0$, we define the space $\mathcal{H}^s(\mathbb{R}^n, H)$ to be

\[ \left\{ u \in L^2(\mathbb{R}^n, H) : \|u\|_{\mathcal{H}^s(\mathbb{R}^n, H)} := (2\pi)^{-n}\int \|F u(\xi)\|_H^2(1 + |\xi|^2)^s d\xi < \infty \right\}. \]

$\mathcal{H}^s(\mathbb{R}^n, H)$ is a Hilbert space with the scalar product

\[ \langle u, w \rangle_{\mathcal{H}^s(\mathbb{R}^n, H)} := (2\pi)^{-n}\int \langle F u(\xi), F w(\xi) \rangle_H(1 + |\xi|^2)^s d\xi. \]

In the case of a real Hilbert space $H$, we mean by $\mathcal{H}^s(\mathbb{R}^n, H)$ the real Hilbert space

\[ \{ u \in \mathcal{H}^s(\mathbb{R}^n, H + iH) : u(x) \in H \text{ for almost every } x \in \mathbb{R}^n \}. \]

We shall use derivatives of $\mathcal{H}^s(\mathbb{R}^n, H)$ functions in the following weak sense:

**Definition 2.** Let $u \in \mathcal{H}^s(\mathbb{R}^n, H)$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leq s$. We denote by $\partial^\alpha u$ an element of $L^2(\mathbb{R}^n, H)$ such that

(6) \[ \langle \partial^\alpha u(\cdot), h \rangle_H = \partial^\alpha \langle u(\cdot), h \rangle_H \quad \text{for any } h \in H. \]

Note that if $u(x) = \sum_{\gamma \in \Gamma} u_\gamma(x)e_\gamma$, then $\partial^\alpha u(x) = \sum_{\gamma \in \Gamma} \partial^\alpha u_\gamma(x)e_\gamma$.

**3. Existence theorem for a single equation.** We shall construct a solution of an equation of the form (1) in the space $\mathcal{H}^s(\mathbb{R}^n, H)$ by approximation by a sequence of solutions of adapted problems “with values in finite-dimensional spaces”. The following lemma plays the basic role in this construction:

**Lemma 3.** Every sequence $(u_k)$ in $\mathcal{H}^s(\mathbb{R}^n, H)$ weakly convergent to $u \in \mathcal{H}^s(\mathbb{R}^n, H)$ contains a subsequence $(u_{k_l})$ for which the sequences $(\partial^\alpha u_{k_l}(x))$ weakly converge in $H$ to $(\partial^\alpha u(x))$ for a.e. $x$ and $|\alpha| < s - n/2$.

**Proof.** The weak convergence of $(u_k)$ implies its boundedness:

(7) \[ \|u_k\|_{\mathcal{H}^s(\mathbb{R}^n, H)} \leq M \]

for some constant $M$. 
The essential ranges of the functions \( u_k, k = 1, 2, \ldots \) (without the values on a set of measure zero) are contained in a separable subspace of \( H \) (see Section 2). Hence we can assume that \( H \) is separable. Let \( (e_\gamma)_{\gamma \in \Gamma} \) be a complete orthonormal system in \( H \) (at most countable).

The weak convergence of \((u_k)\) in \( H^s(\mathbb{R}^n, H) \) implies that
\[
\langle \partial^\alpha u_k(\cdot), e_\gamma \rangle_H \to \langle \partial^\alpha u(\cdot), e_\gamma \rangle_H \quad \text{weakly in } L^2
\]
for any \(|\alpha| \leq s \) and \( \gamma \in \Gamma \).

From (7), we have
\[
\|\langle u_k(\cdot), e_\gamma \rangle_H \|_s \leq M.
\]
Making use of (9) and Lemma 1, we construct by the diagonal method a subsequence \((u_k)\) such that, for any \( \gamma \in \Gamma \),
\[
\langle u_k(\cdot), e_\gamma \rangle_H \to w_\gamma \in H_{loc}^{s-n/2} \quad \text{in } H_{loc}^{s-n/2}.
\]

For \( |\alpha| \leq s - n/2 \), we have
\[
\langle \partial^\alpha u_k(\cdot), e_\gamma \rangle_H \to \partial^\alpha w_\gamma \quad \text{in } L^2_{loc}, \quad \text{for any } \gamma \in \Gamma.
\]

Comparing (8) and (10) for \( \alpha = (0, \ldots, 0) \), we obtain \( \langle u(x), e_\gamma \rangle_H = w_\gamma(x) \) for a.e. \( x \), hence \( \langle u_k(\cdot), e_\gamma \rangle_H \to \langle u(\cdot), e_\gamma \rangle_H \) in \( H_{loc}^{s-n/2} \). By the diagonal method, we construct a subsequence (denoted again by \( u_k \)) for simplicity of notation such that
\[
\langle \partial^\alpha u_k(\cdot), e_\gamma \rangle_H \to \partial^\alpha w_\gamma \quad \text{for a.e. } x,
\]
for any \(|\alpha| \leq s - n/2 \) and \( \gamma \in \Gamma \). We shall show that this is the desired subsequence. By (11), it is sufficient to show that, for \(|\alpha| < s - n/2 \), the set \( \{\|\partial^\alpha u_k(\cdot)\|\} \) is bounded for a.e. \( x \). From (2) and (7), we have
\[
\|\partial^\alpha u_k(\cdot)\|^2 = \sum_{\gamma \in \Gamma} \|\langle \partial^\alpha u_k(\cdot), e_\gamma \rangle_H \|^2 \leq C^2 \sum_{\gamma \in \Gamma} \|\langle u_k(\cdot), e_\gamma \rangle_H \|^2_s = C^2 \|u_k\|_{H^s(\mathbb{R}^n, H)}^2 \leq C^2 M^2,
\]
which ends the proof.

We now formulate and prove the main

**Theorem 1.** Let \( H \) be a real, infinite-dimensional separable Hilbert space, and \( (e_\gamma)_{\gamma = 1, 2, \ldots} \) a complete orthonormal system in \( H \). Let \( H_p \) denote the space generated by \( \{e_\gamma : \gamma = 1, \ldots, p\} \) and let \( R_p : H \to H_p \) be the orthonormal projector onto \( H_p \). Let \( P \) be a polynomial of \( n \) variables and degree \( T \) such that the polynomial \( P(-i\partial) \) of the variable \( \partial \) has real coefficients and satisfies the condition
\[
1 + |\xi|^T \leq C P(\xi), \quad \xi \in \mathbb{R}^n.
\]

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1 + |\xi|^T \leq C P(\xi), \quad \xi \in \mathbb{R}^n.
\]
Fix $t \in [0,T]$ and set

$$m := \sum_{0 \leq l < t - n/2} n^l.$$  

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to H$ satisfy the Carathéodory condition of the following form: $F(x, \cdot)$ is sequentially continuous in the weak topologies of $H^m$ and $H$ for a.e. $x$, and $F(\cdot, (v_\alpha)_{|\alpha| < t - n/2})$ is measurable for all $(v_\alpha)_{|\alpha| < t - n/2} \in H^m$.

Suppose that for any bounded set $K \subset \mathbb{R}^n \times H^m$ there exists a function $h_K \in L^2(\mathbb{R}^n)$ such that

$$\|F(x, (v_\alpha)_{|\alpha| < t - n/2})\| \leq h_K(x) \quad \text{for a.e. } x$$

for $(x, (v_\alpha)_{|\alpha| < t - n/2}) \in K$ ($\| \cdot \|$ denotes the norm in $H^k$ for any $k \in \mathbb{N}$). Assume that there is a sequence of open bounded sets $U_1 \subset U_2 \subset \ldots$ with $\bigcup U_j = \mathbb{R}^n$ and a constant $M$ such that no equation

$$P(D)u = \lambda R_p F_j(x, (\partial^\alpha u)_{|\alpha| < t - n/2}), \quad j = 1, 2, \ldots, \lambda \in [0,1],$$

with

$$F_j(x, (v_\alpha)_{|\alpha| < t - n/2}) := \begin{cases} F(x, (v_\alpha)_{|\alpha| < t - n/2}) & \text{for } x \in U_j, \\ 0 & \text{for } x \notin U_j \end{cases}$$

has a solution in the set

$$\{ u \in H^t(\mathbb{R}^n, H) : \| u \|_{H^t(\mathbb{R}^n, H)} > M \}, \quad p = 1, 2, \ldots,$$

Under these assumptions the equation

$$P(D)u = F(x, (\partial^\alpha u)_{|\alpha| < t - n/2}),$$

understood in the sense of Definition 2, has a solution $u \in H^t(\mathbb{R}^n, H)$ for which

$$\| u \|_{H^t(\mathbb{R}^n, H)} \leq M.$$

\textbf{Proof.} Consider the equations

$$P(D)u = R_p F(x, (\partial^\alpha u)_{|\alpha| < t - n/2}), \quad p = 1, 2, \ldots$$

Treating $H_p$ as $\mathbb{R}^p$, we conclude, from the assumptions of the theorem, that equation (17) has a solution $u_p \in H^t(\mathbb{R}^n, H_p)$ for any $p$ (see [4], Theorem 2). We have

$$\| u_p \|_{H^t(\mathbb{R}^n, H)} \leq M, \quad p = 1, 2, \ldots$$

By the Eberlein–Shmul’yyan theorem, the sequence $(u_p)$ contains a subsequence $(u_{p_k})$ weakly convergent to some $u \in H^t(\mathbb{R}^n, H)$ and

$$\| u \|_{H^t(\mathbb{R}^n, H)} \leq M.$$

By Lemma 3, we may assume that the sequences $(\partial^\alpha u_{p_k}(x))$, $|\alpha| < t - n/2$, weakly converge in $H$ to the corresponding $\partial^\alpha u(x)$ for a.e. $x$. 


We shall prove that \( u \) is a solution of equation (16). We have to show that
\[
P(D)(u(\cdot), h)_H = \langle F(\cdot, (\partial^a u(\cdot)))_{|a|<t-n/2}, h \rangle_H
\]
in \( \mathcal{D}' \) for any \( h \in H \). We know that
\[
P(D)(u_{p_k}(\cdot), h)_H = \langle R_{p_k} F(\cdot, (\partial^a u_{p_k}(\cdot))_{|a|<t-n/2}), h \rangle_H
\]
in \( \mathcal{D}' \). We shall prove that (17) follows from (19) by passing to the limit in \( \mathcal{D}' \) as \( k \to \infty \).

Let \( \varphi \in C^\infty_0 \). We have
\[
\int \varphi(x) (u_{p_k}(x), h)_H \, dx = \int \langle u_{p_k}(x), \varphi(x)h \rangle_H \, dx
\]
because \( \int \langle \cdot, \varphi(x)h \rangle_H \, dx \) is a continuous linear functional on \( \mathcal{H}'(\mathbb{R}^n, H) \). Now, from the sequential continuity of \( P(D) \) in \( \mathcal{D}' \), we conclude that the left-hand side of (19) converges to the left-hand side of (18) in \( \mathcal{D}' \). We shall prove the same for the right-hand sides, which means that
\[
\int \varphi(x)(R_{p_k} F(x, (\partial^a u_{p_k}(x)))_{|a|<t-n/2}, h)_H \, dx
\]
\[
\to \int \varphi(x)(F(x, (\partial^a u(x)))_{|a|<t-n/2}, h)_H \, dx
\]
for any \( \varphi \in C^\infty_0 \).

We show first that
\[
(R_{p_k} F(x, (\partial^a u_{p_k}(x)))_{|a|<t-n/2}, h)_H
\]
\[
\to (F(x, (\partial^a u(x)))_{|a|<t-n/2}, h)_H \quad \text{for a.e. } x .
\]
Assume that \( h \in H_t \) for some \( t \). Then, for large \( k \),
\[
(R_{p_k} F(x, (\partial^a u_{p_k}(x)))_{|a|<t-n/2}, h)_H = (F(x, (\partial^a u_{p_k}(x)))_{|a|<t-n/2}, h)_H ,
\]
hence (21) is true by the Carathéodory condition.

From (12) and (14), we have
\[
\| F(x, (\partial^a u_{p_k}(x)))_{|a|<t-n/2} \| \leq C(x) < \infty \quad \text{for a.e. } x .
\]
This implies (21) for all \( h \in H \) (see [9], p. 121, Theorem 3). By the Lebesgue convergence theorem, formulas (12), (14) and (21) imply the convergence (20). The proof is complete.

**Example 1.** We define a class of equations satisfying the assumptions of Theorem 1.

Let \( P \) be a polynomial of \( n \) variables and degree \( T \) such that the polynomial \( P(-i\partial) \) of the variable \( \partial \) has real coefficients and satisfies (13). Let \( F : \mathbb{R}^n \times \mathbb{R}^m \to H \) satisfy (14) and the Carathéodory condition in the sense of Theorem 1. Assume that there exist constants \( 0 < a < 2, L > 0 \) and
nonnegative functions \( f \in L^{2/a} \) and \( g \in L^{2/(2-a)} \) such that

\[
\langle v(0,...,0), F(x, (v_\alpha)_{|\alpha|<t-n/2}) \rangle_H \leq 0
\]
for \( \|v(0,...,0)\| \geq g(x) \) and a.e. \( x \),

and

\[
\|F(x, (v_\alpha)_{|\alpha|<t-n/2})\| \leq f(x) + L\|v(0,...,0)\|
\]
for \( \|v(0,...,0)\| \leq g(x) \) and a.e. \( x \).

Treating \( H_p \) as \( \mathbb{R}^p \), we obtain the necessary a priori bounds for solutions of equations (15) as in [4], Example 2.

Example 2. We now describe a more concrete example of the class described above.

Let \( n = 1, T = 2, a = 1, P(\xi) = \xi^2 + b, b > 0, \) and \( A : H \to H \) a linear, continuous, invertible operator. Suppose that \( B : \mathbb{R} \times H \to H \) satisfies the Carathéodory condition in the sense of Theorem 1 and

\[
\|B(x,v)\| \leq h(x), \quad h \in L^2(\mathbb{R}).
\]

Let \( F(x,v) = -A^*Av + B(x,v) \). We have

\[
\langle v, F(x,v) \rangle_H = -\langle v, A^*Av \rangle + \langle v, B(x,v) \rangle
\]
\[
= -\langle Av, Av \rangle + \langle v, B(x,v) \rangle
\]
\[
\leq -C\|v\|^2 + \|v\| h(x) \leq 0
\]
for \( \|v\| \geq g(x) := h(x)/C \) with some constant \( C > 0 \). Then condition (23) is satisfied. Condition (24) is satisfied for \( L = 0 \) and

\[
f(x) = (\|A^*A\|/C + 1)h(x).
\]

Consider, for example, the following problem:

\[
-\frac{d^2u(x,t)}{dx^2} + u(x,t) = -u(x,t) + \psi(x, \int_0^1 K(x,t,\tau)u(x,\tau) d\tau)
\]
where \( K \) is measurable, \( K(x,\cdot,\cdot) \in L^2([0,1] \times [0,1]) \) for a.e. \( x, \psi : \mathbb{R}^2 \to \mathbb{R} \) is continuous and

\[
|\psi(x,y)| \leq h(x), \quad h \in L^2(x).
\]

We look for \( u \in H^1(\mathbb{R}, L^2([0,1])) \) (we treat \( u \) as the mapping \( x \mapsto u(x,\cdot) \)).

We have

\[
P(\xi) = \xi^2 + 1
\]
and

\[
F(x,v) = -v + \psi(x, \int_0^1 K(x,\cdot,\tau)v(\tau) d\tau).
\]
The function $F$ satisfies the Carathéodory condition. In fact, the map

$$L^2([0, 1]) \ni v \mapsto \frac{1}{0} \int K(x, \cdot, \tau) \, d\tau \in L^2([0, 1])$$

is linear and completely continuous (for almost all $x$), hence it transforms weakly convergent sequences to strongly convergent ones. The Nemytskiĭ operator

$$L^2([0, 1]) \ni v \mapsto \psi(x, v(\cdot)) \in L^2([0, 1])$$

is continuous by (25) (see for example [2], Proposition 1).

Remark 2. Note that a Hammerstein operator does not have so good properties as the operator $F$ defined in Example 2. Consider the operator

$$G(v) = \int_0^1 K(\cdot, \tau)\psi(\tau, v(\tau)) \, d\tau,$$

where

$$K \in L^2([0, 1] \times [0, 1]), \quad |\psi(t, y)| \leq h(t) + |y|, \quad h \in L^2([0, 1]).$$

Suppose that $G : L^2([0, 1]) \to L^2([0, 1])$ is sequentially continuous in the sense of the weak topology in $L^2([0, 1])$. Then, for any $w \in L^2([0, 1])$, the map

$$G_w : v \mapsto \langle G(v), w \rangle_{L^2([0, 1])}$$

transforms weakly convergent sequences in $L^2([0, 1])$ to convergent numerical ones.

Suppose that $\psi$ is differentiable with respect to the second variable and that $G_w : L^2([0, 1]) \to \mathbb{R}$ satisfies the assumptions of the following theorem of Palmer (see [7]):

Let $X$ be a reflexive Banach space, $Y$ a Banach space and let $F : X \to Y$ be uniformly Fréchet differentiable on any ball in $X$. Then $F$ is sequentially continuous with the weak topology in $X$ and the strong topology in $Y$ if and only if the following two conditions are satisfied:

(i) for any $v \in X$ the Fréchet derivative $F'(v)$ is a completely continuous linear operator,

(ii) the Fréchet derivative $F' : X \to L(X, Y)$ (the space of linear continuous operators from $X$ into $Y$) is completely continuous.

We have

$$G_w(v) = \int_0^1 \int_0^1 K(t, \tau)\psi(\tau, v(\tau)) w(t) \, d\tau \, dt = \int_0^1 K_w(\tau)\psi(\tau, v(\tau)) \, d\tau,$$
where
\[ K_w(\tau) := \int_0^1 K(t, \tau)w(t)\,dt. \]

Compute the derivative
\[ G'_w(v) \cdot h = \int_0^1 K_w(\tau)\partial_2\psi(\tau, v(\tau))h(\tau)\,d\tau. \]

By the isomorphism \( L(L^2([0, 1]); \mathbb{R}) \cong L^2([0, 1]) \), we have
\[ G'_w(v) = K_w(\cdot)\partial_2\psi(\cdot, v(\cdot)). \]

We conclude that the condition (ii) will not be satisfied if \( G'_w \) is not constant. In fact, any nonconstant superposition operator
\[ L^2([0, 1]) \ni v \mapsto N(v) := \varphi(\cdot, v(\cdot)) \in L^2([0, 1]) \]
does not transform bounded sets onto precompact ones. In fact, let \( N(u_1) \neq N(u_2) \) for some \( u_1, u_2 \in L^2([0, 1]) \).
Let
\[ v_k(x) := \begin{cases} u_1(x) & \text{for } x \in [2^{-k}2p, 2^{-k}(2p + 1)], \\ u_2(x) & \text{for } x \in [2^{-k}(2p + 1), 2^{-k}(2p + 2)], \\ 0 & \text{for } x = 1, p = 0, 1, \ldots, 2^k - 2. \end{cases} \]

The sequence \( (v_k) \) is bounded in \( L^2([0, 1]) \) but \( N(v_k) \) has no subsequence which converges in \( L^2([0, 1]) \).

4. Existence theorem for a system of equations. We formulate a theorem similar to Theorem 1 for systems of equations.

**Theorem 2.** Let \( H \) be a real infinite-dimensional separable Hilbert space, and \((e_\gamma)_{\gamma=1,2,\ldots} \) a complete orthonormal system in \( H \). Let \( H_p \) denote the space generated by the system \( \{e_\gamma: \gamma = 1, \ldots, p\} \) and let \( R_p: H \to H_p \) be the orthonormal projector onto \( H_p \). Let \( P_r \) be polynomials of \( n \) variables and degrees \( T_r \) such that the polynomials \( P_r(-i\partial) \) of the variable \( \partial \) have real coefficients and satisfy
\[ 1 + |\xi|^{T_r} \leq CP_r(\xi), \quad \xi \in \mathbb{R}^n, \quad r = 1, \ldots, k, \]
for some constant \( C \). Let \( t_r \in [0, T_r] \), \( r = 1, \ldots, k \), and
\[ m := k \sum_{r=1}^k \sum_{0 \leq l < t_r - n/2} n_l. \]

Assume that \( F: \mathbb{R}^n \times H^m \to H^k \) satisfies the Carathéodory condition of the following form: \( F(x, \cdot) \) is sequentially continuous in the weak topologies of \( H^m \) and \( H^k \) for a.e. \( x \) and \( F(\cdot, (v^\alpha_r)_{|\alpha|<t_r-n/2,r=1,\ldots,k}) \) is measurable for
all \((v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k} \in H^m\). Assume that for any bounded set \(K \subset \mathbb{R}^n \times H^m\) there exists a function \(h_K \in L^2(\mathbb{R}^n)\) such that
\[
\|F(x,(v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k})\| \leq h_K(x)
\]
for \((x,(v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k}) \in K\) a.e. \(x\). Assume that there is a sequence of open bounded sets \(U_1 \subset U_2 \subset \ldots\) with \(\bigcup U_j = \mathbb{R}^n\) and a constant \(M\) such that no system
\[
P_l(D)u^l = \lambda R_p F^l_j(x,(\partial^\alpha u^r)_{|\alpha|<t_r-n/2, r=1,\ldots,k}),
\]
\(l = 1,\ldots,k\) \((F = (F^1,\ldots,F^k))\),
has a solution in the set
\[
\left\{u = (u^1,\ldots,u^k) \in \prod_{r=1}^k H^r(\mathbb{R}^n,H) : \sum_{r=1}^k \|u^r\|^2_{H^r(\mathbb{R}^n,H)} > M^2 \right\}
\]
for \(j = 1, 2,\ldots, \lambda \in [0,1], p = 1, 2,\ldots\) (The functions \(F^l_j\) are defined as
\[
F^l_j(x,(v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k}) := \begin{cases} 
F^l(x,(v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k}) & \text{for } x \in U_j, \\
0 & \text{for } x \notin U_j. 
\end{cases}
\]
Under these assumptions the system
\[
P_l(D)u^l = F^l(x,(\partial^\alpha u^r)_{|\alpha|<t_r-n/2, r=1,\ldots,k}), \quad l = 1,\ldots,k,
\]
has a solution \(u \in \prod_{r=1}^k H^r(\mathbb{R}^n,H)\) for which
\[
\sum_{r=1}^k \|u^r\|^2_{H^r(\mathbb{R}^n,H)} \leq M.
\]

Proof. Similar to the proof of Theorem 1.

Example 3. One can construct an example analogous to Example 1 with the condition
\[
\langle v_{(0,\ldots,0)}, F(x,(v^r_{\alpha})_{|\alpha|<t_r-n/2, r=1,\ldots,k}) \rangle_{H^k} \leq 0
\]
for \(\|v_{(0,\ldots,0)}\| \geq g(x)\) for a.e. \(x\)

instead of (23).

References


Nonlinear elliptic equations


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