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Remarks concerning Driver's equation

by Gerd Herzog and Roland Lemmert (Karlsruhe)

Abstract. We consider uniqueness for the initial value problem $x' = 1 + f(x) - f(t)$, $x(0) = 0$. Several uniqueness criteria are given as well as an example of non-uniqueness.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We consider the initial value problem

$$\begin{cases} x'(t) = 1 + f(x(t)) - f(t)\,, \quad t \ge 0\,, \\ x(0) = 0\,, \end{cases} \tag{1}$$

which has $x(t) = t$ as a solution. Driver [1] asks whether this is in general the only one and proves [2]:

Proposition 1. *There is no solution x with $x(t) < t$ $(t > 0)$ and $x'(t)$ decreasing and no solution $x(t) > t$ $(t > 0)$ with $x'(t)$ increasing* (*in the wider sense*).

(f is to be substituted by $-g$ in Driver's terminology.)

Nowak [3] remarks that Driver's question is not completely answered yet.

We will sharpen Proposition 1 in several ways and give examples of continuous functions f such that (1) is not uniquely solvable. We also provide conditions on f such that (1) is uniquely solvable.

We begin with

Proposition 2. *For each solution x of* (1) *we have $x(t) \le t$, $t \ge 0$.*

Proof. We rewrite the differential equation as

$$\exp(x'(t) - 1) = \exp(f(x(t))) \cdot \exp(-f(t))\,,$$

and because $\exp(s - 1) \ge s$ $(s \in \mathbb{R})$ we get

$$x'(t)\exp(-f(x(t))) \le \exp(-f(t))$$

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and by integration

$$\int_0^{x(t)} \exp(-f(s))\,ds \le \int_0^t \exp(-f(s))\,ds\,,$$

which gives $x(t) \le t$, $t \ge 0$.

Another way of looking at this problem is the following:

The initial value problem

$$\begin{cases} y'(t) = \exp(f(y(t))) \cdot \exp(-f(t)), \\ y(0) = 0 \end{cases} \tag{2}$$

is uniquely solvable, since it has separated variables with $\exp(f(0)) \neq 0$; its solution is $y(t) = t$, and any solution of (1) is a subsolution to (2), hence $x(t) \le t$. ∎

We next give a necessary condition which solutions of (1) have to satisfy.

Proposition 3. *Let* $d(t) = t - x(t)$, x *any solution of* (1), *denote by* $\sigma_f(t)$ *the oscillation of* f *over the interval* $[0,t]$, *and let* $(\cdot)_+$ *be the positive part of a function. Then*

$$0 \le d(t) \le \sigma_f(t) \int_0^t (d'(s))_+\,ds\,, \qquad t \ge 0\,. \tag{3}$$

Proof. Let $t > 0$ and c a constant which will be determined later. Then d satisfies

$$d'(t) = -(f(x(t)) - c)d'(t) + f(t) - c - (f(x(t)) - c)x'(t)\,.$$

By integration we get

$$\int_{x(t)}^t (1 - (f(s) - c))\,ds = \int_0^t (f(x(s)) - c)(x'(s) - 1)\,ds\,.$$

(This relation is most easily verified by differentiation.)

Now setting $c = \min\{f(s) : 0 \le s \le t\}$ we have

$$\sigma_f(t) \ge f(s) - c \ge 0\,, \qquad \sigma_f(t) \ge f(x(s)) - c \ge 0$$

and therefore

$$(t - x(t))(1 - \sigma_f(t)) \le \sigma_f(t) \int_0^t (-d'(s))_+\,ds\,,$$

from which (3) easily follows by using $t - x(t) = \int_0^t d'(s)ds$. ∎

Proposition 3 shows that (1) cannot have a solution different from t near 0 such that $x'(t) \le 1$ (which in particular holds if x' decreases since $x'(0) = 1$): In this case $(d'(s))_+ = d'(s)$, which implies $d \equiv 0$ for small $t > 0$ (i.e., for those t with $\sigma_f(t) < 1$).

Now fix $t_0 > 0$ and assume there is a solution of (1) such that $x(t_0) = t_0$. Then $z(t) = x(t + t_0) - t_0$ satisfies

$$\begin{cases} z'(t) = 1 + g(z(t)) - g(t)\,, \\ z(0) = 0\,, \end{cases}$$

where g is defined by $g(s) = f(s + t_0) - t_0$. By Proposition 3 we have

$$0 \le t - z(t) \le \sigma_g(t) \int\limits_0^t (1 - z'(s))_+ \, ds\,, \qquad t \ge 0\,,$$

from which we get

$$0 \le t - x(t) \le \sigma_f(t_0, t) \int\limits_{t_0}^t (1 - x'(s))_+ \, ds\,, \qquad t \ge t_0\,,$$

where $\sigma_f(t_0, t)$ denotes the oscillation of f over the interval $[t_0, t]$. This shows that no solution can leave the diagonal at a time $t_0 > 0$ if $x'(t) \le 1$.

Of course, if f is decreasing, (1) is uniquely solvable by standard uniqueness theorems. On the other hand, Proposition 3 implies uniqueness if f is increasing; for in this case we have the inequality $x'(t) \le 1$, $t \ge 0$, because of $x(t) \le t$ $(t \ge 0)$. Remarkably enough, from the above considerations we see that if f is locally of bounded variation, then (1) is uniquely solvable: We write $f = f_1 - f_2$, f_1, f_2 increasing, and get

$$x'(t) \ge 1 + f_1(x(t)) - f_1(t)\,, \qquad t \ge 0\,,$$

so $x(t) \ge t$, and finally $x(t) = t$, $t \ge 0$.

PROPOSITION 4. *Let $f'(s)$ exist for $s > 0$ and let there exist $c < \overline{c} < 1$ such that*

$$f'(s) \le \frac{c}{s} + 1 - \overline{c}\,, \qquad s \in (0, 1]\,.$$

Then problem (1) *is uniquely solvable in* $[0, 1]$.

Proof. From $1 - x'(t) = -f(x(t)) + f(t)$ we get for $t, x(t) > 0$,

$$1 - x'(t) = \int\limits_{x(t)}^t f'(s)\, ds \le (1 - \overline{c})(t - x(t)) + c(\log t - \log x(t))$$

or

$$x'(t) \ge 1 + c \log x(t) - c \log t + (1 - \overline{c})(x(t) - t)\,. \tag{4}$$

If (1) is not uniquely solvable, we may assume by Kneser's theorem that there is a solution $x : [0, 1] \to \mathbb{R}$, positive in $(0, 1]$, such that $1 > x(1) > d > 0$, with d to be determined in a moment.

We now consider the initial value problem

$$\begin{cases} z'(t) = 1 + c \log z(t) - c \log t + (1 - \overline{c})(z(t) - t)\,, \\ z(1) = x(1)\,. \end{cases} \tag{5}$$

By (4), x is a subsolution to the left for (5) in $(0,1]$. Now, because of $c < \bar{c} < 1$, there is $1 > d > 0$ such that

$$\bar{c}(s-1) \le c \log s\,, \qquad d \le s \le 1\,,$$

so that $y(t) = x(1) \cdot t$ satisfies

$$y'(t) \le 1 + c \log y(t) - c \log t + (1-\bar{c})(y(t) - t)\,, \qquad 0 < t \le 1\,.$$

Therefore y is a supersolution of (5) to the left and by standard comparison theorems we get

$$x(t) \le y(t)\,, \qquad t \in (0,1]\,. \tag{6}$$

But then x cannot be a solution of (1), since (6) implies $x'(0) \le x(1) < 1$. ■

PROPOSITION 5. *If f satisfies*

$$f(t) - f(x) \le \frac{1}{t}(t-x)\,, \qquad 0 \le x < t \le 1\,,$$

then (1) *is uniquely solvable in* $[0,1]$.

Proof. The proof follows the same ideas as the proof of Proposition 4. In this case we write

$$1 - x'(t) = f(t) - f(x(t)) \le \frac{1}{t}(t - x(t))\,, \qquad 0 < t \le 1\,.$$

Then x is a subsolution to the left for

$$z'(t) = \frac{1}{t} z(t), \qquad z(1) = x(1)\,.$$

Standard comparison theorems [4] give $x(t) \le x(1) \cdot t$, $0 < t \le 1$, since $z(t) = x(1) \cdot t$ is the solution to this latter problem. Hence again $x'(0) \le x(1)$, which is impossible if $x(1) < 1$. ■

We finally construct an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ such that (1) is not uniquely solvable.

To this end we define by induction a sequence $b_1 = 1, a_1, b_2, a_2, b_3, \ldots$ of numbers which tends strictly monotonically to zero; f will be zero outside $(0,1)$ and on any interval $[b_{n+1}, a_n]$ $(n \ge 1)$, and positive elsewhere.

Let $b_1 = 1 > a_1 > 0$ (the value of a_1 will be fixed later), $\gamma : [0,1] \to [0,1]$ continuous, $\gamma(0) = \gamma(1) = 0$ and $m = \int_0^1 \gamma(s)ds > 0$. We define $\gamma_1(s) = \gamma((s-a_1)/(1-a_1))$, $m_1 = (1/(b_1 - a_1)) \int_{a_1}^{b_1} \gamma_1(s)ds$ and remark that $m_1 = m$.

By $I(\mu)$ we denote $\int_0^1 ds/(1 + \mu\gamma(s))$. Let (μ_n) be a sequence of positive numbers tending to zero, $\mu_1 = 1$, $\mu_n \le 1$.

In $[a_1, b_1]$ we solve $x'(t) = 1 - \gamma_1(t)$, $x(b_1) = a_1$ and set $b_2 = x(a_1)$, which gives

$$a_1 - b_2 = b_1 - a_1 - \int_{a_1}^{b_1} \gamma_1(s)\, ds$$

or

$$a_1 - b_2 = (1-m)(1-a_1)\,.$$

Therefore $b_2 < a_1$, and we define $f = \gamma_1$ in $[a_1, b_1]$, $f \equiv 0$ in $[b_2, a_1]$.

Next we choose a_2 such that

$$b_2 - a_2 = (a_1 - b_2) \cdot \frac{1}{I(\mu_2)} = (1-m)(1-a_1) \cdot \frac{1}{I(\mu_2)}$$

and solve $x'(t) = 1 + \gamma_2(x(t))$, $x(a_1) = b_2$, where

$$f(s) := \gamma_2(s) := \mu_2 \gamma_1\left(\frac{b_1 - a_1}{b_2 - a_2}(s - a_2) + a_1\right), \qquad a_2 \le s \le b_2\,.$$

Since the differential equation for x has separated variables, an easy calculation shows $a_2 = x(b_2)$.

Up to now f is defined on $[a_2, 1]$, and x satisfies the differential equation from (1) on $[a_2, 1]$, x being increasing with values in $[a_2, 1]$.

To proceed by induction, let $a_n < b_n$ be defined,

$$f(t) := \gamma_n(t) = \mu_n \gamma\left(\frac{1}{b_n - a_n}(t - a_n)\right), \qquad a_n \le t \le b_n\,.$$

We solve $x'(t) = 1 - \gamma_n(t)$, $t \in [a_n, b_n]$, $x(b_n) = a_n$, set $b_{n+1} = x(a_n) < a_n$, define $f \equiv 0$ in $[b_{n+1}, a_n]$ and a_{n+1} by

$$I(\mu_{n+1})(b_{n+1} - a_{n+1}) = a_n - b_{n+1}\,,$$

so $b_{n+1} > a_{n+1}$. Now we solve $x'(t) = 1 + \gamma_{n+1}(x(t))$, $x(a_n) = b_{n+1}$ on $[a_{n+1}, b_{n+1}]$, the solution of which satisfies $x(b_{n+1}) = a_{n+1}$, f is defined on $[a_{n+1}, 1]$ and x satisfies the differential equation in (1).

By our construction we have, for $n \ge 1$,

$$a_n - b_{n+1} = (1 - \mu_n m)(b_n - a_n)\,,$$
$$b_{n+1} - a_{n+1} = (1 - \mu_n m)(b_n - a_n) \cdot \frac{1}{I(\mu_{n+1})}\,,$$

so for $n \ge 2$,

$$b_n - a_n = (1-a_1)(1-m) \cdot \prod_{k=2}^{n-1}(1 - \mu_k m) \cdot \prod_{k=2}^{n} \frac{1}{I(\mu_k)}\,,$$
$$a_{n-1} - b_n = (1-a_1)(1-m) \cdot \prod_{k=2}^{n-1}(1 - \mu_k m) \cdot \prod_{k=2}^{n-1} \frac{1}{I(\mu_k)}\,.$$

From Jensen's inequality, applied to the convex functions $h(x) = 1/(1+\mu x)$ ($\mu \ge 0$, $x \ge 0$), we have

$$I(\mu) = \frac{1}{b_1 - a_1} \int\limits_{a_1}^{b_1} \frac{ds}{1 + \mu\gamma_1(s)} \ge \frac{1}{1+\mu m};$$

therefore the sequence $b_1, a_1, b_2, a_2, b_3, a_3, \ldots$ is convergent if

$$\sum_{n=3}^{\infty} \prod_{k=2}^{n-1} (1 - \mu_k^2 m^2)$$

converges, which is the case, for example, for $\mu_k = 1/\sqrt[4]{k}$, $k \in \mathbb{N}$, as is easily verified using Raabe's test for convergence. For suitable $a_1 < 1$ we finally get $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. So f is defined everywhere, continuous, bounded, and the solution x solves (1) with $x(t) < t$, $t \in (0, 1]$. ∎

Remarks. 1) For a suitable choice of γ, the function f is C^∞ in $\mathbb{R}\setminus\{0\}$.

2) If we define $F(t, x) = 1 + f(x) - f(t)$, Kamke's or related uniqueness theorems are of course applicable if f satisfies an appropriate condition. Our condition in Proposition 4 cannot be subsumed under this, since, for an autonomous equation $x' = g(x)$, the condition $g'(x) \le c/x$ does not imply uniqueness, as $g(x) = \sqrt{x}$ shows.

3) It would be interesting to know whether the condition $f'(t) \le 1/t + 1$, $0 < t \le 1$, implies uniqueness for (1). This condition would contain the conditions of Propositions 4 and 5.

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MATHEMATISCHES INSTITUT I
UNIVERSITÄT KARLSRUHE
ENGLERSTR. 2
POSTFACH 6380
D-76128 KARLSRUHE 1, GERMANY

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