

**On local motion of a general compressible
 viscous heat conducting fluid bounded by a free surface**

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Abstract. The motion of a viscous compressible heat conducting fluid in a domain in \mathbb{R}^3 bounded by a free surface is considered. We prove local existence and uniqueness of solutions in Sobolev–Slobodetskiĭ spaces in two cases: with surface tension and without it.

1. Introduction. In this paper we consider the motion of a viscous heat conducting fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ with a free boundary S_t . Let $v = v(x, t)$ be the velocity of the fluid (i.e. $v = (v_1, v_2, v_3)$), $\varrho = \varrho(x, t)$ the density, $\vartheta = \vartheta(x, t)$ the temperature, $f = f(x, t)$ the external force field per unit mass, $r = r(x, t)$ the efficiency of heat sources per unit mass, $p = p(\varrho, \vartheta)$ the pressure, μ and ν the constant viscosity coefficients, σ the constant coefficient of surface tension, κ the constant coefficient of heat conductivity, $c_v = c_v(\varrho, \vartheta)$ the specific heat at constant volume, p_0 the external (constant) pressure. Then the problem is described by the following system (see [4], Chs. 2 and 5):

$$\begin{aligned}
 (1.1) \quad & \varrho(v_t + v \cdot \nabla v) + \nabla p - \mu \Delta v - \nu \nabla \operatorname{div} v = \varrho f && \text{in } \tilde{\Omega}^T, \\
 & \varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \tilde{\Omega}^T, \\
 & \varrho c_v(\vartheta_t + v \cdot \nabla \vartheta) + \vartheta p_\vartheta \operatorname{div} v - \kappa \Delta \vartheta \\
 & \quad - \frac{\mu}{2} \sum_{i,j=1}^3 (v_{i,x_j} + v_{j,x_i})^2 - (\nu - \mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \tilde{\Omega}^T, \\
 & \mathbb{T} \cdot \bar{n} - \sigma H \bar{n} = -p_0 \bar{n} && \text{on } \tilde{S}^T,
 \end{aligned}$$

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$$\begin{aligned}
(1.1) \quad & v \cdot \bar{n} = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \tilde{S}^T, \\
[\text{cont.}] \quad & \frac{\partial\vartheta}{\partial n} = \tilde{\vartheta} && \text{on } \tilde{S}^T, \\
& v|_{t=0} = v_0 && \text{in } \Omega, \\
& \varrho|_{t=0} = \varrho_0 && \text{in } \Omega, \\
& \vartheta|_{t=0} = \vartheta_0 && \text{in } \Omega,
\end{aligned}$$

where $\phi(x, t) = 0$ describes S_t , \bar{n} is the unit outward vector normal to the boundary (i.e. $\bar{n} = \nabla\phi/|\nabla\phi|$), $\tilde{\Omega}^T = \bigcup_{t \in (0, T)} \Omega_t \times \{t\}$, Ω_t is the domain of the drop at time t and $\Omega_0 = \Omega$ is its initial domain, $\tilde{S}^T = \bigcup_{t \in (0, T)} S_t \times \{t\}$. Finally, $\mathbb{T} = \mathbb{T}(v, p)$ denotes the stress tensor of the form

$$\begin{aligned}
(1.2) \quad \mathbb{T} &= \{T_{ij}\} = \{-p\delta_{ij} + \mu(v_{i,x_j} + v_{j,x_i}) + (\nu - \mu)\delta_{ij} \operatorname{div} v\} \\
&\equiv \{-p\delta_{ij} + D_{ij}(v)\},
\end{aligned}$$

where $i, j = 1, 2, 3$, $\mathbb{D} = \mathbb{D}(v) = \{D_{ij}\}$ is the deformation tensor. Moreover, thermodynamic considerations imply that $c_v > 0$, $\kappa > 0$, $\nu \geq \frac{1}{3}\mu > 0$. By H we denote the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$(1.3) \quad H\bar{n} = \Delta_{S_t}(t)x, \quad x = (x_1, x_2, x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t . Let S_t be given by $x = x(s_1, s_2, t)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, where U is an open set. Then

$$\begin{aligned}
(1.4) \quad \Delta_{S_t}(t) &= g^{-1/2} \frac{\partial}{\partial s_\alpha} g^{-1/2} \hat{g}_{\alpha\beta} \frac{\partial}{\partial s_\beta} \\
&= g^{-1/2} \frac{\partial}{\partial s_\alpha} g^{1/2} g^{\alpha\beta} \frac{\partial}{\partial s_\beta} \quad (\alpha, \beta = 1, 2),
\end{aligned}$$

where the summation convention over repeated indices is assumed, $g = \det\{g_{\alpha\beta}\}_{\alpha, \beta=1,2}$, $g_{\alpha\beta} = x_\alpha \cdot x_\beta$ ($x_\alpha = \partial x / \partial s_\alpha$), $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\hat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements of $\{g_{\alpha\beta}\}$.

Let the domain Ω be given. Then by (1.1)₅, $\Omega_t = \{x \in \mathbb{R}^3 : x = x(\xi, t), \xi \in \Omega\}$, where $x = x(\xi, t)$ is the solution of the Cauchy problem

$$(1.5) \quad \frac{\partial x}{\partial t} = v(x, t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Therefore the transformation $x = x(\xi, t)$ connects Eulerian x and Lagrangian ξ coordinates of the same fluid particle. Hence

$$(1.6) \quad x = \xi + \int_0^t u(\xi, s) ds \equiv X_u(\xi, t),$$

where $u(\xi, t) = v(X_u(\xi, t), t)$. Moreover, the kinematic boundary condition (1.1)₅ implies that the boundary S_t is a material surface. Thus, if $\xi \in S = S_0$ then $X_u(\xi, t) \in S_t$ and $S_t = \{x : x = X_u(\xi, t), \xi \in S\}$.

By the equation of continuity (1.1)₂ and (1.1)₅ the total mass M of the drop is conserved and the following relation between ϱ and Ω_t holds:

$$(1.7) \quad \int_{\Omega_t} \varrho(x, t) dx = M.$$

The aim of this paper is to prove the local-in-time existence and uniqueness of solutions to problem (1.1) in Sobolev–Slobodetskiĭ spaces (see definition in Sect. 2). In the case of compressible barotropic fluid the corresponding drop problem has been considered by W. M. Zajączkowski in [13] and [16], while papers [14] and [15] refer to the global existence of solution to the same drop problem. Local existence of solutions in the compressible barotropic case was also considered in [5], [6], [12], while in the incompressible barotropic case local existence is proved in [2] and [10].

This paper consists of four sections. In Section 2 notation and auxiliary results are presented. In Section 3 we prove the local existence and uniqueness of solution to problem (1.1) in the case $\sigma = 0$. In this case there is no surface tension. Finally, Section 4 concerns the local existence and uniqueness of solution to problem (1.1) in the case $\sigma \neq 0$, i.e. when the shape of the free boundary S_t of Ω_t is governed by surface tension.

2. Notations and auxiliary results. In Sections 3 and 4 of this paper we use the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l, l/2}(Q_T)$, $l \in \mathbb{R}_+^1$ (see [3]), of functions defined in Q_T where $Q_T = \Omega^T \equiv \Omega \times (0, T)$ ($\Omega \subset \mathbb{R}^3$ is a domain, $T \leq \infty$) or $Q_T = S^T \equiv S \times (0, T)$, $S = \partial\Omega$.

We define $W_2^{l, l/2}(\Omega^T)$ as the space of functions u such that

$$(2.1) \quad \|u\|_{W_2^{l, l/2}(\Omega^T)} = \left[\sum_{|\alpha|+2i \leq [l]} \|D_\xi^\alpha \partial_t^i u\|_{L_2(\Omega^T)}^2 + \sum_{|\alpha|+2i=[l]} \left(\int_0^T \int_\Omega \int_\Omega \frac{|D_\xi^\alpha \partial_t^i u(\xi, t) - D_\xi^\alpha \partial_t^i u(\xi', t)|^2}{|\xi - \xi'|^{3+2(l-[l])}} d\xi d\xi' dt + \int_\Omega \int_0^T \int_0^T \frac{|D_\xi^\alpha \partial_t^i u(\xi, t) - D_\xi^\alpha \partial_t^i u(\xi, t')|^2}{|t - t'|^{1+2(l/2-[l/2])}} dt dt' d\xi \right) \right]^{1/2} \equiv \left[\sum_{|\alpha|+2i \leq [l]} |D_\xi^\alpha \partial_t^i u|_{2, \Omega^T}^2 + \sum_{|\alpha|+2i=[l]} ([D_\xi^\alpha \partial_t^i u]_{l-[l], 2, \Omega^T, \xi}^2 + [D_\xi^\alpha \partial_t^i u]_{l/2-[l/2], 2, \Omega^T, t}^2) \right]^{1/2} < \infty,$$

where we use generalized (Sobolev) derivatives, $D_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3}$, $\partial_{\xi_j}^{\alpha_j} = \partial^{\alpha_j} / \partial \xi_j^{\alpha_j}$ ($j = 1, 2, 3$), $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\partial_t^i = \partial^i / \partial t^i$ and $[l]$ is the integer part of l . In the case when l is an integer the second terms in the above formulae must be omitted, while in the case of $l/2$ being integer the last terms in the above formulae must be omitted as well.

Similarly to $W_2^{l,l/2}(\Omega^T)$, using local mappings and a partition of unity we introduce the normed space $W_2^{l,l/2}(S^T)$ of functions defined on $S^T = S \times (0, T)$, where $S = \partial\Omega$.

We also use the usual Sobolev spaces $W_2^l(Q)$, where $l \in \mathbb{R}_+$, $Q = \Omega$ ($\Omega \subset \mathbb{R}^3$ is a bounded domain) or $Q = S$. In the case $Q = \Omega$ the norm in $W_2^l(\Omega)$ is defined as follows:

$$\begin{aligned} \|u\|_{W_2^l(\Omega)} &= \left(\sum_{|\alpha| \leq [l]} \|D_\xi^\alpha u\|_{L_2(\Omega)}^2 \right. \\ &\quad \left. + \sum_{|\alpha|=[l]} \int_\Omega \int_\Omega \frac{|D_\xi^\alpha u(\xi) - D_\xi^\alpha u(\xi')|^2}{|\xi - \xi'|^{3+2(l-[l])}} d\xi d\xi' \right)^{1/2} \\ &\equiv \left(\sum_{|\alpha| \leq [l]} |D_\xi^\alpha u|_{2,\Omega}^2 + [D_\xi^\alpha u]_{l-[l],2,\Omega}^2 \right)^{1/2}, \end{aligned}$$

where the last term is omitted when l is an integer. Similarly, by using local mappings and a partition of unity we define $W_2^l(S)$.

To simplify notation we write

$$\begin{aligned} \|u\|_{l,Q} &= \|u\|_{W_2^{l,l/2}(Q)} && \text{if } Q = \Omega^T \text{ or } Q = S^T, \quad l \geq 0; \\ \|u\|_{l,Q} &= \|u\|_{W_2^l(Q)} && \text{if } Q = \Omega \text{ or } Q = S, \quad l \geq 0, \end{aligned}$$

and $W_2^{0,0}(Q) = W_2^0(Q) = L_2(Q)$. Moreover,

$$\|u\|_{L_p(Q)} = |u|_{p,Q}, \quad 1 \leq p \leq \infty.$$

Next introduce the space $\Gamma_0^{l,l/2}(\Omega)$ with the norm

$$\|u\|_{\Gamma_0^{l,l/2}(\Omega)} = \sum_{|\alpha|+2i \leq l} \|D_\xi^\alpha \partial_t^i u\|_{0,\Omega} \equiv |u|_{l,0,\Omega}$$

and the space $L_p(0, T; \Gamma_0^{l,l/2}(\Omega))$ with the norm

$$\|u\|_{L_p(0,T;\Gamma_0^{l,l/2}(\Omega))} \equiv |u|_{l,0,p,\Omega^T}, \quad \text{where } 1 \leq p \leq \infty.$$

Moreover, let $C^\alpha(\Omega^T)$ ($\alpha \in (0, 1)$) denote the Hölder space with the norm

$$\|u\|_{C^\alpha(\Omega^T)} = \sup_{\Omega^T} \frac{|u(\xi, t) - u(\xi', t')|}{(\sqrt{|\xi - \xi'|^2 + |t - t'|^2})^\alpha};$$

let $C_B^0(\Omega^T)$ be the space of continuous bounded functions on Ω^T with the norm

$$\|u\|_{C_B^0(\Omega^T)} = \sup_{\Omega^T} |u(\xi, t)|$$

and let $C^{2,1}(Q)$ ($Q \subset \mathbb{R}^3 \times (0, T)$) denote the space of functions u such that $D_\xi^\alpha \partial_t^i u \in C^0(Q)$ for $|\alpha| + 2i \leq 2$.

Finally, the following seminorms are used:

$$\|u\|_{\kappa, Q^T} = \left(\int_0^T \frac{|u|_{2, Q}^2}{t^{2\kappa}} dt \right)^{1/2},$$

where $Q = \Omega$ ($\Omega \subset \mathbb{R}^3$ is a bounded domain) or $Q = S$, and $\kappa \in (0, 1)$;

$$[u]_{l, 2, Q} = [u]_{l, 2, Q, \xi} + [u]_{l, 2, Q, t},$$

where

$$\begin{aligned} [u]_{l, 2, Q, \xi} &= \sum_{|\alpha|+2i=l} [D_\xi^\alpha \partial_t^i u]_{l-|\alpha|, 2, Q, \xi}, \\ [u]_{l, 2, Q, t} &= \sum_{|\alpha|+2i=l} [D_\xi^\alpha \partial_t^i u]_{l/2-|\alpha|/2, 2, Q, t}, \end{aligned}$$

$Q = \Omega \times J$ ($\Omega \subset \mathbb{R}^3$ is a domain, $J = (0, T)$ or $J = (-\infty, T)$) or $Q = S \times J$.

In the case when $J = (0, T)$ the seminorms $[D_\xi^\alpha \partial_t^i u]_{l-|\alpha|, 2, Q, \xi}$ and $[D_\xi^\alpha \partial_t^i u]_{l/2-|\alpha|/2, 2, Q, t}$ are defined in (2.1). In the case when $J = (-\infty, T)$ we define the above seminorms in the same way.

Let X be whichever of the function spaces mentioned above. We say that a vector-valued function $u = (u_1, \dots, u_\nu)$ belongs to X if $u_i \in X$ for any $1 \leq i \leq \nu$.

In the sequel we shall use various notations for derivatives of u (where u is a scalar- or vector-valued function $u = (u_1, u_2, u_3)$). If u is a scalar-valued function we denote by $D_\xi^k u$ (where $\xi \in \Omega \subset \mathbb{R}^3$) the vector of all derivatives of u of order k , i.e. $D_\xi^k u = (D_\xi^\alpha u)_{|\alpha|=k}$. Similarly, if $u = (u_1, u_2, u_3)$ we denote by $D_\xi^k u$ the vector $(D_\xi^\alpha u_j)_{|\alpha|=k, j=1, 2, 3}$. By $D_{\xi, t}^k u$ we denote the vector $(D_\xi^\alpha \partial_t^i u_j)_{|\alpha|+2i=k, j=1, 2, 3}$ in the case when $u = (u_1, u_2, u_3)$ and the vector $(D_\xi^\alpha \partial_t^i u)_{|\alpha|+2i=k}$ in the scalar case. Hence

$$|D_\xi^k u| = \sum_{|\alpha|=k} |D_\xi^\alpha u| \quad \text{and} \quad |D_{\xi, t}^k u| = \sum_{|\alpha|+2i=k} |D_\xi^\alpha \partial_t^i u|.$$

We also use the notation $\nabla_\xi u \equiv D_\xi^1 u$ or $u_\xi \equiv D_\xi^1 u$.

Next, we denote by $u \cdot v$ either the scalar product of vectors u and v , or the product of matrices u and v .

Finally, we denote by $D_{\xi,t}^k u D_{\xi,t}^l v$ the following number:

$$(2.2) \quad D_{\xi,t}^k u D_{\xi,t}^l v = \sum_{\substack{|\alpha|+2i=k \\ |\beta|+2j=l}} \sum_{m=1}^p \sum_{n=1}^s D_{\xi}^{\alpha} \partial_t^i u_m D_{\xi}^{\beta} \partial_t^j v_n,$$

where $u = (u_1, \dots, u_p)$, $v = (v_1, \dots, v_s)$ ($k \geq 0$, $l \geq 0$, $p > 1$, $s > 1$). The product of more than two such factors is defined similarly.

We use the following lemmas.

LEMMA 2.1. *The following imbedding holds:*

$$(2.3) \quad W_r^l(\Omega) \subset L_p^{\alpha}(\Omega) \quad (\Omega \subset \mathbb{R}^3),$$

where $|\alpha| + 3/r - 3 \leq l$, $l \in \mathbb{Z}$, $1 \leq p, r \leq \infty$; $L_p^{\alpha}(\Omega)$ is the space of functions u such that $|D_{\xi}^{\alpha} u|_{p,\Omega} < \infty$; $W_r^l(\Omega)$ is the Sobolev space.

Moreover, the following interpolation inequalities hold:

$$(2.4) \quad |D_{\xi}^{\alpha} u|_{p,\Omega} \leq c\varepsilon^{1-\kappa} |D_{\xi}^l u|_{r,\Omega} + c\varepsilon^{-\kappa} |u|_{r,\Omega},$$

where $\kappa = |\alpha|/l + 3/(lr) - 3/(lp) < 1$, ε is a parameter and $c > 0$ is a constant independent of u and ε ;

$$(2.5) \quad |D_{\xi}^{\alpha} u|_{q,S} \leq c\varepsilon^{1-\kappa} |D_{\xi}^l u|_{r,\Omega} + c\varepsilon^{-\kappa} |u|_{r,\Omega},$$

where $\kappa = |\alpha|/l + 3/(lr) - 2/(lq) < 1$, ε is a parameter and $c > 0$ is a constant independent of u and ε . ■

Lemma 2.1 follows from Theorem 10.2 of [3].

LEMMA 2.2 (see [7]). *For sufficiently regular u we have*

$$\|\partial_t^i u(t)\|_{2l-1-2i,\Omega} \leq c(\|u\|_{2l,\Omega^T} + \|\partial_t^i u(0)\|_{2l-1-2i,\Omega}),$$

where $0 \leq 2i \leq 2l - 1$, $l \in \mathbb{N}$ and $c > 0$ is a constant independent of T . ■

LEMMA 2.3. *Let $u(\xi, t) = 0$ for $t \leq 0$. Then*

$$\begin{aligned} \int_{-\infty}^T dt \int_{-\infty}^T dt' \frac{|u(\xi, t) - u(\xi, t')|_{2,Q}^2}{|t - t'|^{1+2\alpha}} &\leq \int_0^T dt \int_0^T dt' \frac{|u(\xi, t) - u(\xi, t')|_{2,Q}^2}{|t - t'|^{1+2\alpha}} \\ &+ \frac{1}{\alpha} \int_0^T \frac{|u(t)|_{2,Q}^2}{t^{2\alpha}} dt, \end{aligned}$$

where $Q = \Omega$ ($\Omega \subset \mathbb{R}^3$ is a domain) or $Q = S = \partial\Omega$, and $\alpha \in \mathbb{R}$. ■

LEMMA 2.4. *Let $\tau \in (0, 1)$. Then for all $u \in W_2^{0,\tau/2}(\Omega^T)$,*

$$(2.6) \quad \int_0^T |u|_{2,\Omega}^2 \frac{dt}{t^{\tau}} \leq c_1 \int_0^T dt \int_0^T dt' \frac{|u(\cdot, t) - u(\cdot, t')|_{2,\Omega}^2}{|t - t'|^{1+\tau}} + c_2 T^{-\tau} \int_0^T |u|_{2,\Omega}^2 dt,$$

where c_1, c_2 do not depend on T and u . For $T = \infty$ the last term in (2.6) vanishes. ■

This was shown in [11], Lemma 6.3.

3. Local existence in the case $\sigma = 0$. In order to prove local existence of solutions of (1.1) we rewrite it in the Lagrangian coordinates introduced by (1.5) and (1.6):

$$\begin{aligned}
(3.1) \quad & \eta u_t - \mu \nabla_u^2 u - \nu \nabla_u \nabla_u \cdot u + \nabla_u p(\eta, \gamma) = \eta g && \text{in } \Omega^T, \\
& \eta_t + \eta \nabla_u \cdot u = 0 && \text{in } \Omega^T, \\
& \eta c_v(\eta, \gamma) \gamma_t - \kappa \nabla_u^2 \gamma = -\gamma p_\gamma(\eta, \gamma) \nabla_u \cdot u \\
& \quad + \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_j + \xi_{x_j} \cdot \nabla_\xi u_i)^2 \\
& \quad + (\nu - \mu)(\nabla_u \cdot u)^2 + \eta k && \text{in } \Omega^T, \\
& \mathbb{T}_u(u, p) \cdot \bar{n} = -p_0 \bar{n} && \text{on } S^T, \\
& \bar{n} \cdot \nabla_u \gamma = \tilde{\gamma} && \text{on } S^T, \\
& u|_{t=0} = v_0 && \text{in } \Omega, \\
& \eta|_{t=0} = \rho_0 && \text{in } \Omega, \\
& \gamma|_{t=0} = \vartheta_0 && \text{in } \Omega,
\end{aligned}$$

where $u(\xi, t) = v(X_u(\xi, t), t)$, $\gamma(\xi, t) = \vartheta(X_u(\xi, t), t)$, $\eta(\xi, t) = \rho(X_u(\xi, t), t)$, $g(\xi, t) = f(X_u(\xi, t), t)$, $k(\xi, t) = r(X_u(\xi, t), t)$, $\nabla_u = \xi_x \nabla_\xi \equiv \{\xi_{ix} \partial_{\xi_i}\}$,

$$\mathbb{T}_u(u, p) = -p \mathbf{1} + \mathbb{D}_u(u),$$

$$\mathbb{D}_u(u) = \{\mu(\xi_{kx_i} \partial_{\xi_k} u_j + \xi_{kx_j} \partial_{\xi_k} u_i) + (\nu - \mu) \delta_{ij} \nabla_u \cdot u\}$$

(here the summation convention over repeated indices is assumed and $\mathbf{1}$ is the unit matrix) and $\tilde{\gamma}(\xi, t) = \tilde{\vartheta}(X_u(\xi, t), t)$.

Let $A = \{a_{ij}\}$ be the Jacobi matrix of the transformation $x = X_u(\xi, t)$, where $a_{ij} = \delta_{ij} + \int_0^t \partial_{\xi_j} u_i(\xi, \tau) d\tau$. Assuming that $|\nabla_\xi u|_{\infty, \Omega^T} \leq M$ we obtain

$$(3.2) \quad 0 < c_1(1 - Mt)^3 \leq \det\{x_\xi\} \leq c_2(1 + Mt)^3, \quad t \leq T,$$

where $c_1, c_2 > 0$ are constants and $T > 0$ is sufficiently small. Moreover,

$$\det A = \exp\left(\int_0^t \nabla_u \cdot u d\tau\right) = \rho_0 / \eta.$$

Let S_t be determined (at least locally) by the equation $\phi(x, t) = 0$. Then S is described by $\phi(x(\xi, t), t)|_{t=0} \equiv \tilde{\phi}(\xi) = 0$. Thus, we have

$$\bar{n}(x(\xi, t), t) = -\frac{\nabla_x \phi(x, t)}{|\nabla_x \phi(x, t)|} \Big|_{x=x(\xi, t)} \quad \text{and} \quad \bar{n}_0(\xi) = -\frac{\nabla_\xi \tilde{\phi}(\xi)}{|\nabla_\xi \tilde{\phi}(\xi)|}.$$

First we consider the linear problems

$$(3.3) \quad \begin{aligned} Lu &\equiv u_t - \mu \nabla_\xi^2 u - \nu \nabla_\xi \nabla_\xi \cdot u = F && \text{in } \Omega^T, \\ \mathbb{D}_\xi(u) \cdot \bar{n}_0 &= G && \text{on } S^T, \\ u|_{t=0} &= u_0 && \text{in } \Omega \end{aligned}$$

(where $\mathbb{D}_\xi(u) = \{\mu(\partial_{\xi_i} u_j + \partial_{\xi_j} u_i) + (\nu - \mu)\delta_{ij}\partial_{\xi_k} u_k\}$) and

$$(3.4) \quad \begin{aligned} \gamma_t - \kappa \nabla_\xi^2 \gamma &= K && \text{in } \Omega^T, \\ \bar{n}_0 \cdot \nabla_\xi \gamma &= \tilde{\gamma} && \text{on } S^T, \\ \gamma|_{t=0} &= \gamma_0 && \text{in } \Omega. \end{aligned}$$

We assume

$$(3.5) \quad \begin{aligned} F &\in W_2^{2,1}(\Omega^T), \quad G \in W_2^{3-1/2, 3/2-1/4}(S^T), \\ \mathbf{D}_{\xi,t}^2 G|_{1/4, S^T} &< \infty, \quad u_0 \in W_2^3(\Omega) \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} K &\in W_2^{2,1}(\Omega^T), \quad \tilde{\gamma} \in W_2^{3-1/2, 3/2-1/4}(S^T), \\ \mathbf{D}_{\xi,t}^2 \tilde{\gamma}|_{1/4, S^T} &< \infty, \quad \gamma_0 \in W_2^3(\Omega). \end{aligned}$$

Moreover, we assume the following compatibility conditions:

$$(3.7) \quad D_\xi^\alpha (\mathbb{D}_\xi(u(0)) \cdot \bar{n}_0 - G(0)) = 0, \quad |\alpha| \leq 1, \quad \text{on } S,$$

and

$$(3.8) \quad D_\xi^\alpha (\bar{n}_0 \cdot \nabla_\xi \gamma(0) - \tilde{\gamma}(0)) = 0, \quad |\alpha| \leq 1, \quad \text{on } S.$$

First we consider problem (3.3). Define functions ψ_i for $i = 0, 1$ by

$$(3.9) \quad \psi_i = \partial_t^i u|_{t=0} \quad \text{in } \Omega.$$

Hence

$$(3.10) \quad \psi_0 = u_0, \quad \psi_1 = \mu \Delta u_0 + \nu \nabla \operatorname{div} u_0 + F(0).$$

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^3$ be either a halfspace or a bounded domain with smooth boundary $\partial\Omega$ and let $T \leq \infty$. Assume that*

$$(3.11) \quad \psi_0 \in W_2^3(\Omega), \quad \psi_1 \in W_2^1(\Omega).$$

Then there exists $v \in W_2^{4,2}(\Omega^T)$ such that

$$(3.12) \quad \partial_t^i v|_{t=0} = \psi_i \quad \text{in } \Omega \quad (i = 0, 1)$$

and

$$(3.13) \quad \|v\|_{4,\Omega^T} \leq c(\|\psi_0\|_{3,\Omega} + \|\psi_1\|_{1,\Omega}),$$

where $c > 0$ is a constant independent of T .

Proof. Using the Hestenes–Whitney method (see [1]) we can extend ψ_0 and ψ_1 to functions $\tilde{\psi}_0 \in W_2^3(\mathbb{R}^3)$ and $\tilde{\psi}_1 \in W_2^1(\mathbb{R}^3)$ such that

$$\|\tilde{\psi}_0\|_{3,\mathbb{R}^3} \leq c\|\psi_0\|_{3,\Omega}, \quad \|\tilde{\psi}_1\|_{1,\mathbb{R}^3} \leq c\|\psi_1\|_{1,\Omega},$$

where $c = c(\Omega)$. Then from [16] (Lemma 6.5) we deduce that there exists $\tilde{v} \in W_2^{4,2}(\mathbb{R}^3 \times \mathbb{R}_+^1)$ such that

$$(3.14) \quad \partial_t^i \tilde{v}|_{t=0} = \tilde{\psi}_i, \quad i = 0, 1,$$

and

$$(3.15) \quad \|\tilde{v}\|_{4,\mathbb{R}^3 \times \mathbb{R}_+^1} \leq c(\|\tilde{\psi}_0\|_{3,\mathbb{R}^3} + \|\tilde{\psi}_1\|_{1,\mathbb{R}^3}) \leq c(\|\psi_0\|_{3,\Omega} + \|\psi_1\|_{1,\Omega}).$$

Therefore $v = \tilde{v}|_{\Omega^T}$ satisfies conditions (3.12) and (3.13). ■

Now we prove the following theorem.

THEOREM 3.1. *Let $S \in W_2^{4-1/2}$ and let assumptions (3.5) and (3.7) be satisfied ($T < \infty$). Then there exists a solution of (3.3) such that $u \in W_2^{4,2}(\Omega^T)$ and*

$$(3.16) \quad \|u\|_{4,\Omega^T} \leq c(T)(\|F\|_{2,\Omega^T} + \|G\|_{3-1/2,S^T} + \|D_{\xi,t}^2 G\|_{1/4,S^T} + |u(0)|_{3,0,\Omega}),$$

where c is an increasing continuous function of T and

$$(3.17) \quad |u(0)|_{3,0,\Omega} = \|u_0\|_{3,\Omega} + \|\partial_t u|_{t=0}\|_{1,\Omega}.$$

Proof. Introduce the function

$$(3.18) \quad w = u - v,$$

where v is the function from Lemma 3.1. Then instead of (3.3) we obtain the problem

$$(3.19) \quad \begin{aligned} w_t - \mu \Delta w - \nu \nabla \operatorname{div} w &= F - Lv \equiv f && \text{in } \Omega^T, \\ \mathbb{D}_\xi(w) \cdot \bar{n}_0 &= G - \mathbb{D}_\xi(v) \cdot \bar{n}_0 \equiv g && \text{on } S^T, \\ w|_{t=0} &= 0 && \text{in } \Omega, \end{aligned}$$

where in view of the compatibility condition (3.7) we have

$$(3.20) \quad f(0) = g(0) = 0.$$

It is sufficient to consider problem (3.19) in $\Omega^T \equiv R^T \equiv \mathbb{R}_+^3 \times [0, T]$ because using a partition of unity and appropriate norms (see [8], Sects. 20, 21) we obtain the existence and the appropriate estimate of solutions of (3.19) in a bounded domain Ω .

Thus, consider problem (3.19) in R^T and extend functions f , g , w by zero for $t < 0$ to functions f_1 , g_1 , w_1 . Then instead of (3.19) we get the following boundary value problem:

$$(3.21) \quad \begin{aligned} Lw_1 &= f_1 && \text{in } \tilde{D}_4(T) = \mathbb{R}_+^3 \times (-\infty, T], \\ \mathbb{D}_\xi(w_1) \cdot \bar{n}_0 &= g_1 && \text{in } \tilde{E}_3(T) = \mathbb{R}^2 \times (-\infty, T]. \end{aligned}$$

Next using the Hestenes–Whitney method extend f_1 and g_1 to functions f_2 and g_2 defined on $\mathbb{R}_+^3 \times (-\infty, \infty)$ and $\mathbb{R}^2 \times (-\infty, \infty)$, respectively. Then instead of (3.21) we have the problem

$$(3.22) \quad \begin{aligned} Lw_2 &= f_2 && \text{in } \tilde{D}_4 = \mathbb{R}_+^3 \times (-\infty, \infty), \\ \mathbb{D}_\xi(w_2) \cdot \bar{n}_0 &= g_2 && \text{in } \tilde{E}_3 = \mathbb{R}^2 \times (-\infty, \infty), \end{aligned}$$

and the estimate

$$(3.23) \quad \|f_2\|_{2, \tilde{D}_4} \leq c \|f_1\|_{2, \tilde{D}_4(T)}.$$

Next we extend f_2 by the Hestenes–Whitney method to a function $f_3 \in W_2^{4,2}(\mathbb{R}^4)$ such that

$$(3.24) \quad \|f_3\|_{2, \mathbb{R}^4} \leq c \|f_2\|_{2, \tilde{D}_4}.$$

Consider now the system

$$(3.25) \quad Lw_3 = f_3 \quad \text{in } \mathbb{R}^4.$$

By potential techniques (see [8], Sections 12 and 21) and (3.23), (3.24) there exists a solution $w_3 \in W_2^{4,2}(\mathbb{R}^4)$ of (3.25) and

$$(3.26) \quad \|w_3\|_{4, \mathbb{R}^4} \leq c(T) \|f\|_{2, R^T},$$

where $c(T)$ is an increasing function of T .

Introduce the function

$$(3.27) \quad w_4 = w_2 - w_3.$$

By (3.22) and (3.25) we have

$$(3.28) \quad \begin{aligned} Lw_4 &= 0 && \text{in } \tilde{D}_4, \\ \mathbb{D}_\xi(w_4) \cdot \bar{n}_0 &= g_2 - \mathbb{D}_\xi(w_3) \cdot \bar{n}_0 \equiv g_3 && \text{in } \tilde{E}_3. \end{aligned}$$

Again by potential techniques there exists a solution $w_4 \in W_2^{4,2}(\tilde{D}_4)$ of (3.28) and

$$\begin{aligned} \|w_4\|_{4, \tilde{D}_4} &\leq c[g_3]_{3-1/2, 2, \tilde{E}_3} \leq c([g_2]_{3-1/2, 2, \tilde{E}_3} + \|w_3\|_{4, \tilde{D}_4}) \\ &\leq c(T)([g_1]_{3-1/2, 2, \tilde{E}_3}(T) + \|f\|_{2, R^T}), \end{aligned}$$

where we used (3.26) and the Hestenes–Whitney method for g_1 . Hence

$$(3.29) \quad \|w_4\|_{4,R^T} \leq c(T)([g_1]_{3-1/2,2,\tilde{E}_3(T)} + \|f\|_{2,R^T}).$$

Further, using Lemma 2.3 we have

$$(3.30) \quad [g_1]_{3-1/2,2,\tilde{E}_3(T)} = \sum_{|\alpha|=2} [D_\xi^\alpha g_1]_{1/2,2,\tilde{E}_3(T),\xi} + [\partial_t g_1]_{1/2,2,\tilde{E}_3(T),\xi} \\ + \sum_{|\alpha|=2} [D_\xi^\alpha g_1]_{1/4,2,\tilde{E}_3(T),t} + [\partial_t g_1]_{1/4,2,\tilde{E}_3(T),t} \\ \leq c \left\{ \sum_{|\alpha|=2} [D_\xi^\alpha g]_{1/2,2,\mathbb{R}^2 \times [0,T],\xi} + [\partial_t g]_{1/2,2,\mathbb{R}^2 \times [0,T],\xi} \right. \\ + \sum_{|\alpha|=2} [D_\xi^\alpha g]_{1/4,2,\mathbb{R}^2 \times [0,T],t} + [\partial_t g]_{1/4,2,\mathbb{R}^2 \times [0,T],t} \\ \left. + \left(\int_0^T \frac{|D_{\xi,t}^2 g|_{2,\mathbb{R}^2}^2}{t^{1/2}} dt \right)^{1/2} \right\} \\ \leq c(\|g\|_{3-1/2,S^T} + \mathbf{I}D_{\xi,t}^2 g \mathbf{I}_{1/4,S^T}),$$

where $S^T = \mathbb{R}^2 \times [0, T]$.

Taking into account the right-hand side of (3.19) and (3.18), (3.21), (3.22), (3.25)–(3.30) we conclude that there exists a solution $u \in W_2^{4,2}(\Omega^T)$ of (3.3) satisfying

$$(3.31) \quad \|u\|_{4,R^T} \\ \leq c(T)(\|F\|_{2,R^T} + \|v\|_{4,R^T} + \|G\|_{3-1/2,S^T} \\ + \mathbf{I}D_{\xi,t}^2 G \mathbf{I}_{1/4,S^T} + \|v_\xi \bar{n}_0\|_{3-1/2,S^T} + \mathbf{I}D_{\xi,t}^2(v_\xi \bar{n}_0) \mathbf{I}_{1/4,S^T}),$$

where

$$v_\xi \bar{n}_0 = \sum_{i,j,k=1}^3 v_{i\xi_j} n_{0k}.$$

By Lemma 3.1 we have

$$(3.32) \quad \|v\|_{4,R^T} \leq c(\|\phi_0\|_{3,\mathbb{R}_+^3} + \|\phi_1\|_{1,\mathbb{R}_+^3})$$

and

$$(3.33) \quad \|v_\xi \bar{n}_0\|_{3-1/2,S^T} \leq \|\tilde{v}_\xi \bar{n}_0\|_{3-1/2,S \times (0,\infty)} \leq c\|\tilde{v}_\xi\|_{3,\mathbb{R}_+^3 \times (0,\infty)} \\ \leq c\|v\|_{4,R^T} \leq c(\|\phi_0\|_{3,\mathbb{R}_+^3} + \|\phi_1\|_{1,\mathbb{R}_+^3}),$$

where we have used the fact that $S \in W_2^{4-1/2}$.

It remains to estimate $\mathbf{I}D_{\xi,t}^2(v_\xi \bar{n}_0) \mathbf{I}_{1/4,S^T}$. We have

$$\begin{aligned}
(3.34) \quad & \mathbf{I}D_{\xi,t}^2(v_\xi \bar{n}_0)\mathbf{I}_{1/4,S^T} \\
& = \left(\int_0^T \frac{|D_\xi^2 \bar{n}_0 D_\xi^1 v + D_\xi^1 \bar{n}_0 D_\xi^2 v + \bar{n}_0 D_\xi^3 v + \bar{n}_0 \partial_t D_\xi^1 v|_{2,S}^2}{t^{1/2}} dt \right)^{1/2} \\
& \leq c \left(\int_0^T \frac{|D_\xi^2 \bar{n}_0|_{4,S}^2 |D_\xi^1 v|_{4,S}^2 + |D_\xi^2 v|_{2,S}^2 + |D_\xi^3 v|_{2,S}^2 + |\partial_t D_\xi^1 v|_{2,S}^2}{t^{1/2}} dt \right)^{1/2},
\end{aligned}$$

where we have used the fact that $S \in W_2^{4-1/2}$ and Lemma 2.1, and where the products are understood in the sense of (2.2).

Next, by Lemma 2.1 we get

$$\int_0^T \frac{|D_\xi^2 \bar{n}_0|_{4,S}^2 |D_\xi^1 v|_{4,S}^2}{t^{1/2}} dt \leq c \int_0^T \frac{\|D_\xi^1 v\|_{1,S}^2}{t^{1/2}} dt.$$

Hence in view of Lemma 2.4, (3.34) yields

$$\begin{aligned}
& \mathbf{I}D_{\xi,t}^2(v_\xi \bar{n}_0)\mathbf{I}_{1/4,S^T} \\
& \leq c \left(\int_0^\infty \frac{|D_\xi^1 \tilde{v}|_{2,S}^2 + |D_\xi^2 \tilde{v}|_{2,S}^2 + |D_\xi^3 \tilde{v}|_{2,S}^2 + |D_\xi^1 \partial_t \tilde{v}|_{2,S}^2}{t^{1/2}} dt \right)^{1/2} \\
& \leq c_1 \left[\int_0^\infty dt \int_0^\infty dt' \left(\frac{|D_\xi^1 \tilde{v}(\xi, t) - D_\xi^1 \tilde{v}(\xi, t')|_{2,S}^2}{|t - t'|^{1+1/2}} \right. \right. \\
& \quad + \frac{|D_\xi^2 \tilde{v}(\xi, t) - D_\xi^2 \tilde{v}(\xi, t')|_{2,S}^2}{|t - t'|^{1+1/2}} + \frac{|\partial_t D_\xi^1 \tilde{v}(\xi, t) - \partial_{t'} D_\xi^1 \tilde{v}(\xi, t')|_{2,S}^2}{|t - t'|^{1+1/2}} \\
& \quad \left. \left. + \frac{|D_\xi^3 \tilde{v}(\xi, t) - D_\xi^3 \tilde{v}(\xi, t')|_{2,S}^2}{|t - t'|^{1+1/2}} \right) \right]^{1/2}.
\end{aligned}$$

Therefore

$$(3.35) \quad \mathbf{I}D_{\xi,t}^2(v_\xi \bar{n}_0)\mathbf{I}_{1/4,S^T} \leq c \|v\|_{4,R^T},$$

where we have used Theorem 5.1 from [8].

Taking into account (3.31)–(3.33) and (3.35) we get (3.16). This completes the proof of the theorem. ■

In the same way we can prove

THEOREM 3.2. *Let $S \in W_2^{4-1/2}$ and let assumptions (3.6) and (3.8) be satisfied ($T < \infty$). Then there exists a solution of (3.4) such that $\gamma \in W_2^{4,2}(\Omega^T)$ and*

$$\begin{aligned}
(3.36) \quad & \|\gamma\|_{4,\Omega^T} \\
& \leq c(T) (\|K\|_{2,\Omega^T} + \|\tilde{\gamma}\|_{3-1/2,S^T} + \mathbf{I}D_{\xi,t}^2 \tilde{\gamma}\mathbf{I}_{1/4,S^T} + |\gamma(0)|_{3,0,\Omega}),
\end{aligned}$$

where c is an increasing function of T and

$$(3.37) \quad |\gamma(0)|_{3,0,\Omega} = \|\gamma_0\|_{3,\Omega} + \|\partial_t^1 \gamma|_{t=0}\|_{1,\Omega} \cdot \blacksquare$$

Now we have to consider the following problems:

$$(3.38) \quad \begin{aligned} \eta u_t - \mu \nabla_\xi^2 u - \nu \nabla_\xi \nabla_\xi \cdot u &= F && \text{in } \Omega^T, \\ \mathbb{D}_\xi(u) \cdot \bar{n}_0 &= G && \text{on } S^T, \\ u|_{t=0} &= u_0 && \text{in } \Omega \end{aligned}$$

and

$$(3.39) \quad \begin{aligned} \eta c_v(\eta, \beta) \gamma_t - \kappa \nabla_\xi^2 \gamma &= K && \text{in } \Omega^T, \\ \bar{n}_0 \cdot \nabla_\xi \gamma &= \tilde{\gamma} && \text{on } S^T, \\ \gamma|_{t=0} &= \gamma_0 && \text{in } \Omega. \end{aligned}$$

First we consider (3.38). The following theorem is proved in [15].

THEOREM 3.3. *Assume that*

$$\begin{aligned} S \in W_2^{4-1/2}, F \in W_2^{2,1}(\Omega^T), G \in W_2^{3-1/2,3/2-1/4}(S^T), \\ \mathbf{D}_{\xi,t}^2 G|_{1/4,S^T} < \infty, u_0 \in W_2^3(\Omega), \\ \eta \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)) \cap C^\alpha(\Omega^T) \ (\alpha \in (0, 1)), 1/\eta \in L_\infty(\Omega^T). \end{aligned}$$

Moreover, let the compatibility condition (3.7) be satisfied. Then there exists a unique solution $u \in W_2^{4,2}(\Omega^T)$ to problem (3.38) satisfying the estimate

$$(3.40) \quad \begin{aligned} \|u\|_{4,\Omega^T} \leq \phi_1(\|1/\eta\|_{\infty,\Omega^T}, \|\eta\|_{\infty,\Omega^T}, T) [\|F\|_{2,\Omega^T} + \|G\|_{3-1/2,S^T} \\ + \mathbf{D}_{\xi,t}^2 G|_{1/4,S^T} + \phi_2(\|\eta\|_{2,0,\infty,\Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}) \|u\|_{2,\Omega^T}] \\ + \phi_3(\|1/\eta\|_{\infty,\Omega^T}, T) \|u(0)\|_{3,0,\Omega}, \end{aligned}$$

where ϕ_i ($i = 1, 2, 3$) are nonnegative increasing continuous functions of their arguments. \blacksquare

Now we consider problem (3.39). Assume that $\eta \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega))$ and $\beta \in W_2^{4,2}(\Omega^T)$. Applying Theorems 10.2 and 10.4 of [3] we find that $\eta \in C_B^0(\Omega^T)$ and $\beta \in C_B^0(\Omega^T)$. Therefore, there exists a bounded domain $V \subset \mathbb{R}^2$ such that $(\eta(\xi, t), \beta(\xi, t)) \in V$ for any $(\xi, t) \in \Omega^T$.

LEMMA 3.2. *Assume that $\eta \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)) \cap C^\alpha(\Omega^T)$ (where $\alpha \in (0, 1/2)$), $1/\eta \in L_\infty(\Omega^T)$, $\eta > 0$, $\beta \in W_2^{4,2}(\Omega^T)$, $1/\beta \in L_\infty(\Omega^T)$, $\beta > 0$, $c_v \in C^2(\mathbb{R}_+^2)$, $c_v > 0$. Then $\eta c_v(\eta, \beta) \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega))$, $\eta c_v \in C^\alpha(\Omega^T)$, $1/(\eta c_v(\eta, \beta)) \in L_\infty(\Omega^T)$ and*

$$(3.41) \quad \begin{aligned} \|\eta c_v(\eta, \beta)\|_{2,0,\infty,\Omega^T} \\ \leq \psi_1(\|c_v\|_{C^2(\bar{V})}, \|1/c_v\|_{C^0(\bar{V})}, \|\eta\|_{2,0,\infty,\Omega^T}, \|\beta\|_{4,\Omega^T}, \|\beta(0)\|_{3,0,\Omega}), \end{aligned}$$

$$(3.42) \quad \|\eta c_v(\eta, \beta)\|_{\infty,\Omega^T} \leq \|c_v\|_{C^0(\bar{V})} \|\eta\|_{\infty,\Omega^T},$$

$$(3.43) \quad |1/(\eta c_v(\eta, \beta))|_{\infty, \Omega^T} \leq \|1/c_v\|_{C^0(\bar{V})} |1/\eta|_{\infty, \Omega^T},$$

$$(3.44) \quad \|\eta c_v(\eta, \beta)\|_{C^\alpha(\Omega^T)} \leq \psi_2(\|c_v\|_{C^2(\bar{V})}, \|\eta\|_{C^\alpha(\Omega^T)}, \|\beta\|_{4, \Omega^T}, |\beta(0)|_{3,0,\Omega}, T),$$

where ψ_i ($i = 1, 2$) are nonnegative increasing continuous functions of their arguments.

Proof. By the assumptions we can choose V such that $V \subset \mathbb{R}_+^2$. Thus (3.42) and (3.43) are obviously satisfied. In order to obtain (3.41) we calculate the derivatives $D_\xi^\alpha \partial_t^i(\eta c_v)$ (where $|\alpha| + 2i \leq 2$) and next we apply Lemmas 2.1 and 2.2. To prove (3.44) we also use Lemmas 2.1 and 2.2 and the Sobolev imbedding theorem. ■

THEOREM 3.4. *Assume that*

$$\begin{aligned} S &\in W_2^{4-1/2}, K \in W_2^{2,1}(\Omega^T), \tilde{\gamma} \in W_2^{3-1/2, 3/2-1/4}(S^T), \gamma_0 \in W_2^3(\Omega), \\ \eta &\in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)), 1/\eta \in L_\infty(\Omega^T), \eta \in C^\alpha(\Omega^T) \quad (0 < \alpha < 1/2), \\ \eta &> 0, \beta \in W_2^{4,2}(\Omega^T), 1/\beta \in L_\infty(\Omega^T), \beta > 0, c_v \in C^2(\mathbb{R}_+^2), c_v > 0. \end{aligned}$$

Moreover, let the compatibility condition (3.8) be satisfied. Then there exists a unique solution $\gamma \in W_2^{4,2}(\Omega^T)$ to problem (3.39) satisfying the estimate

$$(3.45) \quad \begin{aligned} \|\gamma\|_{4, \Omega^T} &\leq \phi_4(\|c_v\|_{C^0(\bar{V})}, \|1/c_v\|_{C^0(\bar{V})}, |\eta|_{\infty, \Omega^T}, |1/\eta|_{\infty, \Omega^T}, T) \\ &\quad \times [\|K\|_{2, \Omega^T} + \|\tilde{\gamma}\|_{3-1/2, S^T} + \|D_{\xi,t}^2 \tilde{\gamma}\|_{1/4, S^T} \\ &\quad + \phi_5(\|c_v\|_{C^2(\bar{V})}, \|1/c_v\|_{C^0(\bar{V})}, |\eta|_{2,0,\infty, \Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}, \\ &\quad \|\beta\|_{4, \Omega^T}, |\beta(0)|_{3,0,\Omega}, T) \|\gamma\|_{2, \Omega^T}] \\ &\quad + \phi_6(\|1/c_v\|_{C^0(\bar{V})}, |1/\eta|_{\infty, \Omega^T}, T) |\gamma(0)|_{3,0,\Omega}, \end{aligned}$$

where ϕ_i ($i = 4, 5, 6$) are nonnegative increasing continuous functions of their arguments.

Proof. Since by Lemma 3.2, $\eta c_v(\eta, \beta) \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega))$, $\eta c_v(\eta, \beta) \in C^\alpha(\Omega^T)$ ($\alpha \in (0, 1/2)$) and $1/(\eta c_v(\eta, \beta)) \in L_\infty(\Omega^T)$, using the same argument as in Theorem 3.3 and inequalities (3.41)–(3.44) we prove the existence of a unique solution $\gamma \in W_2^{4,2}(\Omega^T)$ of problem (3.39) satisfying the estimate (3.45). ■

Now consider the problems

$$(3.46) \quad \begin{aligned} \eta u_t - \mu \nabla_w^2 u - \nu \nabla_w \nabla_w \cdot u &= F && \text{in } \Omega^T, \\ \mathbb{D}_w(u) \cdot \bar{n} &= G && \text{on } S^T, \\ u|_{t=0} &= u_0 && \text{in } \Omega, \end{aligned}$$

and

$$(3.47) \quad \begin{aligned} \eta c_v(\eta, \beta) \gamma_t - \kappa \nabla_w^2 \gamma &= K && \text{in } \Omega^T, \\ \bar{n} \cdot \nabla_w \gamma &= \tilde{\gamma} && \text{on } S^T, \\ \gamma|_{t=0} &= \gamma_0 && \text{in } \Omega, \end{aligned}$$

where $\bar{n} = \bar{n}(X_w(\xi, t), t)$.

The following theorem concerns problem (3.46).

THEOREM 3.5. *Assume that*

$$\begin{aligned} S &\in W_2^{4-1/2}, F \in W_2^{2,1}(\Omega^T), G \in W_2^{3-1/2, 3/2-1/4}(S^T), \\ u_0 &\in W_2^3(\Omega), \eta \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)), 1/\eta \in L_\infty(\Omega^T), \\ \eta &\in C^\alpha(\Omega^T) \ (\alpha \in (0, 1)), w \in W_2^{4,2}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega)), T < \infty. \end{aligned}$$

Moreover, let the compatibility condition (3.7) be satisfied. Assume that T is so small that

$$(3.48) \quad \begin{aligned} T^a (\|w\|_{4, \Omega^T} + |w(0)|_{3,0,\Omega}) \\ \times \phi_7 \left(\int_0^T |w_\xi|_{\infty, \Omega} dt, |1/\eta|_{\infty, \Omega^T}, |\eta|_{\infty, \Omega^T}, \|w\|_{4, \Omega^T}, |w(0)|_{3,0,\Omega}, T \right) \leq \delta, \end{aligned}$$

where ϕ_7 is a nonnegative nondecreasing continuous function of its arguments, $a > 0$ is a constant and $\delta > 0$ is sufficiently small. Then there exists a unique solution $u \in W_2^{4,2}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega))$ to problem (3.46) satisfying the estimate

$$(3.49) \quad \begin{aligned} \|u\|_{4, \Omega^T} &\leq \phi_1(|1/\eta|_{\infty, \Omega^T}, |\eta|_{\infty, \Omega^T}, T) [\|F\|_{2, \Omega^T} + \|G\|_{3-1/2, S^T} \\ &\quad + \|D_{\xi,t}^2 G\|_{1/4, S^T} + \phi_2(|\eta|_{2,0,\infty, \Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}) \|u\|_{2, \Omega^T}] \\ &\quad + \phi_3'(|1/\eta|_{\infty, \Omega^T}, T) |u(0)|_{3,0,\Omega}, \end{aligned}$$

where ϕ_i ($i = 1, 2$) are the functions from Theorem 3.3, and ϕ_3' is a nonnegative increasing function.

Proof. In order to prove the existence of solutions to (3.46) we rewrite it as

$$(3.50) \quad \begin{aligned} \eta u_t - \mu \nabla_\xi^2 u - \nu \nabla_\xi \nabla_\xi \cdot u \\ &= F + \mu (\nabla_w^2 u - \nabla_\xi^2 u) + \nu (\nabla_w \nabla_w \cdot u - \nabla_\xi \nabla_\xi \cdot u) \\ &\equiv F + \tilde{F} && \text{in } \Omega^T, \\ \mathbb{D}_\xi(u) \cdot \bar{n}_0 &= G + (\mathbb{D}_\xi(u) \cdot \bar{n}_0 - \mathbb{D}_w(u) \cdot \bar{n}) \\ &= \mathbb{D}_\xi(u) \cdot (\bar{n}_0 - \bar{n}) + (\mathbb{D}_\xi(u) - \mathbb{D}_w(u)) \cdot \bar{n} \equiv G + \tilde{G} && \text{on } S^T, \\ u|_{t=0} &= u && \text{in } \Omega. \end{aligned}$$

Using Theorem 3.3 we have the following estimate for solutions of (3.50):

$$(3.51) \quad \begin{aligned} \|u\|_{4,\Omega^T} &\leq \phi_1(|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, T) [\|F + \tilde{F}\|_{2,\Omega^T} \\ &\quad + \|G + \tilde{G}\|_{3-1/2,S^T} + \|D_{\xi,t}^2 G + D_{\xi,t}^2 \tilde{G}\|_{1/4,S^T} \\ &\quad + \phi_2(|\eta|_{2,0,\infty,\Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}) \|u\|_{2,\Omega^T}] \\ &\quad + \phi_3(|1/\eta|_{\infty,\Omega^T}, T) |u(0)|_{3,0,\Omega}. \end{aligned}$$

First we estimate $\|\tilde{F}\|_{2,\Omega^T}$. By the form of the operator ∇_w we obtain

$$(3.52) \quad \begin{aligned} &\|\tilde{F}\|_{2,\Omega^T} \\ &\leq c \|(\xi_x \nabla_\xi)^2 u - \nabla_\xi^2 u\|_{2,\Omega^T} + c \|\xi_x \nabla_\xi (\xi_x \nabla_\xi \cdot u) - \nabla_\xi \nabla_\xi \cdot u\|_{2,\Omega^T} \\ &\leq c \|(\xi_x \nabla_\xi)^2 u - \nabla_\xi^2 u\|_{2,\Omega^T} \\ &\quad + c \|\xi_x \nabla_\xi (\xi_x) \nabla_\xi \cdot u\|_{2,\Omega^T} + c \|(\xi_x^2 - \mathbf{1}) D_\xi^2 u\|_{2,\Omega^T} \\ &\leq c \left[\left\| f_1 \int_0^t D_\xi^2 w \, d\tau D_\xi^1 u \right\|_{2,\Omega^T} + \left\| f_2 \int_0^t D_\xi^1 w \, d\tau D_\xi^2 u \right\|_{2,\Omega^T} \right], \end{aligned}$$

where $f_i = f_i(\mathbf{1} + \int_0^t w_\xi \, d\tau)$ ($i = 1, 2$) are smooth functions of their arguments; $\xi_x = x_\xi^{-1} = \tilde{x}_\xi / \det\{x_\xi\}$, \tilde{x}_ξ is the matrix of algebraic complements of $\{x_\xi\}$ and $x_\xi = \mathbf{1} + \int_0^t w_\xi \, d\tau$. The products on the right-hand side of (3.52) (and below) are in the sense of (2.2).

Now, applying Hölder and Minkowski inequalities and Lemmas 2.1–2.2 we get

$$(3.53) \quad \begin{aligned} &\left\| f_1 \int_0^t D_\xi^2 w \, d\tau u_\xi \right\|_{2,\Omega^T} + \left\| f_2 \int_0^t w_\xi \, d\tau D_\xi^2 u \right\|_{2,\Omega^T} \\ &\leq c [T \|w\|_{4,\Omega^T}^2 \|u\|_{4,\Omega^T} + T^{3/2} \|w\|_{4,\Omega^T}^3 \|u\|_{4,\Omega^T} \\ &\quad + T^{1/2} \|w\|_{4,\Omega^T} \|u\|_{4,\Omega^T} \\ &\quad + T^{1/2} \|w\|_{4,\Omega^T}^2 (\|u\|_{4,\Omega^T} + \|u(0)\|_{3,\Omega}) \\ &\quad + T^{1/2} (\|w\|_{4,\Omega^T} + \|w(0)\|_{3,\Omega}) (\|u\|_{4,\Omega^T} + \|u(0)\|_{3,\Omega})], \end{aligned}$$

where $c = c(\int_0^T |w_\xi|_{\infty,\Omega} \, dt)$.

Next consider

$$(3.54) \quad \begin{aligned} \|\tilde{G}\|_{3-1/2,S^T} &\leq c \left\| f_3 \int_0^t w_\xi \, d\tau u_\xi \tilde{\phi}_\xi \right\|_{3-1/2,S^T} \\ &\leq c \left(\sum_{|\alpha|+2i \leq 2} \left| D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi \, d\tau u_\xi \tilde{\phi}_\xi \right) \right|_{2,S^T} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/2, S^T, \xi} \\
& + \sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/4, S^T, t},
\end{aligned}$$

where $f_3 = f_3(\mathbf{1} + \int_0^t w_\xi d\tau)$ is a smooth function of its arguments.

By the Hölder and Minkowski inequalities, Lemma 2.1 (inequality (2.5)) and Lemma 2.2 we have

$$\begin{aligned}
(3.55) \quad & \sum_{|\alpha|+2i \leq 2} \left| D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right|_{2, S^T} \\
& \leq c [T^{1/2} \|w\|_{4, \Omega^T} \|u\|_{4, \Omega^T} + T \|w\|_{4, \Omega^T}^2 \|u\|_{4, \Omega^T} + T^{3/2} \|w\|_{4, \Omega^T}^3 \|u\|_{4, \Omega^T} \\
& \quad + T^{1/2} \|w\|_{4, \Omega^T}^2 (\|u\|_{4, \Omega^T} + |u(0)|_{3, 0, \Omega}) \\
& \quad + T^{1/2} (\|w\|_{4, \Omega^T} + |w(0)|_{3, 0, \Omega}) (\|u\|_{4, \Omega^T} + |u(0)|_{3, 0, \Omega})] \equiv I,
\end{aligned}$$

where $c = c(\int_0^T |w_\xi|_{\infty, \Omega} dt)$.

Now consider

$$I_1 \equiv \sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/2, S^T, \xi}.$$

We shall only estimate some terms of I_1 . For example we have

$$\begin{aligned}
I_2 \equiv & \left\{ \int_0^T \int_S \int_S \frac{|[f_3''(\int_0^t D_\xi^2 w d\tau)]^2 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi](\xi) \dots}{\dots} \dots \right. \\
& \left. \dots \frac{-[f_3''(\int_0^t D_{\xi'}^2 w d\tau)]^2 \int_0^t w_{\xi'} d\tau u_{\xi'} \tilde{\phi}_{\xi'}](\xi')^2}{|\xi - \xi'|^3} d\xi d\xi' dt \right\}^{1/2} \\
& \leq c \left\{ \int_0^T \int_S \int_S \frac{|[f_3''(\int_0^t w_\xi d\tau)](\xi) - [f_3''(\int_0^t w_{\xi'} d\tau)](\xi')|^2}{|\xi - \xi'|^3} \right. \\
& \quad \times \left. \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{| \int_0^t [D_\xi^2 w(\xi) - D_{\xi'}^2 w(\xi')] d\tau |^2}{|\xi - \xi'|^3} \right. \\
& \quad \times \left. \left| \int_0^t D_\xi^2 w d\tau \right|^2 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + c \left\{ \int_0^T \int_S \int_S \frac{|\int_0^t [w_\xi(\xi) - w_{\xi'}(\xi')] d\tau|^2}{|\xi - \xi'|^3} \left| \int_0^t D_\xi^2 w d\tau \right|^4 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|u_\xi(\xi) - u_{\xi'}(\xi')|^2}{|\xi - \xi'|^3} \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|\tilde{\phi}_\xi(\xi) - \tilde{\phi}_{\xi'}(\xi')|^2}{|\xi - \xi'|^3} \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 d\xi d\xi' dt \right\}^{1/2}.
\end{aligned}$$

Using the Taylor formula we get

$$\begin{aligned}
(3.56) \quad I_2 & \leq c \left\{ \int_0^T \int_S \int_S \frac{|\int_0^t D_\xi^2 w(\tilde{\xi}) d\tau|^2}{|\xi - \xi'|} \right. \\
& \times \left. \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|\int_0^t [D_\xi^2 w(\xi) - D_{\xi'}^2 w(\xi')] d\tau|^2}{|\xi - \xi'|^3} \right. \\
& \times \left. \left| \int_0^t D_\xi^2 w d\tau \right|^2 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|\int_0^t D_\xi^2 w(\tilde{\xi}) d\tau|^2}{|\xi - \xi'|} \left| \int_0^t D_\xi^2 w d\tau \right|^4 |u_\xi|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|D_\xi^2 u(\tilde{\xi})|^2}{|\xi - \xi'|} \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |\tilde{\phi}_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& + c \left\{ \int_0^T \int_S \int_S \frac{|D_\xi^2 \tilde{\phi}(\tilde{\xi})|^2}{|\xi - \xi'|} \left| \int_0^t D_\xi^2 w d\tau \right|^4 \left| \int_0^t w_\xi d\tau \right|^2 |u_\xi|^2 d\xi d\xi' dt \right\}^{1/2} \\
& \equiv I_2^1 + I_2^2 + I_2^3 + I_2^4 + I_2^5,
\end{aligned}$$

where $\tilde{\xi} = \theta\xi + (1 - \theta)\xi'$, $0 < \theta < 1$.

Obviously, we have

$$I_2^1 \leq cT^2 \|w\|_{4, \Omega^T}^4 \|u\|_{4, \Omega^T}.$$

For I_2^2 , we obtain

$$I_2^2 \leq c \left\{ \int_0^T \sup_S \left| \int_0^t D_\xi^2 w d\tau \right|^2 \sup_S \left| \int_0^t w_\xi d\tau \right|^2 \sup_S |u_\xi|^2 \sup_S |\tilde{\phi}_\xi|^2 \right\}^{1/2}$$

$$\begin{aligned}
& \times \left\{ \int_S \int_S \frac{|\int_0^t [D_\xi^2 w(\xi) - D_{\xi'}^2 w(\xi')] d\tau|^2}{|\xi - \xi'|^3} d\xi d\xi' dt \right\}^{1/2} \\
& \leq cT^{1/2} \|w\|_{4,\Omega^T} T^{1/2} \|w\|_{4,\Omega^T} T^{1/2} \\
& \times \left\{ \int_0^T \int_S \int_S \frac{|D_\xi^2 w(\xi) - D_{\xi'}^2 w(\xi')|^2}{|\xi - \xi'|^3} d\xi d\xi' dt \right\}^{1/2} \|u\|_{4,\Omega^T} \\
& \leq cT^{3/2} \|w\|_{4,\Omega^T}^3 \|u\|_{4,\Omega^T},
\end{aligned}$$

where we have used the fact that $D_\xi^2 w \in W_2^{1/2,1/4}(S^T)$ and Lemma 2.1 (inequality (2.5)).

Next we estimate

$$\begin{aligned}
I_2^3 & \leq c \left\{ \int_0^T \sup_S \left| \int_0^t D_\xi^2 w d\tau \right|^4 \sup_S |u_\xi|^2 \sup_S |\tilde{\phi}_\xi|^2 \right. \\
& \quad \times \left. \sup_S \left| \int_0^t D_\xi^2 w(\tilde{\xi}) d\tau \right|^2 \left(\int_S \int_S \frac{d\xi d\xi'}{|\xi - \xi'|} dt \right) \right\}^{1/2} \\
& \leq cT^{3/2} \|w\|_{4,\Omega^T}^3 \|u\|_{4,\Omega^T}.
\end{aligned}$$

The other terms both of (3.56) and of (3.55) can be estimated in the same way.

Therefore, by the above considerations

$$\sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/2, S^T, \xi} \leq (1 + T \|w\|_{4,\Omega^T}) I,$$

where I is defined in (3.55).

Consider now

$$\begin{aligned}
& \sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/4, S^T, t} \\
& \leq c \left[\int_S \int_0^T \int_0^T \frac{A(\xi, t, t')}{|t - t'|^{3/2}} d\xi dt dt' \right]^{1/2} \equiv I_3,
\end{aligned}$$

where

$$\begin{aligned}
A(\xi, t, t') & \equiv \left| D_{\xi,t}^2 \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) (t) \right. \\
& \quad \left. - D_{\xi,t'}^2 \left(f_3 \int_0^t w_\xi d\tau u_\xi \tilde{\phi}_\xi \right) (t') \right|^2.
\end{aligned}$$

As before we only estimate some terms of I_3 . Set

$$B(\xi, t, t') \equiv \left| \left(f_3'' \left(\int_0^t D_\xi^2 w \, d\tau \right)^2 \int_0^t w_\xi \, d\tau u_\xi \tilde{\phi}_\xi \right) (t) \right. \\ \left. - \left(f_3'' \left(\int_0^{t'} D_\xi^2 w \, d\tau \right)^2 \int_0^{t'} w_\xi \, d\tau u_\xi \tilde{\phi}_\xi \right) (t') \right|^2.$$

We have

$$(3.57) \quad I_5 \equiv \left[\int_S \int_0^T \int_0^T \frac{B(\xi, t, t')}{|t-t'|^{3/2}} \, d\xi \, dt \, dt' \right]^{1/2} \\ \leq c \left(\int_S \int_0^T \int_0^T \frac{|f_3''(\int_0^t w_\xi \, d\tau) - f_3''(\int_0^{t'} w_\xi \, d\tau)|^2}{|t-t'|^{3/2}} \right. \\ \times \left. \left| \int_0^t D_\xi^2 w \, d\tau \right|^4 \left| \int_0^t w_\xi \, d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 \, d\xi \, dt \, dt' \right)^{1/2} \\ + c \left(\int_S \int_0^T \int_0^T \frac{|\int_0^t D_\xi^2 w \, d\tau - \int_0^{t'} D_\xi^2 w \, d\tau|^2}{|t-t'|^{3/2}} \right. \\ \times \left. \left| \int_0^t D_\xi^2 w \, d\tau \right|^2 \left| \int_0^t w_\xi \, d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 \, d\xi \, dt \, dt' \right)^{1/2} \\ + c \left(\int_S \int_0^T \int_0^T \frac{|u_\xi(t) - u_\xi(t')|^2}{|t-t'|^{3/2}} \left| \int_0^t D_\xi^2 w \, d\tau \right|^4 \left| \int_0^t w_\xi \, d\tau \right|^2 |\tilde{\phi}_\xi|^2 \, d\xi \, dt \, dt' \right)^{1/2} \\ + c \left(\int_S \int_0^T \int_0^T \frac{|\int_0^t w_\xi \, d\tau - \int_0^{t'} w_\xi \, d\tau|^2}{|t-t'|^{3/2}} \left| \int_0^t D_\xi^2 w \, d\tau \right|^4 |u_\xi|^2 |\tilde{\phi}_\xi|^2 \, d\xi \, dt \, dt' \right)^{1/2} \\ \equiv I_4^1 + I_4^2 + I_4^3 + I_4^4.$$

Using the Taylor formula we obtain

$$I_4^2 \equiv c \left(\int_S \int_0^T \int_0^T \frac{|\int_0^t D_\xi^2 w \, d\tau - \int_0^{t'} D_\xi^2 w \, d\tau|^2}{|t-t'|^{3/2}} \right. \\ \times \left. \left| \int_0^t D_\xi^2 w \, d\tau \right|^2 \left| \int_0^t w_\xi \, d\tau \right|^2 |u_\xi|^2 |\tilde{\phi}_\xi|^2 \, d\xi \, dt \, dt' \right)^{1/2} \\ \leq c \left(\int_0^T \int_0^T |t-t'|^{1/2} \left(\int_S |D_\xi^2 w(\tilde{t})|^2 \, d\xi \right) \right.$$

$$\begin{aligned}
& \times \sup_S \left| \int_0^t D_\xi^2 w \, d\tau \right|^2 \sup_S \left| \int_0^t w_\xi \, d\tau \right|^2 \sup_S |u_\xi|^2 \sup_S |\tilde{\phi}_\xi|^2 \, dt \, dt')^{1/2} \\
& \leq cT \|w\|_{4,\Omega^T}^2 \sup_{\tilde{t}} \|w(\tilde{t})\|_{3,\Omega}^2 \left(\int_0^T \int_0^T T^{1/2} \sup_S |u_\xi|^2 \, dt \, dt' \right)^{1/2} \\
& \leq cT^{5/4} \|w\|_{4,\Omega^T}^2 (\|w\|_{4,\Omega^T} + |w(0)|_{3,0,\Omega}) \|u\|_{4,\Omega^T},
\end{aligned}$$

where $\tilde{t} = \theta t + (1 - \theta)t'$, $0 < \theta < 1$.

In the same way the other terms both of (3.57) and of

$$\sum_{|\alpha|+2i=2} \left[D_\xi^\alpha \partial_t^i \left(f_3 \int_0^t w_\xi \, d\tau u_\xi \tilde{\phi}_\xi \right) \right]_{1/4,S^T,t}$$

can be estimated.

Thus, from the above considerations we obtain

$$\begin{aligned}
(3.58) \quad \|\tilde{G}\|_{3-1/2,S^T} & \leq T^{1/2} (\|w\|_{4,\Omega^T} + |w(0)|_{3,0,\Omega}) \\
& \quad \times \phi \left(\int_0^T |w|_{\infty,\Omega} \, dt, \|w\|_{4,\Omega^T}, |w(0)|_{3,0,\Omega}, T \right) \\
& \quad \times (\|u\|_{4,\Omega^T} + |u(0)|_{3,0,\Omega}),
\end{aligned}$$

where ϕ is a nonnegative nondecreasing continuous function of its arguments.

It remains to estimate $\mathbf{I} D_{\xi,t}^2 \tilde{G} \mathbf{I}_{1/4,S^T}$. We get

$$\begin{aligned}
(3.59) \quad \mathbf{I} D_{\xi,t}^2 \tilde{G} \mathbf{I}_{1/4,S^T} & = \left(\int_0^T \frac{|D_{\xi,t}^2 (f_3 \int_0^t w_\xi \, d\tau u_\xi \tilde{\phi}_\xi)|_{2,S^T}^2}{t^{1/2}} \, dt \right)^{1/2} \\
& \leq c[T^{1/4} \|w\|_{4,\Omega^T} \|u\|_{4,\Omega^T} + T^{3/4} \|w\|_{4,\Omega^T} \|u\|_{4,\Omega^T} \\
& \quad + T^{5/4} \|w\|_{4,\Omega^T}^3 \|u\|_{4,\Omega^T} + T^{1/4} \|w\|_{4,\Omega^T} (\|u\|_{4,\Omega^T} + |u(0)|_{3,0,\Omega}) \\
& \quad + T^{1/4} (\|w\|_{4,\Omega^T} + |w(0)|_{3,0,\Omega}) (\|u\|_{4,\Omega^T} + |u(0)|_{3,0,\Omega})].
\end{aligned}$$

By (3.51)–(3.59) we get

$$\begin{aligned}
\|u\|_{4,\Omega^T} & \leq \phi_1 (|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, T) [\|F\|_{2,\Omega^T} + \|G\|_{3-1/2,S^T} \\
& \quad + \mathbf{I} D_{\xi,t}^2 \tilde{G} \mathbf{I}_{1/4,S^T} + \phi_2 (|\eta|_{2,0,\infty,\Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}) \|u\|_{2,\Omega^T} \\
& \quad + \phi_3 (|1/\eta|_{\infty,\Omega^T}, T) |u(0)|_{3,0,\Omega} + T^a (\|w\|_{4,\Omega^T} + |w(0)|_{3,0,\Omega}) \\
& \quad \times \phi_7 \left(|1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, \int_0^T |w|_{\infty,\Omega} \, dt, \|w\|_{4,\Omega^T}, |w(0)|_{3,0,\Omega}, T \right) \\
& \quad \times (\|u\|_{4,\Omega^T} + |u(0)|_{3,0,\Omega}),
\end{aligned}$$

where $a > 0$ is a constant.

Therefore if $\delta > 0$ is sufficiently small (3.49) follows from (3.48). To prove the existence and uniqueness of solutions to problem (3.46) we use the method of successive approximations. Put u_m in the right-hand side of (3.50) in place of u and replace u by u_{m+1} in the left-hand side of (3.50). Then by the contraction theorem we have the existence of solutions of problem (3.46) for sufficiently small T . Therefore the theorem is proved. ■

In the same way the following theorem can be proved:

THEOREM 3.6. *Assume that*

$$\begin{aligned} S &\in W_2^{4-1/2}, K \in W_2^{4,2}(\Omega^T), \tilde{\gamma} \in W_2^{3-1/2,3/2-1/4}(S^T), \\ \mathbf{I}D_{\xi,t}^2 \tilde{\gamma} \mathbf{I}_{1/4,S^T} &< \infty, \gamma_0 \in W_2^3(\Omega), \eta \in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)), \\ 1/\eta &\in L_\infty(\Omega^T), \eta > 0, \eta \in C^\alpha(\Omega^T) \ (\alpha \in (0, 1/2)), \\ \beta &\in W_2^{4,2}(\Omega^T), 1/\beta \in L_\infty(\Omega^T), \beta > 0, c_v \in C^2(\mathbb{R}_+^2), c_v > 0, \\ w &\in W_2^{4,2}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega)), T < \infty. \end{aligned}$$

Moreover, let the compatibility condition (3.8) be satisfied. Assume that T is so small that

$$(3.60) \quad T^a (\|w\|_{4,\Omega^T} + |w(0)|_{3,0,\Omega}) \phi_8 \left(\int_0^T |w_\xi|_{\infty,\Omega} dt, \|c_v\|_{C^2(\bar{V})}, \right. \\ \left. \|1/c_v\|_{C^0(\bar{V})}, |1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, \|\beta\|_{4,\Omega^T}, |\beta(0)|_{3,0,\Omega}, \right. \\ \left. \|w\|_{4,\Omega^T}, |w(0)|_{3,0,\Omega}, T \right) \leq \delta,$$

where ϕ_8 is a nonnegative nondecreasing continuous function of its arguments, $a > 0$ is a constant and $\delta > 0$ is sufficiently small. Then there exists a unique solution $\gamma \in W_2^{4,2}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega))$ to problem (3.47) satisfying the estimate

$$(3.61) \quad \|\gamma\|_{4,\Omega^T} \leq \phi_4(\|c_v\|_{C^2(\bar{V})}, \|1/c_v\|_{C^0(\bar{V})}, |1/\eta|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}, T) \\ \times [\|K\|_{2,\Omega^T} + \|\tilde{\gamma}\|_{3-1/2,S^T} + \mathbf{I}D_{\xi,t}^2 \tilde{\gamma} \mathbf{I}_{1/4,S^T} \\ + \phi_5(\|c_v\|_{C^2(\bar{V})}, \|1/c_v\|_{C^0(\bar{V})}, |\eta|_{2,0,\infty,\Omega^T}, \\ \|\eta\|_{C^\alpha(\Omega^T)}, \|\beta\|_{4,\Omega^T}, |\beta(0)|_{3,0,\Omega}) \|\gamma\|_{2,\Omega^T}] \\ + \phi'_6(\|1/c_v\|_{C^0(\bar{V})}, |1/\eta|_{\infty,\Omega^T}, T) |\gamma(0)|_{3,0,\Omega},$$

where ϕ_i ($i = 4, 5$) are the functions from Theorem 3.4, and ϕ'_6 is a nonnegative increasing function. ■

Now consider the continuity equation (3.1)₂. Integrating it we have

$$(3.62) \quad \eta(\xi, t) = \varrho_0(\xi) \exp \left[- \int_0^t \nabla_u \cdot u(\xi, \tau) d\tau \right].$$

The following lemma holds (see [15]).

LEMMA 3.3. Let $\varrho_0 \in W_2^3(\Omega)$, $u \in C^0(0, T; W_2^3(\Omega)) \cap W_2^{4,2}(\Omega^T)$. Then the solution (3.62) of the continuity equation satisfies $\eta \in C^0(0, T; W_2^3(\Omega))$, $\eta_t \in C^0(0, T; W_2^2(\Omega)) \cap L_2(0, T; W_2^3(\Omega))$, $\eta_{tt} \in L_2(0, T; W_2^1(\Omega))$ and the following estimates hold:

$$(3.63) \quad \sup_t \|\eta\|_{3,\Omega} \leq \tilde{\psi}_1(T^{1/2}\|u\|_{4,\Omega^T})(T^{1/2}\|u\|_{4,\Omega^T} + 1)\|\varrho_0\|_{3,\Omega},$$

$$(3.64) \quad \|\eta_t\|_{L_2(0,T;W_2^3(\Omega))} \leq \tilde{\psi}_2(T^{1/2}\|u\|_{4,\Omega^T})T^{1/2}\|u\|_{4,\Omega^T}\|\varrho_0\|_{3,\Omega},$$

$$(3.65) \quad \|\eta_{tt}\|_{L_2(0,T;W_2^1(\Omega))} \leq \tilde{\psi}_3(T^{1/2}\|u\|_{4,\Omega^T})(T^{1/2} + 1)\|u\|_{4,\Omega^T}\|\varrho_0\|_{3,\Omega},$$

$$(3.66) \quad \sup_t \|\eta_t\|_{2,\Omega} \leq \tilde{\psi}_4(T^{1/2}\|u\|_{4,\Omega^T}) \sup_t \|u\|_{3,\Omega} \|\varrho_0\|_{3,\Omega},$$

$$(3.67) \quad |1/\eta|_{\infty,\Omega^T} \leq \tilde{\psi}_5(T^{1/2}\|u\|_{4,\Omega^T})|1/\varrho_0|_{\infty,\Omega},$$

$$|\eta|_{\infty,\Omega^T} \leq \tilde{\psi}_6(T^{1/2}\|u\|_{4,\Omega^T})|\varrho_0|_{\infty,\Omega},$$

$$(3.68) \quad \|\eta\|_{C^\alpha(\Omega^T)} \leq \tilde{\psi}_7(T^{1/2}\|u\|_{4,\Omega^T})[\tilde{\psi}_8(T^{1/2}\|u\|_{4,\Omega}) \\ + T^{1-\alpha}(\|u\|_{4,\Omega^T} + \|u(0)\|_{3,\Omega})]\|\varrho_0\|_{C^\alpha(\Omega)},$$

where $\tilde{\psi}_i$ ($i = 1, \dots, 8$) are nonnegative nondecreasing continuous functions. ■

Now we are able to prove the existence of solution to problem (3.1).

THEOREM 3.7. Let $S \in W_2^{4-1/2}$, $f \in C^{2,1}(\mathbb{R}^3 \times [0, T])$, $r \in C^{2,1}(\mathbb{R}^3 \times [0, T])$, $\tilde{\vartheta} \in C^{2,1}(\mathbb{R}^3 \times [0, T])$, $v_0 \in W_2^3(\Omega)$, $\vartheta_0 \in W_2^3(\Omega)$, $1/\vartheta_0 \in L_\infty(\Omega)$, $\vartheta_0 > 0$, $\varrho_0 \in W_2^3(\Omega)$, $1/\varrho_0 \in L_\infty(\Omega)$, $\varrho_0 > 0$, $c_v \in C^2(\mathbb{R}^2_+)$, $c_v > 0$, $p \in C^3(\mathbb{R}_+^2)$. Moreover, assume that the following compatibility conditions are satisfied:

$$(3.69) \quad D_\xi^\alpha(\mathbb{D}_\xi(v_0) \cdot \bar{n}_0 - p(\varrho_0, \vartheta_0)\bar{n}_0 + p_0\bar{n}_0) = 0, \quad |\alpha| \leq 1, \quad \text{on } S,$$

and

$$(3.70) \quad D_\xi^\alpha(\bar{n}_0 \cdot \nabla_\xi \vartheta_0 - \tilde{\vartheta}(\xi, 0)) = 0, \quad |\alpha| \leq 1, \quad \text{on } S.$$

Let $T^* > 0$ be so small that $0 < c_1(1 - CK_0T^*)^3 \leq \det\{x_\xi\} \leq c_2(1 + CK_0T^*)^3$ (where $x(\xi, t) = \xi + \int_0^t u_0(\xi, \tau) d\tau$ for $t \leq T^*$, u_0 is given by (3.74), K_0 is given by (3.91) and the constant $C = C(K_0)$, which is a nondecreasing continuous function of K_0 , is given by (3.94)). Then there exists T^{**} with $0 < T^{**} \leq T^*$ such that for $T \leq T^{**}$ there exists a unique solution $(u, \gamma, \eta) \in W_2^{4,2}(\Omega^T) \times W_2^{4,2}(\Omega^T) \times C^0(0, T; \Gamma_0^{3,3/2}(\Omega))$ of problem (3.1) and

$$\|u\|_{4,\Omega^T} + \|\gamma\|_{4,\Omega^T} \leq CK_0, \\ \sup_t \|\eta\|_{3,\Omega} + \sup_t \|\eta_t\|_{2,\Omega} + \|\eta_t\|_{L_2(0,T;W_2^3(\Omega))} + \|\eta_{tt}\|_{L_2(0,T;W_2^1(\Omega))} \\ \leq \Phi_1(T, T^a K_0)\|\varrho_0\|_{3,\Omega},$$

$|1/\eta|_{\infty, \Omega^T} + |\eta|_{\infty, \Omega^T} \leq \Phi_2(T^{1/2}K_0)|1/\varrho_0|_{\infty, \Omega} + \Phi_3(T^{1/2}K_0)|\varrho_0|_{\infty, \Omega}$,
 where Φ_i ($i = 1, 2, 3$) are increasing continuous functions of their arguments,
 $a > 0$.

Proof. We prove the existence of solutions of problem (3.1) applying the method of successive approximations. To do this consider the problems

$$(3.71) \quad \begin{aligned} \eta_m \partial_t u_{m+1} - \mu \nabla_{u_m}^2 u_{m+1} - \nu \nabla_{u_m} \nabla_{u_m} \cdot u_{m+1} \\ = -\nabla_{u_m} p(\eta_m, \gamma_m) + \eta_m g & \quad \text{in } \Omega^T, \\ \mathbb{D}_{u_m}(u_{m+1}) \cdot \bar{n}(u_m) = [p(\eta_m, \gamma_m) - p_0] \bar{n}(u_m) & \quad \text{on } S^T, \\ u_{m+1}|_{t=0} = v_0 & \quad \text{in } \Omega; \end{aligned}$$

$$(3.72) \quad \begin{aligned} \eta_m c_v(\eta_m, \gamma_m) \partial_t \gamma_{m+1} - \kappa \nabla_{u_m}^2 \gamma_{m+1} \\ = -\gamma_m p_\gamma(\eta_m, \gamma_m) \nabla_{u_m} \cdot u_m \\ + \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_{mj} + \xi_{x_j} \cdot \nabla_\xi u_{mi})^2 \\ + (\nu - \mu)(\nabla_{u_m} \cdot u_m)^2 + \eta_m k & \quad \text{in } \Omega^T, \\ \bar{n}(u_m) \cdot \nabla_{u_m} \gamma_{m+1} = \tilde{\gamma} & \quad \text{on } S^T, \\ \gamma_{m+1}|_{t=0} = \vartheta_0 & \quad \text{in } \Omega \end{aligned}$$

(where $\tilde{\gamma}(\xi, t) = \tilde{\vartheta}(X_{u_m}(\xi, t), t) = \tilde{\vartheta}(\xi + \int_0^t u_m d\tau, t)$) and

$$(3.73) \quad \begin{aligned} \partial_t \eta_m + \eta_m \nabla_{u_m} \cdot u_m = 0 & \quad \text{in } \Omega^T, \\ \eta_m|_{t=0} = \varrho_0 & \quad \text{in } \Omega. \end{aligned}$$

For u_0 we take a function from Lemma 3.1 such that

$$(3.74) \quad \partial_t^i u_0|_{t=0} = \psi_i \quad \text{in } \Omega \quad (i = 0, 1)$$

and

$$(3.75) \quad \|u_0\|_{4, \Omega^T} \leq c(\|\psi_0\|_{3, \Omega} + \|\psi_1\|_{1, \Omega}),$$

where $\psi_i = \partial_t^i u|_{t=0}$ are calculated from problem (3.1), i.e.

$$(3.76) \quad \psi_0 = v_0, \quad \varrho_0 \psi_1 - \mu \nabla_\xi^2 v_0 - \nu \nabla_\xi \nabla_\xi \cdot v_0 + \nabla_\xi p(\varrho_0, \vartheta_0) = \varrho_0 g(0).$$

Similarly, for γ_0 we take a function from Lemma 3.1 such that

$$(3.77) \quad \partial_t^i \gamma_0|_{t=0} = \sigma_i \quad \text{in } \Omega \quad (i = 0, 1)$$

and

$$(3.78) \quad \|\gamma_0\|_{4, \Omega^T} \leq c(\|\sigma_0\|_{3, \Omega} + \|\sigma_1\|_{1, \Omega}),$$

where $\sigma_i = \partial_t^i \gamma|_{t=0}$ ($i = 0, 1$), i.e.

$$(3.79) \quad \begin{aligned} \sigma_0 &= \vartheta_0, \\ \eta_0 c_v(\eta_0, \vartheta_0) \sigma_1 - \kappa \nabla_\xi^2 \vartheta_0 &= -\vartheta_0 p_\gamma(\eta_0, \vartheta_0) \nabla_\xi \cdot v_0 \\ &+ \frac{\mu}{2} \sum_{i,j=1}^3 (\partial_{\xi_i} v_{0j} + \partial_{\xi_j} v_{0i})^2 + (\nu - \mu) (\nabla_\xi \cdot v_0)^2 + \eta_0 k(0), \end{aligned}$$

and η_0 is a solution of the problem

$$(3.80) \quad \partial_t \eta_0 + \eta_0 \nabla_{u_0} u_0 = 0, \quad \eta_0|_{t=0} = \varrho_0.$$

Assume that $u_m \in W_2^{4,2}(\Omega^T)$ and $\gamma_m \in W_2^{4,2}(\Omega^T)$. Then by Lemma 3.3, $1/\eta_m \in L_\infty(\Omega^T)$, $\eta_m > 0$ and $\eta_m \in C^\alpha(\Omega^T)$. Next, since $\gamma_m(t) = \int_0^t \partial_\tau \gamma_m(\tau) d\tau + \gamma_m(0)$ we have

$$\gamma_m(t) \geq \vartheta_0 - T^{1/2} \left(\int_0^T |\partial_\tau \gamma_m|^2 d\tau \right)^{1/2}.$$

Thus, from the assumption that $1/\vartheta_0 \in L_\infty(\Omega)$ and $\vartheta_0 > 0$ it follows that for T sufficiently small, $1/\gamma_m \in L_\infty(\Omega^T)$ and $\gamma_m > 0$.

Now assume that inequalities (3.48) and (3.60) with $w = u_m$, $\eta = \eta_m$ and $\beta = \gamma_m$ are satisfied with sufficiently small δ . Then by Theorem 3.5 there exists a unique solution $u_{m+1} \in W_2^{4,2}(\Omega^T)$ (where $T = T(\delta)$ is also small) of (3.71) such that

$$(3.81) \quad \begin{aligned} \|u_{m+1}\|_{4,\Omega^T} &\leq \phi_1(|1/\eta_m|_{\infty,\Omega^T}, |\eta_m|_{\infty,\Omega^T}, T) \\ &\times [\|\nabla_{u_m} p(\eta_m, \gamma_m)\|_{2,\Omega^T} + \|\eta_m g\|_{2,\Omega^T} \\ &+ \|[p(\eta_m, \gamma_m) - p_0] \bar{n}(u_m)\|_{3-1/2,S^T} \\ &+ \mathbf{I} D_{\xi,t}^2 \{ [p(\eta_m, \gamma_m) - p_0] \bar{n}(u_m) \} \mathbf{I}_{1/4,S^T} \\ &+ \phi_2(\|\eta_m\|_{2,0,\infty,\Omega^T}, \|\eta_m\|_{C^\alpha(\Omega^T)}) \|u_m\|_{2,\Omega^T} \\ &+ \phi_3'(|1/\eta_m|_{\infty,\Omega^T}, T) \|u_m(0)\|_{3,0,\Omega}. \end{aligned}$$

Similarly, by Theorem 3.6 there is a unique solution $\gamma_{m+1} \in W_2^{4,2}(\Omega^T)$ ($T = T(\delta)$) of (3.72) such that

$$(3.82) \quad \begin{aligned} \|\gamma_{m+1}\|_{4,\Omega^T} &\leq \phi_4(\|c_v\|_{C^2(\bar{V}_m)}, \|1/c_v\|_{C^0(\bar{V}_m)}, |1/\eta_m|_{\infty,\Omega^T}, |\eta_m|_{\infty,\Omega^T}, T) \\ &\times [\|\gamma_m p_\gamma(\eta_m, \gamma_m) \nabla_{u_m} u_m\|_{2,\Omega^T} \\ &+ \|(\nabla_{u_m} u_m)^2\|_{2,\Omega^T} + \|\eta_m k\|_{2,\Omega^T} + \|\tilde{\gamma}\|_{3-1/2,S^T} + \mathbf{I} D_{\xi,t}^2 \tilde{\gamma} \mathbf{I}_{1/4,S^T} \end{aligned}$$

$$\begin{aligned}
& + \phi_5(\|c_v\|_{C^2(\bar{V}_m)}, \|1/c_v\|_{C^0(\bar{V}_m)}, |\eta_m|_{2,0,\infty,\Omega^T}, \\
& \quad \|\eta_m\|_{C^\alpha(\Omega^T)}, \|\gamma_m\|_{4,\Omega^T}, |\gamma_m(0)|_{3,0,\Omega}) \|\gamma_m\|_{2,\Omega^T} \\
& + \phi_6'(\|1/c_v\|_{C^0(\bar{V}_m)}, |1/\eta_m|_{\infty,\Omega^T}, T) |\gamma_m(0)|_{3,0,\Omega},
\end{aligned}$$

where $V_m \subset \mathbb{R}_+^2$ is a bounded domain such that $(\eta_m(\xi, t), \gamma_m(\xi, t)) \in V_m$ for any $(\xi, t) \in \Omega^T$.

In order to estimate the terms on the right-hand sides of (3.81) and (3.82) we use the same methods as in Theorem 3.5. Thus, we get

$$\begin{aligned}
(3.83) \quad & \|\nabla_{u_m} p(\eta_m, \gamma_m)\|_{2,\Omega^T} \\
& = \|\xi_x \nabla_\xi p(\eta_m, \gamma_m)\|_{2,\Omega^T} \leq \|\xi_x (p_\eta D_\xi^1 \eta_m + p_\gamma D_\xi^1 \gamma_m)\|_{2,\Omega^T} \\
& \leq \sum_{|\alpha|+2i \leq 2} |D_\xi^\alpha \partial_t^i [\tilde{f}_1 (p_\eta D_\xi^1 \eta_m + p_\gamma D_\xi^1 \gamma_m)]|_{2,\Omega^T} \\
& \leq c \|\tilde{f}_1\|_{C^2(\bar{G}_m)} \|p\|_{C^2(\bar{V}_m)} \alpha_1 (T^a (\sup_t \|\partial_t \eta_m\|_{1,\Omega} + \sup_t \|\eta_m\|_{3,\Omega}), \\
& \quad T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega} + \|\gamma_m\|_{4,\Omega^T} + \sup_t |\gamma_m(t)|_{3,0,\Omega}), T^{a_1}), \\
(3.84) \quad & \|\gamma_m p_\gamma(\eta_m, \gamma_m) \nabla_{u_m} \cdot u_m\|_{2,\Omega^T} = \|\gamma_m p_\gamma(\eta_m, \gamma_m) \tilde{f}_1 \cdot D_\xi^1 u_m\|_{2,\Omega^T} \\
& \leq c \|\tilde{f}_1\|_{C^2(\bar{G}_m)} \|p\|_{C^3(\bar{V}_m)} \alpha_2 (T^a (\sup_t \|\partial_t \eta_m\|_{1,\Omega} + \sup_t \|\eta_m\|_{3,\Omega}), \\
& \quad T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega} + \|\gamma_m\|_{4,\Omega^T} + \sup_t |\gamma_m(t)|_{3,0,\Omega}), T^{a_2})
\end{aligned}$$

and

$$\begin{aligned}
(3.85) \quad & \|(\nabla_{u_m} \cdot u_m)^2\|_{2,\Omega^T} + \left\| \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_{mj} + \xi_{x_j} \cdot \nabla_\xi u_{mi})^2 \right\|_{2,\Omega^T} \\
& \leq \|(\tilde{f}_2 \nabla_\xi u_m)^2\|_{2,\Omega^T} \\
& \leq c \|\tilde{f}_2\|_{C^2(\bar{G}_m)} \alpha_3 (T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega}), T^{a_3}).
\end{aligned}$$

In (3.83)–(3.85), $\tilde{f}_i = \tilde{f}_i(\int_0^t D_\xi^1 u_m d\tau)$ ($i = 1, 2$) are smooth functions, $G_m \subset \mathbb{R}^9$ is a bounded domain such that $\int_0^T D_\xi^1 u_m d\tau \in G_m$ for any $(\xi, t) \in \Omega^T$, α_i ($i = 1, 2, 3$) are nonnegative increasing continuous functions of their arguments which are polynomials such that $\alpha_i(0, 0, 0) = 0$, $a > 0$ and $a_i > 0$ ($i = 1, 2, 3$) are constants.

Next, we obtain

$$\begin{aligned}
(3.86) \quad & \|\eta_m g\|_{2,\Omega^T} \leq c \|f\|_{C^{2,1}(\bar{Q}_m)} \alpha_4 (T^a (\sup_t \|\eta_m\|_{3,\Omega} + \sup_t \|\partial_t \eta_m\|_{1,\Omega}), \\
& \quad T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega}), T^{a_4}),
\end{aligned}$$

$$(3.87) \quad \|\eta_m k\|_{2,\Omega^T} \leq c \|r\|_{C^{2,1}(\bar{Q}_m)} \alpha_5 (T^a (\sup_t \|\eta_m\|_{3,\Omega} + \sup_t \|\partial_t \eta_m\|_{1,\Omega}), \\ T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega}), T^{a_5})$$

and

$$(3.88) \quad \|\tilde{\gamma}\|_{3-1/2,S^T} + \|D_{\xi,t}^2 \tilde{\gamma}\|_{1/4,S^T} \\ \leq c \|\tilde{\vartheta}\|_{C^{2,1}(\bar{Q}_m)} \alpha_6 (T^a (\|u_m\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega}), T^{a_6}),$$

where $Q_m \subset \mathbb{R}^3 \times [0, T]$ is a bounded domain such that $(\xi + \int_0^t u_m d\tau, t) \in Q_m$ for any $(\xi, t) \in \Omega^T$, α_i ($i = 4, 5, 6$) are nonnegative increasing continuous functions of their arguments which are polynomials such that $\alpha_i(0, 0, 0) = 0$, and $a_i > 0$ ($i = 4, 5, 6$) are constants.

Finally, we have

$$(3.89) \quad \|[p(\eta_m, \gamma_m) - p_0] \bar{n}(u_m)\|_{3-1/2,S^T} \\ + \|D_{\xi,t}^2 \{[p(\eta_m, \gamma_m) - p_0] \bar{n}(u_m)\}\|_{1/4,S^T} \\ \leq c \left\{ \sum_{|\alpha|+2i \leq 2} |D_\xi^\alpha \partial_t^i \{[p(\eta_m, \gamma_m) - p_0] \tilde{f}\}|_{2,S^T} \right. \\ + \sum_{|\alpha|+2i=2} [D_\xi^\alpha \partial_t^i \{[p(\eta_m, \gamma_m) - p_0] \tilde{f}\}]_{1/2,S^T,\xi} \\ + \sum_{|\alpha|+2i=2} [D_\xi^\alpha \partial_t^i \{[p(\eta_m, \gamma_m) - p_0] \tilde{f}\}]_{1/4,S^T,t} \\ \left. + \left(\int_0^T \frac{|D_{\xi,t}^2 \{[p(\eta_m, \gamma_m) - p_0] \tilde{f}\}|_{2,S}^2}{t^{1/2}} dt \right)^{1/2} \right\} \\ \leq c \|\tilde{f}\|_{C^3(\bar{G}_m)} \|p\|_{C^3(\bar{G}_m)} \alpha_7 (T^a (\sup_t \|\eta_m\|_{3,\Omega} + \sup_t \|\partial_t \eta_m\|_{1,\Omega}), \\ T^a (\|u\|_{4,\Omega^T} + \sup_t |u_m(t)|_{3,0,\Omega} + \|\gamma_m\|_{4,\Omega^T} + \sup_t |\gamma_m(t)|_{3,0,\Omega}), T^{a_7}),$$

where α_7 has the same properties as α_1 and α_2 , $a_7 > 0$ is a constant, and $\tilde{f} = \tilde{f}(\int_0^t D_\xi^1 u_m d\tau)$ is a smooth function.

Let

$$y_m(t) = \|u_m\|_{4,\Omega^t}^2 + \|\gamma_m\|_{4,\Omega^t}^2 + \sup_{\tau \in (0,t)} |u_m(\tau)|_{3,0,\Omega}^2 + \sup_{\tau \in (0,t)} |\gamma_m(\tau)|_{3,0,\Omega}^2, \\ b = \|\varrho_0\|_{3,\Omega} + |1/\varrho_0|_{\infty,\Omega} + |\varrho_0|_{\infty,\Omega} + |u_m(0)|_{3,0,\Omega} + |\gamma_m(0)|_{3,0,\Omega}, \\ c_m = \|p\|_{C^3(\bar{V}_m)} + \|f\|_{C^{2,1}(\bar{Q}_m)} + \|r\|_{C^{2,1}(\bar{Q}_m)} + \|\tilde{\vartheta}\|_{C^{2,1}(\bar{Q}_m)} + \|c_v\|_{C^2(\bar{V}_m)} \\ + \|1/c_v\|_{C^0(\bar{V}_m)} + \|\tilde{f}\|_{C^2(\bar{G}_m)} + \sum_{i=1}^2 \|\tilde{f}_i\|_{C^2(\bar{G}_m)}.$$

Hence, taking into account inequalities (3.81)–(3.89) and using Lemma 3.3 we obtain

$$y_{m+1}(t) \leq \beta_1(t^a y_m(t), b, c_m, t) + \beta_2(t^a y_m(t), y_m(t), b, c_m) \int_0^t y_{m+1}(\tau) d\tau,$$

where β_i ($i = 1, 2$) are nonnegative nondecreasing continuous functions of their arguments and

$$\beta_1(0, b, c_m, 0) = [\phi'_3(\tilde{\psi}_5(0)|1/\varrho_0|_{\infty, \Omega}, T) + \phi'_6(c_m, \tilde{\psi}_5(0)|1/\varrho_0|_{\infty, \Omega}, T)].$$

In view of the Gronwall lemma we have

$$(3.90) \quad y_{m+1}(t) \leq \beta_1(t^a y_m(t), b, c_m, t) \times \exp[t\beta_2(t^a y_m(t), y_m(t), b, c_m)].$$

Using (3.75) and (3.78) we get

$$(3.91) \quad y_0(t) \leq c(\|\psi_0\|_{3, \Omega}^2 + \|\psi_1\|_{1, \Omega}^2 + \|\sigma_0\|_{3, \Omega}^2 + \|\sigma_1\|_{1, \Omega}^2) < cb \equiv K_0.$$

Assume that

$$(3.92) \quad y_m(t) \leq CK_0 \quad \forall t \leq T^*$$

(where $C > 0$ is a constant which will be chosen later, and $T^* > 0$). We shall prove that

$$(3.93) \quad c_m \leq \tilde{c},$$

where \tilde{c} is a constant independent of m and C .

In fact, by Lemma 3.3 and (3.90) we have

$$|\eta_m|_{\infty, \Omega T^*} \leq \tilde{\psi}_6(T^{*1/2}CK_0)|\varrho_0|_{\infty, \Omega} \leq K$$

for sufficiently small $T^* = T^*(CK_0)$ (where K is a constant independent of CK_0). Similarly

$$|\gamma_m|_{\infty, \Omega T^*} \leq \vartheta_0 + T^{*1/2}CK_0 \leq K \quad \text{for sufficiently small } T^*.$$

Hence there exists a bounded domain $V \subset \mathbb{R}_+^2$ such that $V_m \subset V$ for any m . Thus, we proved that $\|p\|_{C^3(\bar{V}_m)} + \|c_v\|_{C^2(\bar{V}_m)}$ is estimated by a constant independent of m and K_0 . In the same way we can prove that there exist bounded domains $G \subset \mathbb{R}^9$ and $Q \subset \mathbb{R}^3 \times (0, T)$ such that $G_m \subset G$ and $Q_m \subset Q$ for any m . Thus we have proved (3.93).

Now inequalities (3.92)–(3.93) yield that for $t \leq T^*$ we have

$$y_{m+1}(t) \leq \beta_1(t^a CK_0, b, \tilde{c}, t) \exp[t\beta_2(t^a CK_0, CK_0, b, \tilde{c})] \leq CK_0,$$

if $C = C(K_0)$ is chosen so large that

$$(3.94) \quad \beta_1(0, b, \tilde{c}, 0) < CK_0$$

and if T^* is sufficiently small ($C(K_0)$ in (3.94) is a nondecreasing continuous function of K_0).

In this way we have shown that there exists a sufficiently large $C = C(K_0)$ (satisfying (3.94)) such that $C(K_0)$ is a nondecreasing continuous function of K_0 , and a sufficiently small $T^* > 0$ such that for $t \leq T^*$ and for $m = 0, 1, \dots$ we have

$$(3.95) \quad y_m(t) \leq CK_0.$$

(By (3.95) and (3.93) inequalities (3.48) and (3.60) with $w = u_m$, $\eta = \eta_m$ and $\beta = \gamma_m$ are satisfied with sufficiently small δ and sufficiently small T^* independent of m .)

Now we prove the convergence of the sequence $\{u_m, \gamma_m, \eta_m\}$. Consider the following system of problems for the differences $U_{m+1} = u_{m+1} - u_m$, $\Gamma_{m+1} = \gamma_{m+1} - \gamma_m$, $H_m = \eta_m - \eta_{m-1}$:

$$(3.96) \quad \begin{aligned} & \eta_m \partial_t U_{m+1} - \mu \nabla_{u_m}^2 U_{m+1} - \nu \nabla_{u_m} \nabla_{u_m} \cdot U_{m+1} \\ & = -H_m \partial_t u_m - \mu (\nabla_{u_m}^2 - \nabla_{u_{m-1}}^2) u_m \\ & \quad - \nu (\nabla_{u_m} \nabla_{u_m} \cdot - \nabla_{u_{m-1}} \nabla_{u_{m-1}} \cdot) u_m \\ & \quad + \nabla_{u_m} p(\eta_m, \gamma_m) - \nabla_{u_{m-1}} p(\eta_{m-1}, \gamma_{m-1}) + H_m g \\ & \equiv F_1 + F_2, \end{aligned}$$

$$\begin{aligned} & \mathbb{D}_{u_m}(U_{m+1}) \cdot \bar{n}(u_m) \\ & = -[\mathbb{D}_{u_m}(u_m) \cdot \bar{n}(u_m) - \mathbb{D}_{u_{m-1}}(u_m) \cdot \bar{n}(u_{m-1})] \\ & \quad + [p(\eta_m, \gamma_m) \bar{n}(u_m) - p(\eta_{m-1}, \gamma_{m-1}) \bar{n}(u_{m-1})] \\ & \quad - p_0 [\bar{n}(u_m) - \bar{n}(u_{m-1})] \equiv G_1 + G_2, \end{aligned}$$

$$U_{m+1}|_{t=0} = 0;$$

$$(3.97) \quad \begin{aligned} & \eta_m c_v(\eta_m, \gamma_m) \partial_t \Gamma_{m+1} - \kappa \nabla_{u_m}^2 \Gamma_{m+1} = -H_m c_v(\eta_m, \gamma_m) \partial_t \gamma_m \\ & \quad + \eta_{m-1} \partial_t \gamma_m [c_v(\eta_{m-1}, \gamma_{m-1}) - c_v(\eta_m, \gamma_m)] \\ & \quad + \frac{\mu}{2} \sum_{i,j=1}^3 [(\xi_{x_i} \cdot \nabla_{\xi} u_{mj} + \xi_{x_j} \cdot \nabla_{\xi} u_{mi})^2 \\ & \quad - (\xi_{x_i} \cdot \nabla_{\xi} u_{m-1,j} + \xi_{x_j} \cdot \nabla_{\xi} u_{m-1,i})^2] \\ & \quad + (\nu - \mu) [(\nabla_{u_m} \cdot u_m)^2 - (\nabla_{u_{m-1}} \cdot u_{m-1})^2] \\ & \quad - \gamma_m p_{\gamma}(\eta_m, \gamma_m) \nabla_{u_m} \cdot u_m \\ & \quad + \gamma_{m-1} p_{\gamma}(\eta_{m-1}, \gamma_{m-1}) \nabla_{u_{m-1}} \cdot u_{m-1} + H_m k \equiv I_1 + I_2, \end{aligned}$$

$$(3.97) \quad \begin{aligned} & \bar{n}(u_m) \cdot \nabla_{u_m} (\Gamma_{m+1}) \\ \text{[cont.]} \quad & = -[\bar{n}(u_m) \cdot \nabla_{u_m} \gamma_m - \bar{n}(u_{m-1}) \cdot \nabla_{u_{m-1}} \gamma_m] \equiv J, \\ & \Gamma_m|_{t=0} = 0; \end{aligned}$$

$$(3.98) \quad \begin{aligned} & \partial_t H_m + H_m \operatorname{div}_{u_m} u_m = -\eta_{m-1} (\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1}), \\ & H_m|_{t=0} = 0, \end{aligned}$$

where

$$\begin{aligned} & \operatorname{div}_{u_i} u_i = \nabla_{u_i} \cdot u_i, \\ F_2 & = -H_m \partial_t u_m + H_m g + \nabla_{u_m} p(\eta_m, \gamma_m) - \nabla_{u_{m-1}} p(\eta_{m-1}, \gamma_{m-1}), \\ G_2 & = p(\eta_m, \gamma_m) \bar{n}(u_m) - p(\eta_{m-1}, \gamma_{m-1}) \bar{n}(u_{m-1}), \\ I_2 & = -H_m c_v(\eta_m, \gamma_m) \partial_t \gamma_m + \eta_{m-1} \partial_t \gamma_m [c_v(\eta_{m-1}, \gamma_{m-1}) - c_v(\eta_m, \gamma_m)] \\ & \quad + H_m k + \gamma_{m-1} p_\gamma(\eta_{m-1}, \gamma_{m-1}) \nabla_{u_{m-1}} \cdot u_{m-1} \\ & \quad - \gamma_m p_\gamma(\eta_m, \gamma_m) \nabla_{u_m} \cdot u_m \end{aligned}$$

and F_1 , G_1 , I_1 are determined by the remaining terms on the right-hand sides.

Applying Theorems 3.5, 3.6 and Lemma 3.3 to problems (3.96) and (3.97) respectively and using inequalities (3.93), (3.95) we obtain

$$(3.99) \quad \begin{aligned} & \|U_{m+1}\|_{4, \Omega^t} + \sup_t |U_{m+1}|_{3,0, \Omega} + \|\Gamma_{m+1}\|_{4, \Omega^t} + \sup_t |\Gamma_{m+1}|_{3,0, \Omega} \\ & \leq \tilde{\phi}(t, K_0) [\|U_{m+1}\|_{2, \Omega^t} + \|F_1\|_{2, \Omega^t} + \|F_2\|_{2, \Omega^t} + \|I_1\|_{2, \Omega^t} \\ & \quad + \|I_2\|_{2, \Omega^t} + \|G_1\|_{3-1/2, S^t} + \|G_2\|_{3-1/2, S^t} + \|D_{\xi, t}^2 G_1\|_{1/4, S^t} \\ & \quad + \|D_{\xi, t}^2 G_2\|_{1/4, S^t} + \|J\|_{3-1/2, S^t} + \|D_{\xi, t}^2 J\|_{1/4, S^t}], \end{aligned}$$

where $\tilde{\phi}$ is a nonnegative increasing continuous function of its arguments.

To estimate the right-hand sides of (3.99) we shall consider the following functions connected with the qualitative forms of F_i , I_i , G_i ($i = 1, 2$) and J :

$$(3.100) \quad \begin{aligned} F'_1 & = f_1 \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \int_0^t D_\xi^1 U_m d\tau D_\xi^2 u_m \\ & \quad + f_2 \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\ & \quad \times \int_0^t D_\xi^1 U_m d\tau \int_0^t D_\xi^2 \tilde{u}_m d\tau D_\xi^1 u_m \end{aligned}$$

$$\begin{aligned}
& + f_3 \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) \int_0^t D_\xi^1 U_m d\tau \\
& \times \int_0^t D_\xi^2 u_{m-1} d\tau D_\xi^1 u_m ; \\
F'_2 = & - H_m \sum_{i=1}^3 \partial_t u_{mi} + H_m \sum_{i=1}^3 g_i \\
& + f_4 \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\
& \times \int_0^t D_\xi^1 U_m d\tau [p_\eta(\eta_m, \gamma_m) D_\xi^1 \eta_m + p_\gamma(\eta_m, \gamma_m) D_\xi^1 \gamma_m] \\
& + f_5 \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) [p_{\eta\eta}(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 \tilde{\eta}_m H_m \\
& + p_{\eta\gamma}(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 \tilde{\gamma}_m H_m + p_{\gamma\eta}(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 \tilde{\eta}_m \Gamma_m \\
& + p_{\gamma\gamma}(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 \tilde{\gamma}_m \Gamma_m \\
(3.100) \quad & + p_\eta(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 H_m + p_\gamma(\tilde{\eta}_m, \tilde{\gamma}_m) D_\xi^1 \Gamma_m] ; \\
\text{[cont.]} \\
I'_1 = & f_6 \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau \right) D_\xi^1 u_m D_\xi^1 U_m \\
& + f_7 \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\
& \times D_\xi^1 u_m \int_0^t D_\xi^1 U_m d\tau D_\xi^1 u_{m-1} \\
& + f_8 \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) D_\xi^1 U_m D_\xi^1 u_{m-1} \\
& + f_9 \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) \\
& \times \int_0^t D_\xi^1 U_m d\tau D_\xi^1 u_{m-1} ; \\
I'_2 = & - H_m c_v(\eta_m, \gamma_m) \partial_t \gamma_m + H_m k \\
& + \eta_{m-1} \partial_t \gamma_m [c_{v\eta}(\tilde{\eta}_m, \tilde{\gamma}_m) H_m + c_{v\gamma}(\tilde{\eta}_m, \tilde{\gamma}_m) \Gamma_m]
\end{aligned}$$

$$\begin{aligned}
& + f_{10} \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) p_\gamma(\eta_{m-1}, \gamma_{m-1}) D_\xi^1 u_{m-1} \Gamma_m \\
& + f_{11} \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) \\
& \times \gamma_m [p_{\gamma\eta}(\tilde{\eta}_m, \tilde{\gamma}_m) H_m + p_{\gamma\gamma}(\tilde{\eta}_m, \tilde{\gamma}_m) \Gamma_m] D_\xi^1 u_{m-1} \\
& + f_{12} \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau \right) \gamma_m p_\gamma(\eta_m, \gamma_m) D_\xi^1 U_m \\
& + f_{13} \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \gamma_m p_\gamma(\eta_m, \gamma_m) \int_0^t D_\xi^1 U_m d\tau D_\xi^1 u_m ; \\
G'_1 & = f_{14} \left(\mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau, \mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\
& \times \int_0^t D_\xi^1 U_m d\tau D_\xi^1 u_m \\
(3.100) \quad & + f_{15} \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\
[\text{cont.}] \quad & \times \left(\int_0^t D_\xi^1 U_m d\tau \right)^2 (D_\xi^1 u_m)^2 \\
& + f_{16} \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \int_0^t D_\xi^1 U_m d\tau ; \\
G'_2 & = f_{17} \left(\mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) p(\eta_m, \gamma_m) \int_0^t D_\xi^1 U_m d\tau \\
& + f_{18} \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau \right) \\
& \times [p_\eta(\tilde{\eta}_m, \tilde{\gamma}_m) H_m + p_\gamma(\tilde{\eta}_m, \tilde{\gamma}_m) \Gamma_m] \mathbf{1} ; \\
J' & = f_{19} \left(\mathbf{1} + \int_0^t D_\xi^1 u_m d\tau, \mathbf{1} + \int_0^t D_\xi^1 u_{m-1} d\tau, \mathbf{1} + \int_0^t D_\xi^1 \tilde{u}_m d\tau \right) \\
& \times \int_0^t D_\xi^1 U_m d\tau D_\xi^1 \gamma_m .
\end{aligned}$$

Here f_i ($i = 1, \dots, 19$) are certain smooth vector-valued functions of their arguments, $\tilde{u}_m = \theta_1 u_m + (1 - \theta_1) u_{m-1}$, $\tilde{\eta}_m = \theta_2 \eta_m + (1 - \theta_2) \eta_{m-1}$, $\tilde{\gamma}_m = \theta_3 \gamma_m + (1 - \theta_3) \gamma_{m-1}$, $0 < \theta_i < 1$ ($i = 1, 2, 3$) and the products of vectors in (3.100) are understood in the sense of (2.2).

In order to estimate the functions in (3.100) we use the Hölder and Minkowski inequalities, Lemma 2.1, Lemma 3.3, (3.91) and (3.92). Since the estimates of $\|F_i\|_{2,\Omega^t}$, $\|G_i\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 G_i \mathbf{I}_{1/4,S^t}$, $\|I_i\|_{2,\Omega^t}$ (where $i = 1, 2$) and $\|J\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 J \mathbf{I}_{1/4,S^t}$ are the same as those of $\|F'_i\|_{2,\Omega^t}$, $\|G'_i\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 G'_i \mathbf{I}_{1/4,S^t}$, $\|I'_i\|_{2,\Omega^t}$ (where $i = 1, 2$) and $\|J'\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 J' \mathbf{I}_{1/4,S^t}$ we get

$$\begin{aligned}
\|F_1\|_{2,\Omega^t} &\leq c_1(t, K_0) t^{1/2} Y_m(t), \\
\|F_2\|_{2,\Omega^t} &\leq c_2(K_0) \sup_{\tau} |H_m|_{3,0,\Omega} \\
&\quad + c_3(t, K_0) t^{1/2} (Y_m(t) + \sup_{\tau} |H_m|_{3,0,\Omega}), \\
\|I_1\|_{2,\Omega^t} &\leq c_4(t, K_0) t^{1/2} Y_m(t), \\
\|I_2\|_{2,\Omega^t} &\leq c_5(K_0) (\sup_{\tau} |H_m|_{\infty,\Omega} + \sup_{\tau} |\Gamma_m|_{\infty,\Omega}) \\
(3.101) \quad &\quad + c_6(t, K_0) t^{1/2} Y_m(t) \\
&\quad + c_7(K_0) t^{1/2} (\sup_{\tau} |H_m|_{3,0,\Omega} + \sup_{\tau} |\Gamma_m|_{3,0,\Omega}), \\
\|G_1\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 G_1 \mathbf{I}_{1/4,S^t} &\leq c_8(t, K_0) t^{1/4} Y_m(t), \\
\|G_2\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 G_2 \mathbf{I}_{1/4,S^t} &\leq c_9(t, K_0) t^{1/4} (Y_m(t) + \sup_{\tau} |H_m|_{3,0,\Omega}), \\
\|J\|_{3-1/2,S^t} + \mathbf{I}D_{\xi,t}^2 J \mathbf{I}_{1/4,S^t} &\leq c_{10}(t, K_0) t^{1/4} Y_m(t),
\end{aligned}$$

where $Y_m(t) = \|U_m\|_{4,\Omega^T} + \sup_{\tau} |U_m|_{3,0,\Omega} + \|\Gamma_m\|_{4,\Omega^T} + \sup_{\tau} |\Gamma_m|_{3,0,\Omega}$ and c_i ($i = 1, \dots, 10$) are nonnegative increasing continuous functions of their arguments.

Now inequalities (3.99)–(3.101) yield

$$\begin{aligned}
(3.102) \quad Y_{m+1}(t) &\leq \tilde{\phi}(t, K_0) \|U_{m+1}\|_{2,\Omega^t} + \tilde{\phi}_1(t, K_0) t^{1/4} Y_m(t) \\
&\quad + \tilde{\phi}_2(t, K_0) (\sup_{\tau} |H_m|_{3,0,\Omega} + \sup_{\tau} |H_m|_{\infty,\Omega} + \sup_{\tau} |\Gamma_m|_{\infty,\Omega}).
\end{aligned}$$

By Lemma 3.3,

$$(3.103) \quad \sup_{\tau} |H_m|_{\infty,\Omega} \leq c \sup_{\tau} |H_m|_{3,0,\Omega}$$

and since $\Gamma_m(\tau) = \int_0^\tau \partial_s \Gamma_m(s) ds$ we have

$$(3.104) \quad \sup_{\tau} |\Gamma_m|_{\infty, \Omega} \leq ct^{1/2} \|\Gamma_m\|_{4, \Omega^t}.$$

Next, consider problem (3.98). Integrating (3.98)₁ with respect to time we obtain

$$\begin{aligned} H_m(\xi, t) &= - \exp \left(- \int_0^t \operatorname{div}_{u_m} u_m d\tau \right) \\ &\quad \times \int_0^t \left[\eta_{m-1} (\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1}) \exp \left(\int_0^{t'} \operatorname{div}_{u_m} u_m dt'' \right) \right] dt'. \end{aligned}$$

Therefore, we have (see [15])

$$(3.105) \quad \sup_{\tau} |H_m|_{3,0, \Omega} \leq \tilde{\phi}_3(t, K_0) t^{1/2} \|U_m\|_{4, \Omega^t}.$$

Taking into account (3.102)–(3.105) and the Gronwall inequality we get

$$(3.106) \quad Y_{m+1}(t) \leq \tilde{\phi}_4(t, K_0) t^{1/4} Y_m(t).$$

Hence by (3.106) and (3.105) for $t \leq T^{**} \leq T^*$ (where T^{**} is sufficiently small) the sequence $\{(u_m, \gamma_m, \eta_m)\}$ converges to a limit $(u, \gamma, \eta) \in W_2^{4,2}(\Omega^t) \times W_2^{4,2}(\Omega^t) \times C^0(0, t; \Gamma_0^{3,3/2}(\Omega))$ which is a solution of (3.1). Moreover, from Lemma 3.3 it follows that $\eta_t \in L_2(0, t; W_2^3(\Omega))$ and $\eta_{tt} \in L_2(0, t; W_2^1(\Omega))$.

Uniqueness of (u, γ, η) can be proved in the standard way.

This completes the proof of the theorem. ■

4. Local existence in the case $\sigma \neq 0$. In the case $\sigma \neq 0$ problem (1.1) in the Lagrangian coordinates takes the form

$$\begin{aligned} \eta u_t - \mu \nabla_u^2 u - \nu \nabla_u \nabla_u \cdot u + \nabla_u p(\eta, \gamma) &= \eta g && \text{in } \Omega^T, \\ \eta_t + \eta \nabla_u \cdot u &= 0 && \text{in } \Omega^T, \\ \eta c_v(\eta, \gamma) \gamma_t - \kappa \nabla_u^2 \gamma &= -\gamma p_\gamma(\eta, \gamma) \nabla_u \cdot u \\ (4.1) \quad + \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_j + \xi_{x_j} \cdot \nabla_\xi u_i)^2 + (\nu - \mu)(\nabla_u \cdot u)^2 + \eta k && \text{in } \Omega^T, \\ \mathbb{T}_u(u, p) \cdot \bar{n} - \sigma \Delta_{S_t}(t) X_u(\xi, t) &= -p_0 \bar{n} && \text{on } S^T, \\ \bar{n} \cdot \nabla_u \gamma &= \tilde{\gamma} && \text{on } S^T, \\ u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0, \quad \gamma|_{t=0} = \vartheta_0 && \text{in } \Omega. \end{aligned}$$

The argument used to prove local existence of problem (4.1) is the same as in the case $\sigma = 0$ (see Sect. 3). We consider the problem

$$(4.2) \quad \begin{aligned} \eta u_t - \mu \nabla_w^2 u - \nu \nabla_w \nabla_w \cdot u &= F && \text{in } \Omega^T, \\ \Pi_0 \Pi \mathbb{D}_w(u) \cdot \bar{n}_0 &= G_1 && \text{on } S^T, \\ \bar{n}_0 \cdot \mathbb{D}_w(u) \cdot \bar{n}_0 - \sigma \bar{n}_0 \cdot \Delta_{S_t}(t) \int_0^t u(\tau) d\tau \\ &= G_2 + \sigma \int_0^t H(\tau) d\tau && \text{on } S^T, \\ u|_{t=0} &= u_0 && \text{in } \Omega, \end{aligned}$$

where Π_0 and Π are the projections defined by $\Pi_0 g = g - (g \cdot \bar{n}_0) \bar{n}_0$ and $\Pi g = g - (g \cdot \bar{n}) \bar{n}$, respectively.

The following theorem holds (see [15]).

THEOREM 4.1. *Let*

$$\begin{aligned} S &\in W_2^{4+1/2}, F \in W_2^{2,1}(\Omega^T), G_i \in W_2^{3-1/2, 3/2-1/4}(S^T), \\ \mathbf{D}_{\xi,t}^2 G_i \mathbf{1}_{1/4, S^T} &< \infty \quad (i = 1, 2), \\ D_\xi^\alpha G_1(0) = 0, D_\xi^\alpha (\bar{n}_0 \cdot \mathbb{D}_\xi(u(0)) \cdot \bar{n}_0 - G_2(0)) &= 0 \quad (|\alpha| \leq 1), \\ u_0 &\in W_2^3(\Omega), H \in W_2^{2-1/2, 1-1/4}(S^T), \\ \eta &\in L_\infty(0, T; \Gamma_0^{2,1}(\Omega)) \cap C^\alpha(\Omega) \quad (\alpha \in (0, 1)), 1/\eta \in L_\infty(\Omega^T), \\ w &\in W_2^{4,2}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega)), T < \infty. \end{aligned}$$

Assume that T is so small that

$$(4.3) \quad T^a (\|w\|_{4, \Omega^T} + |w(0)|_{3,0,\Omega}) \times \tilde{\phi} \left(\int_0^T |w_\xi|_{\infty, \Omega} dt, |1/\eta|_{\infty, \Omega^T}, |\eta|_{\infty, \Omega^T}, \|w\|_{4, \Omega^T}, |w(0)|_{3,0,\Omega}, T \right) \leq \delta,$$

where $\tilde{\phi}$ is a nonnegative nondecreasing continuous function of its arguments, $a > 0$ is a constant and $\delta > 0$ is sufficiently small. Then there exists a unique solution $u \in W_2^{4,2}(\Omega) \cap C^0(0, T; \Gamma_0^{3,3/2}(\Omega))$ to problem (4.2) satisfying the estimate

$$(4.4) \quad \begin{aligned} \|u\|_{4, \Omega^T} &\leq \phi_1(|1/\eta|_{\infty, \Omega^T}, |\eta|_{\infty, \Omega^T}, T) \left[\|F\|_{2, \Omega^T} + \sum_{i=1}^2 \|G_i\|_{3-1/2, S^T} \right. \\ &\quad \left. + \sum_{i=1}^2 \mathbf{D}_{\xi,t}^2 G_i \mathbf{1}_{1/4, S^T} + \|H\|_{2-1/2, S^T} \right. \\ &\quad \left. + \phi_2(|\eta|_{2,0,\infty, \Omega^T}, \|\eta\|_{C^\alpha(\Omega^T)}) \|u\|_{2, \Omega^T} \right] \\ &\quad + \phi_3(|1/\eta|_{\infty, \Omega^T}, T) |u(0)|_{3,0,\Omega}, \end{aligned}$$

where ϕ_i ($i = 1, 2, 3$) are nonnegative increasing continuous functions of their arguments. ■

Now consider the problems

$$\begin{aligned}
 & \eta_m \partial_t u_{m+1} - \mu \nabla_{u_m}^2 u_{m+1} - \nu \nabla_{u_m} \nabla_{u_m} \cdot u_{m+1} \\
 & \quad = -\nabla_{u_m} p(\eta_m, \gamma_m) + \eta_m g && \text{in } \Omega^T, \\
 & \Pi_0 \Pi_{u_m} \mathbb{D}_{u_m}(u_{m+1}) \cdot \bar{n}(u_m) = 0 && \text{on } S^T, \\
 (4.5) \quad & \bar{n}_0 \cdot \mathbb{D}_{u_m}(u_{m+1}) \cdot \bar{n}(u_m) - \sigma \bar{n}_0 \cdot \Delta_m(t) \int_0^t u_{m+1}(\tau) d\tau \\
 & \quad = \bar{n}_0 \cdot \bar{n}(u_m) [p(\eta_m, \gamma_m) - p_0] + \sigma \bar{n}_0 \cdot \Delta_m(t) \xi && \text{on } S^T, \\
 & u_{m+1}|_{t=0} = v_0 && \text{in } \Omega;
 \end{aligned}$$

$$\begin{aligned}
 & \eta_m c_v(\eta_m, \gamma_m) \partial_t \gamma_{m+1} - \kappa \nabla_{u_m}^2 \gamma_{m+1} \\
 & \quad = -\gamma_m p_\gamma(\eta_m, \gamma_m) \nabla_{u_m} \cdot u_m \\
 & \quad \quad + \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_{mj} + \xi_{x_j} \cdot \nabla_\xi u_{mi})^2 \\
 (4.6) \quad & \quad + (\nu - \mu)(\nabla_{u_m} \cdot u_m)^2 + \eta_m k && \text{in } \Omega^T, \\
 & \bar{n}(u_m) \cdot \nabla_{u_m} \gamma_{m+1} = \tilde{\gamma} && \text{on } S^T, \\
 & \gamma_{m+1}|_{t=0} = \vartheta_0 && \text{in } \Omega;
 \end{aligned}$$

$$\begin{aligned}
 (4.7) \quad & \partial_t \eta_m + \eta_m \nabla_{u_m} \cdot u_m = 0 && \text{in } \Omega^T, \\
 & \eta_m|_{t=0} = \varrho_0 && \text{in } \Omega.
 \end{aligned}$$

For the zero step functions u_0 , γ_0 and η_0 we take the functions given by formulae (3.74)–(3.80).

Using Theorems 4.1 and 3.6 to problems (4.5) and (4.6), respectively and repeating the argument from the proof of Theorem 3.7 we obtain

THEOREM 4.2. *Let*

$$\begin{aligned}
 & S \in W_2^{4+1/2}, f \in C^{2,1}(\mathbb{R}^3 \times [0, T]), r \in C^{2,1}(\mathbb{R}^3 \times [0, T]), \\
 & \tilde{\vartheta} \in C^{2,1}(\mathbb{R}^3 \times [0, T]), v_0 \in W_2^3(\Omega), \vartheta_0 \in W_2^3(\Omega), \\
 & 1/\vartheta_0 \in L_\infty(\Omega), \vartheta_0 > 0, \varrho_0 \in W_2^3(\Omega), 1/\varrho_0 \in L_\infty(\Omega), \varrho_0 > 0, \\
 & c_v \in C^2(\mathbb{R}_+^2), c_v > 0, p \in C^3(\mathbb{R}_+^2).
 \end{aligned}$$

Moreover, assume that the following compatibility conditions are satisfied:

$$(4.8) \quad D_\xi^\alpha [\mathbb{D}_\xi(v_0) \cdot \bar{n}_0 - (p_0(\varrho_0, \vartheta_0) - p_0) \bar{n}_0 - \sigma \Delta_{S_t}(0) \xi] = 0, \quad |\alpha| \leq 1, \text{ on } S,$$

and

$$(4.9) \quad D_\xi^\alpha(\bar{n}_0 \cdot \nabla_\xi \vartheta_0 - \tilde{\vartheta}(\xi, 0)) = 0, \quad |\alpha| \leq 1, \text{ on } S.$$

Let T^* be so small that $0 < c_1(1 - CK_0T^*)^3 \leq \det\{x_\xi\} \leq c_2(1 + CK_0T^*)^3$ (where $x(\xi, t) = \xi + \int_0^t u_0(\xi, \tau) d\tau$ for $t \leq T^*$, u_0 is given by (3.74), K_0 is given by (3.91) and the constant $C = C(K_0)$ which is a nondecreasing continuous function of K_0 is given by (3.94)). Then there exists T^{**} , $0 < T^{**} \leq T^*$ such that for $T \leq T^{**}$ there exists a unique solution $(u, \gamma, \eta) \in W_2^{4,2}(\Omega^T) \times W_2^{4,2}(\Omega^T) \times C^0(0, t; \Gamma_0^{3,3/2}(\Omega))$ of problem (4.1) and

$$\begin{aligned} \|u\|_{4,\Omega^T} + \|\gamma\|_{4,\Omega^T} &\leq CK_0, \\ \sup_t \|\eta\|_{3,\Omega} + \sup_t \|\eta_t\|_{2,\Omega} + \|\eta_t\|_{L_2(0,T;W_2^3(\Omega))} + \|\eta_{tt}\|_{L_2(0,T;W_2^1(\Omega))} \\ &\leq \Phi_1(T, T^a K_0) \|\varrho_0\|_{3,\Omega}, \\ |1/\eta|_{\infty,\Omega^T} + |\eta|_{\infty,\Omega^T} &\leq \Phi_2(T^{1/2} K_0) |1/\varrho_0|_{\infty,\Omega} + \Phi_3(T^{1/2} K_0) |\varrho_0|_{\infty,\Omega}, \end{aligned}$$

where Φ_i ($i = 1, 2, 3$) are increasing continuous functions of their arguments, $a > 0$. ■

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