ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

On a class of nonlinear elliptic equations in Hilbert spaces

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Abstract. We consider elliptic nonlinear equations in a separable Hilbert space and their solutions in spaces of Sobolev type.

1. Introduction. We study equations of the form

$$P(D)u = F(x, (\partial^\alpha u)) \qquad (D = -i\partial)\,, \tag{1}$$

with a strongly elliptic polynomial P of n variables, defined for $u : \mathbb{R}^n \to H$, where H is a separable Hilbert space. The equations are understood in a weak sense (see Definition 2). We make assumptions giving an a priori bound for solutions in a space of Sobolev type. As an example, we consider assumptions of Bernstein type. Assumptions of this kind appear in the papers [1], [5] concerning equations on a bounded interval, and in [8], [3] and [4] concerning equations on the half-line, on the line and in $\mathbb{R}^n$, respectively.

2. Spaces of Sobolev type

Definition 1. We denote by $\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^n)$, for $s \in \mathbb{R}$, the Sobolev space of real tempered distributions u such that

$$\|u\|_s^2 := (2\pi)^{-n} \int |\mathcal{F}u(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty$$

where $\mathcal{F}$ stands for the Fourier transform.

Remark 1. Definition 1 may be used for both real and complex Sobolev spaces, depending on whether we consider real or complex functions and distributions.

$\mathcal{H}^s$ is a Hilbert space with the scalar product

$$\langle u, w\rangle_s := (2\pi)^{-n} \int (\mathcal{F}u)(\xi)\overline{(\mathcal{F}w)(\xi)}(1 + |\xi|^2)^s \, d\xi$$

1991 *Mathematics Subject Classification*: Primary 35J65.
Key words and phrases: nonlinear elliptic boundary value problems.

in the complex case or

$$\langle u,w\rangle_s := \operatorname{Re}(2\pi)^{-n}\int (\mathcal{F}u)(\xi)\overline{(\mathcal{F}w)(\xi)}(1+|\xi|^2)^s\,d\xi$$

in the real case.

We denote the local Sobolev space by $\mathcal{H}^s_{\mathrm{loc}} = \mathcal{H}^s_{\mathrm{loc}}(\mathbb{R}^n)$ and treat it as a Fréchet space in the standard way (see for example [6]).

Note two important lemmas.

LEMMA 1. *The embedding $\mathcal{H}^s_{\mathrm{loc}} \to \mathcal{H}^{s'}_{\mathrm{loc}}$, for $s > s'$, is completely continuous.*

The proof is in [6], Theorem 10.1.27.

LEMMA 2. *If $u \in \mathcal{H}^s$ then any $\partial^\alpha u$, for $|\alpha| < s - n/2$, is a continuous bounded function and there exists a constant C such that*

$$\sup_{x\in\mathbb{R}^n}\sup_{|\alpha|<s-n/2} |\partial^\alpha u(x)| \le C\|u\|_s\,. \tag{2}$$

Proof. See [6], Corollary 7.9.4. One can obtain inequality (2) by a standard calculation.

Assume that $(H,\langle\cdot,\cdot\rangle_H)$ is a complex Hilbert space. Let $L^2(\mathbb{R}^n,H)$ be the Hilbert space of measurable functions $u:\mathbb{R}^n\to H$ for which

$$\|u\|^2_{L^2(\mathbb{R}^n,H)} := \int \|u(x)\|^2_H\,dx < \infty\,.$$

The scalar product in $L^2(\mathbb{R}^n,H)$ is defined by

$$\langle u,w\rangle_{L^2(\mathbb{R}^n,H)} := \int \langle u(x),w(x)\rangle_H\,dx\,.$$

Let $(e_\gamma)_{\gamma\in\Gamma}$ be a complete orthonormal system in H. For $u\in L^2(\mathbb{R}^n,H)$, let

$$u_\gamma(x) := \langle u(x),e_\gamma\rangle_H\,.$$

By the Bessel inequality, for any finite set $\Gamma'\subset\Gamma$, we have

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^n,H)} &= \int \|u(x)\|^2_H\,dx \ge \int \sum_{\gamma\in\Gamma'} |u_\gamma(x)|^2\,dx\\ &= \sum_{\gamma\in\Gamma'}\int |u_\gamma(x)|^2\,dx = \sum_{\gamma\in\Gamma'}\|u_\gamma\|^2_0\,.\end{aligned}$$

Hence at most countable many u_γ are nonzero outside a set of measure zero. From the Lebesgue theorem and the Parseval equality, we have

$$\|u\|^2_{L^2(\mathbb{R}^n,H)} = \sum_{\gamma\in\Gamma}\|u_\gamma\|^2_0\,. \tag{3}$$

We define the Fourier transform for $L^2(\mathbb{R}^n, H)$ by means of the Fourier transform in $L^2(\mathbb{R}^n, \mathbb{C})$:

$$\mathcal{F}u := \sum_{\gamma\in\Gamma}(\mathcal{F}u_\gamma)e_\gamma\,. \tag{4}$$

One can verify that this definition is independent of the choice of a complete orthonormal system $(e_\gamma)_{\gamma\in\Gamma}$ and that $\mathcal{F}$ is an isomorphism of $L^2(\mathbb{R}^n, H)$ onto itself and, by (3),

$$\|u\|^2_{L^2(\mathbb{R}^n,H)} = (2\pi)^{-n}\|\mathcal{F}u\|^2_{L^2(\mathbb{R}^n,H)} \tag{5}$$

for any $u \in L^2(\mathbb{R}^n, H)$.

For $s \geq 0$, we define the space $\mathcal{H}^s(\mathbb{R}^n, H)$ to be

$$\Big\{u \in L^2(\mathbb{R}^n, H) : \|u\|^2_{\mathcal{H}^s(\mathbb{R}^n,H)} := (2\pi)^{-n}\int \|\mathcal{F}u(\xi)\|^2_H(1+|\xi|^2)^s\,d\xi < \infty\Big\}\,.$$

$\mathcal{H}^s(\mathbb{R}^n, H)$ is a Hilbert space with the scalar product

$$\langle u, w\rangle_{\mathcal{H}^s(\mathbb{R}^n,H)} := (2\pi)^{-n}\int \langle\mathcal{F}u(\xi), \mathcal{F}w(\xi)\rangle_H(1+|\xi|^2)^s\,d\xi\,.$$

In the case of a real Hilbert space H, we mean by $\mathcal{H}^s(\mathbb{R}^n, H)$ the real Hilbert space

$$\{u \in \mathcal{H}^s(\mathbb{R}^n, H + iH) : u(x) \in H \text{ for almost every } x \in \mathbb{R}^n\}\,.$$

We shall use derivatives of $\mathcal{H}^s(\mathbb{R}^n, H)$ functions in the following weak sense:

DEFINITION 2. Let $u \in \mathcal{H}^s(\mathbb{R}^n, H)$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leq s$. We denote by $\partial^\alpha u$ an element of $L^2(\mathbb{R}^n, H)$ such that

$$\langle\partial^\alpha u(\cdot), h\rangle_H = \partial^\alpha\langle u(\cdot), h\rangle_H \quad \text{for any } h \in H\,. \tag{6}$$

Note that if $u(x) = \sum_{\gamma\in\Gamma} u_\gamma(x)e_\gamma$, then $\partial^\alpha u(x) = \sum_{\gamma\in\Gamma}\partial^\alpha u_\gamma(x)e_\gamma$.

3. Existence theorem for a single equation. We shall construct a solution of an equation of the form (1) in the space $\mathcal{H}^t(\mathbb{R}^n, H)$ by approximation by a sequence of solutions of adapted problems "with values in finite-dimensional spaces". The following lemma plays the basic role in this construction:

LEMMA 3. *Every sequence (u_k) in $\mathcal{H}^s(\mathbb{R}^n, H)$ weakly convergent to $u \in \mathcal{H}^s(\mathbb{R}^n, H)$ contains a subsequence (u_{k_l}) for which the sequences $(\partial^\alpha u_{k_l}(x))$ weakly converge in H to $(\partial^\alpha u)(x)$ for a.e. x and $|\alpha| < s - n/2$.*

Proof. The weak convergence of (u_k) implies its boundedness:

$$\|u_k\|_{\mathcal{H}^s(\mathbb{R}^n,H)} \leq M \tag{7}$$

for some constant M.

The essential ranges of the functions $u_k,\ k = 1, 2, \ldots$ (without the values on a set of measure zero) are contained in a separable subspace of H (see Section 2). Hence we can assume that H is separable. Let $(e_\gamma)_{\gamma\in\Gamma}$ be a complete orthonormal system in H (at most countable).

The weak convergence of (u_k) in $\mathcal{H}^s(\mathbb{R}^n, H)$ implies that

$$\langle\partial^\alpha u_k(\cdot), e_\gamma\rangle_H \to \langle\partial^\alpha u(\cdot), e_\gamma\rangle_H \quad \text{weakly in } L^2 \tag{8}$$

for any $|\alpha| \le s$ and $\gamma \in \Gamma$.

From (7), we have

$$\|\langle u_k(\cdot), e_\gamma\rangle_H\|_s \le M\,. \tag{9}$$

Making use of (9) and Lemma 1, we construct by the diagonal method a subsequence (u_{k_l}) such that, for any $\gamma \in \Gamma$,

$$\langle u_{k_l}(\cdot), e_\gamma\rangle_H \to w_\gamma \in \mathcal{H}^{s-n/2}_{\mathrm{loc}} \quad \text{in } \mathcal{H}^{s-n/2}_{\mathrm{loc}}\,.$$

For $|\alpha| \le s - n/2$, we have

$$\langle\partial^\alpha u_{k_l}(\cdot), e_\gamma\rangle_H \to \partial^\alpha w_\gamma \quad \text{in } L^2_{\mathrm{loc}},\ \text{for any } \gamma \in \Gamma\,. \tag{10}$$

Comparing (8) and (10) for $\alpha = (0, \ldots, 0)$, we obtain $\langle u(x), e_\gamma\rangle_H = w_\gamma(x)$ for a.e. x, hence $\langle u_{k_l}(\cdot), e_\gamma\rangle_H \to \langle u(\cdot), e_\gamma\rangle_H$ in $\mathcal{H}^{s-n/2}_{\mathrm{loc}}$. By the diagonal method, we construct a subsequence (denoted again by (u_{k_l}) for simplicity of notation) such that

$$\langle\partial^\alpha u_{k_l}(x), e_\gamma\rangle_H \to \langle\partial^\alpha u(x), e_\gamma\rangle_H \quad \text{for a.e. } x\,, \tag{11}$$

for any $|\alpha| \le s - n/2$ and $\gamma \in \Gamma$. We shall show that this is the desired subsequence. By (11), it is sufficient to show that, for $|\alpha| < s - n/2$, the set $\{\|\partial^\alpha u_{k_l}(x)\|\}$ is bounded for a.e. x. From (2) and (7), we have

$$\begin{aligned}\|\partial^\alpha u_{k_l}(x)\|^2 &= \sum_{\gamma\in\Gamma} |\langle\partial^\alpha u_{k_l}(x), e_\gamma\rangle_H|^2 \\ &\le C^2 \sum_{\gamma\in\Gamma} \|\langle u_{k_l}(\cdot), e_\gamma\rangle_H\|_s^2 \\ &= C^2\|u_{k_l}\|^2_{\mathcal{H}^s(\mathbb{R}^n,H)} \le C^2M^2\,,\end{aligned} \tag{12}$$

which ends the proof.

We now formulate and prove the main

Theorem 1. *Let H be a real, infinite-dimensional separable Hilbert space, and $(e_\gamma)_{\gamma=1,2,\ldots}$ a complete orthonormal system in H. Let H_p denote the space generated by $\{e_\gamma : \gamma = 1, \ldots, p\}$ and let $R_p : H \to H_p$ be the orthonormal projector onto H_p. Let P be a polynomial of n variables and degree T such that the polynomial $P(-i\partial)$ of the variable ∂ has real coefficients and satisfies the condition*

$$1 + |\xi|^T \le CP(\xi)\,, \quad \xi \in \mathbb{R}^n\,. \tag{13}$$

Fix $t \in [0, T[$ *and set*

$$m := \sum_{0 \le l < t - n/2} n^l \,.$$

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to H$ *satisfy the Carathéodory condition of the following form*: $F(x, \cdot)$ *is sequentially continuous in the weak topologies of* H^m *and* H *for a.e.* x, *and* $F(\cdot, (v_\alpha)_{|\alpha|<t-n/2})$ *is measurable for all* $(v_\alpha)_{|\alpha|<t-n/2} \in H^m$.

Suppose that for any bounded set $K \subset \mathbb{R}^n \times H^m$ *there exists a function* $h_K \in L^2(\mathbb{R}^n)$ *such that*

$$\|F(x, (v_\alpha)_{|\alpha|<t-n/2})\| \le h_K(x) \quad \textit{for a.e. } x \tag{14}$$

for $(x, (v_\alpha)_{|\alpha|<t-n/2}) \in K$ ($\|\cdot\|$ *denotes the norm in* H^k *for any* $k \in \mathbb{N}$). *Assume that there is a sequence of open bounded sets* $U_1 \subset U_2 \subset \ldots$ *with* $\bigcup U_j = \mathbb{R}^n$ *and a constant* M *such that no equation*

$$P(D)u = \lambda R_p F_j(x, (\partial^\alpha u)_{|\alpha|<t-n/2}) \,, \quad j = 1, 2, \ldots, \ \lambda \in [0, 1] \,, \tag{15}$$

with

$$F_j(x, (v_\alpha)_{|\alpha|<t-n/2}) := \begin{cases} F(x, (v_\alpha)_{|\alpha|<t-n/2}) & \textit{for } x \in U_j \,, \\ 0 & \textit{for } x \notin U_j \end{cases}$$

has a solution in the set

$$\{u \in \mathcal{H}^t(\mathbb{R}^n, H_p) : \|u\|_{\mathcal{H}^t(\mathbb{R}^n, H)} > M\} \,, \quad p = 1, 2, \ldots$$

Under these assumptions the equation

$$P(D)u = F(x, (\partial^\alpha u)_{|\alpha|<t-n/2}) \,, \tag{16}$$

understood in the sense of Definition 2, *has a solution* $u \in \mathcal{H}^t(\mathbb{R}^n, H)$ *for which*

$$\|u\|_{\mathcal{H}^t(\mathbb{R}^n, H)} \le M \,.$$

Proof. Consider the equations

$$P(D)u = R_p F(x, (\partial^\alpha u)_{|\alpha|<t-n/2}) \,, \quad p = 1, 2, \ldots \tag{17}$$

Treating H_p as $\mathbb{R}^p$, we conclude, from the assumptions of the theorem, that equation (17) has a solution $u_p \in \mathcal{H}^t(\mathbb{R}^n, H_p)$ for any p (see [4], Theorem 2). We have

$$\|u_p\|_{\mathcal{H}^t(\mathbb{R}^n, H)} \le M \,, \quad p = 1, 2, \ldots$$

By the Eberlein–Shmul'yan theorem, the sequence (u_p) contains a subsequence (u_{p_k}) weakly convergent to some $u \in \mathcal{H}^t(\mathbb{R}^n, H)$ and

$$\|u\|_{\mathcal{H}^t(\mathbb{R}^n, H)} \le M \,.$$

By Lemma 3, we may assume that the sequences $(\partial^\alpha u_{p_k}(x))$, $|\alpha| < t - n/2$, weakly converge in H to the corresponding $\partial^\alpha u(x)$ for a.e. x.

We shall prove that u is a solution of equation (16). We have to show that

$$P(D)\langle u(\cdot),h\rangle_H=\langle F(\cdot,(\partial^\alpha u(\cdot))_{|\alpha|<t-n/2}),h\rangle_H \tag{18}$$

in $\mathcal{D}'$ for any $h\in H$. We know that

$$P(D)\langle u_{p_k}(\cdot),h\rangle_H=\langle R_{p_k}F(\cdot,(\partial^\alpha u_{p_k}(\cdot))_{|\alpha|<t-n/2}),h\rangle_H \tag{19}$$

in $\mathcal{D}'$. We shall prove that (17) follows from (19) by passing to the limit in $\mathcal{D}'$ as $k\to\infty$.

Let $\varphi\in\mathcal{C}_0^\infty$. We have

$$\begin{aligned}\int\varphi(x)\langle u_{p_k}(x),h\rangle_H\,dx&=\int\langle u_{p_k}(x),\varphi(x)h\rangle_H\,dx\\&\to\int\langle u(x),\varphi(x)h\rangle_H\,dx\,,\end{aligned}$$

because $\int\langle\cdot(x),\varphi(x)h\rangle_H\,dx$ is a continuous linear functional on $\mathcal{H}^t(\mathbb{R}^n,H)$. Now, from the sequential continuity of $P(D)$ in $\mathcal{D}'$, we conclude that the left-hand side of (19) converges to the left-hand side of (18) in $\mathcal{D}'$. We shall prove the same for the right-hand sides, which means that

$$\begin{aligned}&\int\varphi(x)\langle R_{p_k}F(x,(\partial^\alpha u_{p_k}(x))_{|\alpha|<t-n/2}),h\rangle_H\,dx\\&\qquad\to\int\varphi(x)\langle F(x,(\partial^\alpha u(x))_{|\alpha|<t-n/2}),h\rangle_H\,dx\end{aligned} \tag{20}$$

for any $\varphi\in\mathcal{C}_0^\infty$.

We show first that

$$\begin{aligned}&\langle R_{p_k}F(x,(\partial^\alpha u_{p_k}(x))_{|\alpha|<t-n/2}),h\rangle_H\\&\qquad\to\langle F(x,(\partial^\alpha u(x))_{|\alpha|<t-n/2}),h\rangle_H\quad\text{for a.e. } x\,.\end{aligned} \tag{21}$$

Assume that $h\in H_l$ for some l. Then, for large k,

$$\langle R_{p_k}F(x,(\partial^\alpha u_{p_k}(x))_{|\alpha|<t-n/2}),h\rangle_H=\langle F(x,(\partial^\alpha u_{p_k}(x))_{|\alpha|<t-n/2}),h\rangle_H\,,$$

hence (21) is true by the Carathéodory condition.

From (12) and (14), we have

$$\|F(x,(\partial^\alpha u_{p_k}(x))_{|\alpha|<t-n/2})\|\le C(x)<\infty\quad\text{for a.e. } x\,. \tag{22}$$

This implies (21) for all $h\in H$ (see [9], p. 121, Theorem 3). By the Lebesgue convergence theorem, formulas (12), (14) and (21) imply the convergence (20). The proof is complete.

Example 1. We define a class of equations satisfying the assumptions of Theorem 1.

Let P be a polynomial of n variables and degree T such that the polynomial $P(-i\partial)$ of the variable ∂ has real coefficients and satisfies (13). Let $F:\mathbb{R}^n\times\mathbb{R}^m\to H$ satisfy (14) and the Carathéodory condition in the sense of Theorem 1. Assume that there exist constants $0<a<2$, $L>0$ and nonnegative functions $f \in L^{2/a}$ and $g \in L^{2/(2-a)}$ such that

$$\langle v_{(0,\ldots,0)}, F(x,(v_\alpha)_{|\alpha|<t-n/2})\rangle_H \le 0 \quad \text{for } \|v_{(0,\ldots,0)}\| \ge g(x) \text{ and a.e. } x\,, \tag{23}$$

and

$$\|F(x,(v_\alpha)_{|\alpha|<t-n/2})\| \le f(x) + L\|(v_\alpha)_{|\alpha|<t-n/2}\|^a \quad \text{for } \|v_{(0,\ldots,0)}\| \le g(x) \text{ and a.e. } x\,. \tag{24}$$

Treating H_p as $\mathbb{R}^p$, we obtain the necessary a priori bounds for solutions of equations (15) as in [4], Example 2.

Example 2. We now describe a more concrete example of the class described above.

Let $n = 1$, $T = 2$, $a = 1$, $P(\xi) = \xi^2 + b$, $b > 0$, and $A : H \to H$ a linear, continuous, invertible operator. Suppose that $B : \mathbb{R} \times H \to H$ satisfies the Carathéodory condition in the sense of Theorem 1 and

$$\|B(x,v)\| \le h(x)\,, \qquad h \in L^2(\mathbb{R})\,.$$

Let $F(x,v) = -A^*Av + B(x,v)$. We have

$$\begin{aligned}\langle v, F(x,v)\rangle_H &= -\langle v, A^*Av\rangle + \langle v, B(x,v)\rangle\\ &= -\langle Av, Av\rangle + \langle v, B(x,v)\rangle\\ &\le -C\|v\|^2 + \|v\|h(x) \le 0\end{aligned}$$

for $\|v\| \ge g(x) := h(x)/C$ with some constant $C > 0$. Then condition (23) is satisfied. Condition (24) is satisfied for $L = 0$ and

$$f(x) = (\|A^*A\|/C + 1)h(x)\,.$$

Consider, for example, the following problem:

$$-\frac{d^2u(x,t)}{dx^2} + u(x,t) = -u(x,t) + \psi\Big(x, \int_0^1 K(x,t,\tau)u(x,\tau)\,d\tau\Big)\,,$$

where K is measurable, $K(x,\cdot,\cdot) \in L^2([0,1]\times[0,1])$ for a.e. x, $\psi : \mathbb{R}^2 \to \mathbb{R}$ is continuous and

$$|\psi(x,y)| \le h(x)\,, \qquad h \in L^2(x)\,. \tag{25}$$

We look for $u \in \mathcal{H}^1(\mathbb{R}, L^2([0,1]))$ (we treat u as the mapping $x \mapsto u(x,\cdot)$).

We have

$$P(\xi) = \xi^2 + 1$$

and

$$F(x,v) = -v + \psi\Big(x, \int_0^1 K(x,\cdot,\tau)v(\tau)\,d\tau\Big)\,.$$

The function F satisfies the Carathéodory condition. In fact, the map

$$L^2([0,1]) \ni v \mapsto \int_0^1 K(x,\cdot,\tau)\,d\tau \in L^2([0,1])$$

is linear and completely continuous (for almost all x), hence it transforms weakly convergent sequences to strongly convergent ones. The Nemytskiĭ operator

$$L^2([0,1]) \ni v \mapsto \psi(x, v(\cdot)) \in L^2([0,1])$$

is continuous by (25) (see for example [2], Proposition 1).

Remark 2. Note that a Hammerstein operator does not have so good properties as the operator F defined in Example 2. Consider the operator

$$G(v) = \int_0^1 K(\cdot,\tau)\psi(\tau, v(\tau))\,d\tau\,,$$

where

$$K \in L^2([0,1]\times[0,1]), \quad |\psi(t,y)| \le h(t) + |y|\,, \quad h \in L^2([0,1])\,.$$

Suppose that $G : L^2([0,1]) \to L^2([0,1])$ is sequentially continuous in the sense of the weak topology in $L^2([0,1])$. Then, for any $w \in L^2([0,1])$, the map

$$G_w : v \mapsto \langle G(v), w\rangle_{L^2([0,1])}$$

transforms weakly convergent sequences in $L^2([0,1])$ to convergent numerical ones.

Suppose that ψ is differentiable with respect to the second variable and that $G_w : L^2([0,1]) \to \mathbb{R}$ satisfies the assumptions of the following theorem of Palmer (see [7]):

Let X be a reflexive Banach space, Y a Banach space and let $F : X \to Y$ be uniformly Fréchet differentiable on any ball in X. Then F is sequentially continuous with the weak topology in X and the strong topology in Y if and only if the following two conditions are satisfied:

(i) for any $v \in X$ the Fréchet derivative $F'(v)$ is a completely continuous linear operator,

(ii) the Fréchet derivative $F' : X \to L(X,Y)$ (the space of linear continuous operators from X into Y) is completely continuous.

We have

$$G_w(v) = \int_0^1 \int_0^1 K(t,\tau)\psi(\tau, v(\tau))w(t)\,d\tau\,dt = \int_0^1 K_w(\tau)\psi(\tau, v(\tau))\,d\tau\,,$$

where

$$K_w(\tau) := \int\limits_0^1 K(t,\tau)w(t)\,dt\,.$$

Compute the derivative

$$G'_w(v)\cdot h = \int\limits_0^1 K_w(\tau)\partial_2\psi(\tau,v(\tau))h(\tau)\,d\tau\,.$$

By the isomorphism $L(L^2([0,1]);\mathbb{R}) \cong L^2([0,1])$, we have

$$G'_w(v) = K_w(\cdot)\partial_2\psi(\cdot,v(\cdot))\,.$$

We conclude that the condition (ii) will not be satisfied if G'_w is not constant. In fact, any nonconstant superposition operator

$$L^2([0,1]) \ni v \mapsto N(v) := \varphi(\cdot,v(\cdot)) \in L^2([0,1])$$

does not transform bounded sets onto precompact ones. In fact, let $N(u_1) \neq N(u_2)$ for some $u_1, u_2 \in L^2([0,1])$. Let

$$v_k(x) := \begin{cases} u_1(x) & \text{for } x \in [2^{-k}2p, 2^{-k}(2p+1)[,\\ u_2(x) & \text{for } x \in [2^{-k}(2p+1), 2^{-k}(2p+2)[,\\ 0 & \text{for } x = 1,\ p = 0,1,\ldots,2^{k-2}. \end{cases}$$

The sequence (v_k) is bounded in $L^2([0,1])$ but $N(v_k)$ has no subsequence which converges in $L^2([0,1])$.

4. Existence theorem for a system of equations. We formulate a theorem similar to Theorem 1 for systems of equations.

THEOREM 2. *Let H be a real infinite-dimensional separable Hilbert space, and $(e_\gamma)_{\gamma=1,2,\ldots}$ a complete orthonormal system in H. Let H_p denote the space generated by the system $\{e_\gamma : \gamma = 1,\ldots,p\}$ and let $R_p : H \to H_p$ be the orthonormal projector onto H_p. Let P_r be polynomials of n variables and degrees T_r such that the polynomials $P_r(-i\partial)$ of the variable ∂ have real coefficients and satisfy*

$$1 + |\xi|^{T_r} \le CP_r(\xi)\,, \qquad \xi \in \mathbb{R}^n\,,\ r = 1,\ldots,k\,,$$

for some constant C. Let $t_r \in [0, T_r[$, $r = 1,\ldots,k$, and

$$m := \sum_{r=1}^{k} \sum_{0\le l<t_r-n/2} n^l\,.$$

Assume that $F : \mathbb{R}^n \times H^m \to H^k$ satisfies the Carathéodory condition of the following form: $F(x,\cdot)$ is sequentially continuous in the weak topologies of H^m and H^k for a.e. x and $F(\cdot,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k})$ is measurable for

all $(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k} \in H^m$. *Assume that for any bounded set* $K \subset \mathbb{R}^n \times H^m$ *there exists a function* $h_K \in L^2(\mathbb{R}^n)$ *such that*

$$\|F(x,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k})\| \le h_K(x)$$

for $(x,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k}) \in K$ *a.e.* x. *Assume that there is a sequence of open bounded sets* $U_1 \subset U_2 \subset \ldots$ *with* $\bigcup U_j = \mathbb{R}^n$ *and a constant* M *such that no system*

$$P_l(D)u^l = \lambda R_p F^l_j(x,(\partial^\alpha u^r)_{|\alpha|<t_r-n/2,r=1,\ldots,k}), \quad l=1,\ldots,k \ (F=(F^1,\ldots,F^k)),$$

has a solution in the set

$$\Big\{u=(u^1,\ldots,u^k) \in \bigtimes_{r=1}^{k} \mathcal{H}^{t_r}(\mathbb{R}^n,H_p) : \sum_{r=1}^{k} \|u^r\|^2_{\mathcal{H}^{t_r}(\mathbb{R}^n,H)} > M^2\Big\}$$

for $j=1,2,\ldots$, $\lambda \in [0,1]$, $p=1,2,\ldots$ (*The functions* F^l_j *are defined as*

$$F^l_j(x,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k}) := \begin{cases} F^l(x,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k}) & \text{for } x \in U_j, \\ 0 & \text{for } x \notin U_j.) \end{cases}$$

Under these assumptions the system

$$P_l(D)u^l = F^l(x,(\partial^\alpha u^r)_{|\alpha|<t_r-n/2,r=1,\ldots,k}), \quad l=1,\ldots,k,$$

has a solution $u \in \bigtimes_{r=1}^{k} \mathcal{H}^{t_r}(\mathbb{R}^n,H)$ *for which*

$$\sum_{r=1}^{k} \|u^r\|_{\mathcal{H}^{t_r}(\mathbb{R}^n,H)} \le M.$$

Proof. Similar to the proof of Theorem 1.

EXAMPLE 3. One can construct an example analogous to Example 1 with the condition

$$\langle v_{(0,\ldots,0)}, F(x,(v^r_\alpha)_{|\alpha|<t_r-n/2,r=1,\ldots,k})\rangle_{H^k} \le 0 \quad \text{for } \|v_{(0,\ldots,0)}\| \ge g(x) \text{ for a.e. } x$$

instead of (23).

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Reçu par la Rédaction le 28.10.1992
Révisé le 3.3.1993 et 5.7.1993