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Decompositions of hypersurface singularities of type $J_{k,0}$

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Abstract. Applications of singularity theory give rise to many questions concerning deformations of singularities. Unfortunately, satisfactory answers are known only for simple singularities and partially for unimodal ones. The aim of this paper is to give some insight into decompositions of multi-modal singularities with unimodal leading part. We investigate the $J_{k,0}$ singularities which have modality $k-1$ but the quasihomogeneous part of their normal form only depends on one modulus.

1. Introduction. Let $0 \in W \subset \mathbb{C}^n$ be an analytic hypersurface with an isolated singular point at the origin. When we deform the hypersurface W then the singular point may decompose, i.e. the deformed hypersurface W' may have near the origin several less complicated singular points. The investigation of such decompositions has important applications, e.g. in the theory of Legendrian singularities, i.e. the singularities of wave fronts (compare [1], §21).

In this paper we deal with hypersurfaces given by the equation

$$W = f^{-1}(0)\,,$$

where

$$f(x,y,z) = y^3 + \lambda y^2 x^k + x^{3k} + \Big(\sum_{i=1}^{k-2} c_i x^i\Big) y x^{2k} + \sum_{i=1}^{n-2} z_i^2, \qquad k = 2,3,\ldots,$$

$c_1,\ldots,c_{k-2},\lambda$ are complex parameters, and $4\lambda^3+27 \neq 0$. The hypersurface W has an isolated singular point at the origin. In Arnold's classification (see [1], §15) such singularities are called $J_{k,0}$. We investigate how the decompositions depend on the modulus λ. It turns out that the exceptional values of λ are 0 and e where $2e^3+27=0$. Moreover, these values are universal.

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Namely, every decomposition of any $J_{k,0}$ hypersurface singularity occurs (up to topological type) for $c = 0$ and $\lambda = 0$ or $\lambda = e$. From this we deduce that for a given k, the Legendrian singularities $J_{k,0}(\lambda, 0)$, $\lambda = 0, e$, are universal for all $J_{k,0}$ Legendrian singularities.

2. Notation

2.1. V-*equivalence.* In this paper we shall base on the following notion of equivalence, the so-called V-equivalence. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of analytic functions and let W and U be the germs of the hypersurfaces $f^{-1}(0)$ and $g^{-1}(0)$. We say that the germs W and U are *equivalent* (resp. the germs f and g are V-*equivalent*) if there exists an analytic change of the coordinate system ϕ (and an invertible germ h resp.) such that

$$\phi(U) = W \qquad (\text{and respectively } f(\phi(z)) = h(z)g(z)).$$

If ϕ is only a homeomorphism then we say that W and U (resp. f and g) are *topologically equivalent*, i.e. have the same topological type.

We remark that the topological type of a singular point coincides with the notion of μ-stratum (for the definition see e.g. [1], § 15.0), i.e. singularities belonging to one μ-stratum are topologically equivalent. Therefore to denote the topological type we shall use the symbol of the corresponding μ-stratum.

2.2. *Decompositions.* Let W be the germ of the hypersurface $f^{-1}(0)$, where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function with an isolated critical point at the origin. Let S be the analytic type of this point. Let $S_1, \ldots, S_m$ be topological types of singular points of analytic hypersurfaces. We say that *there exists a decomposition*

$$S \to S_1, \ldots, S_m$$

if there exists a one-parameter analytic deformation F_ε of f ($F_0 = f$) such that for ε close enough to 0 (but $\varepsilon \neq 0$) the hypersurface $W = F_\varepsilon^{-1}(0)$ has just m singular points, of types $S_1, \ldots, S_m$, which tend to the origin as $\varepsilon \to 0$. We remark that the coordinates of the critical points of the function F_ε are Puiseux series of ε. Hence the notion of limit is well-defined.

If the deformation F_ε has only one singular point tending to the origin and moreover it is topologically equivalent to S then we say that the deformation is *trivial*, otherwise is *nontrivial*.

We recall that analytically equivalent singularites have the same decompositions.

3. Main results. The case $k = 2$ is well known. The decompositions of $J_{2,0}$ do not depend on the modulus (see [6]). We shall deal with the next cases; $k \geq 3$ (cf. [8, 2, 3] for $k = 3$). We denote by $J_{k,0}(\lambda, c)$ the singular point of the hypersurface $f_{\lambda,c}^{-1}(0)$ where

$$f_{\lambda,c}(x,y,z) = y^3 + \lambda y^2 x^k + x^{3k} + \Big(\sum_{i=1}^{k-2} c_i x^i\Big) y x^{2k} + \sum_{i=1}^{n-2} z_i^2, \qquad k = 2,3,\ldots$$

First we reduce the problem to the quasihomogeneous case, i.e. to the case $c_1 = \ldots = c_{k-2} = 0$.

THEOREM 1. *If there exists a decomposition*

$$J_{k,0}(\lambda, c) \to S_1, \ldots, S_m$$

then there exists a decomposition

$$J_{k,0}(\lambda, 0) \to S_1, \ldots, S_m\,.$$

The analytic type of the quasihomogeneous part of f,

$$y^3 + \lambda y^2 x^k + x^{3k} + \sum_{i=1}^{n-2} z_i^2\,,$$

is classified by the so-called j-*invariant*,

$$j = j(\lambda) = \frac{-4\lambda^6}{27(4\lambda^3 + 27)}.$$

We recall that the j-invariant classifies the quasihomogeneous polynomials

$$H(x,y) = Ay^3 + By^2x^k + Cyx^{2k} + Dx^{3k}, \qquad A \neq 0\,,$$

up to:

- the change of the coordinate system

$$y \to ay + bx^k, \quad a \neq 0, \qquad x \to cx, \quad c \neq 0\,;$$

- multiplication by a nonzero constant, $H \to dH$, $d \neq 0$.

If $H(x,y) = h(y - \alpha x^k)(y - \beta x^k)(y - \gamma x^k)$ then

$$j = \frac{4(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)^3}{27(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2}\,.$$

For more details about j see [4], §IV.4.

We shall show that there are only two exceptional values of j, namely 0 and 1 (the harmonic and anharmonic cases).

THEOREM 2. *For λ_1, λ_2 such that $j(\lambda_1), j(\lambda_2) \neq 0, 1, \infty$ there exists a decomposition*

$$J_{k,0}(\lambda_1, 0) \to S_1, \ldots, S_m$$

if and only if there exists a decomposition

$$J_{k,0}(\lambda_2, 0) \to S_1, \ldots, S_m\,.$$

For $k \geq 3$ the cases $j = 0, 1$ are distinguished.

THEOREM 3. *There exist decompositions*

$$J_{k,0}(\lambda, 0) \to E_6, E_{6k-10}\,, \qquad J_{k,0}(\lambda, 0) \to E_7, E_{6k-9}$$

only if respectively $j(\lambda) = 0$ *or* 1.

Moreover, the cases $j = 0, 1$ are universal.

THEOREM 4. *For any* λ *with* $j(\lambda) \neq 0, 1, \infty$, *there exists a decomposition*

$$J_{k,0}(\lambda, 0) \to S_1, \ldots, S_m$$

if and only if there exist decompositions

$$J_{k,0}(0, 0) \to S_1, \ldots, S_m, \quad J_{k,0}(e, 0) \to S_1, \ldots, S_m, \quad j(e) = 1\,.$$

Remark. Since the variables $z_1, \ldots, z_{n-2}$ play no role in our investigations, we shall omit them (i.e. we shall consider the two-dimensional case).

The crucial point of the proofs of Theorems 1, 2 and 4 is to consider the following group actions on deformations of $J_{k,0}$ singularities:

• the quasihomogeneous $\mathbb{C}^*$ action: $t * F(x, y) = \widetilde{F}(x, y)$ where $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and

$$\widetilde{F}(x, y) = t^{3k} F\left(\frac{x}{t}, \frac{y}{t^k}\right),$$

• the shift $\mathbb{C}$ action: $t \circ F(x, y) = \widetilde{F}(x, y)$ where $t \in \mathbb{C}$ and

$$\widetilde{F}(x, y) = (1 + tx)^{3k} F\left(\frac{x}{1 + tx}, \frac{y}{(1 + tx)^k}\right).$$

Both these actions preserve the types of singular points of $F^{-1}(0)$. Moreover, they transform diagonal deformations to underdiagonal ones. By abuse of language we shall mean by an *underdiagonal deformation* of the quasihomogeneous singularity $J_{k,0}(*, 0)$ the polynomial

$$F(x, y) = \sum a_{i,j} x^i y^j \quad \text{where } i + kj \leq 3k\,.$$

In contrast to the quasihomogeneous $\mathbb{C}^*$ action the shift action changes the j-invariant of the leading part

$$\Gamma F(x, y) = \sum a_{i,j} x^i y^j \quad \text{where } i + kj = 3k.$$

The proof of Theorem 3 is based on reduction to the well-known case $k = 3$ (see [8, 3, 2]) obtained with the help of a σ-process.

4. Proof of Theorem 1. Let $F_\varepsilon(x,y)$ be an analytic deformation of the $J_{k,0}(\lambda,c)$ singularity

$$F_0(x,y)=y^3+\lambda y^2x^k+x^{3k}+\Big(\sum_{i=1}^{k-2}c_ix^i\Big)yx^{2k}\,.$$

Let $F_\varepsilon(x,y)=\sum a_{i,j}(\varepsilon)x^iy^j$ be the Taylor expansion of F_ε ($a_{i,j}(\varepsilon)$ are germs of analytic functions of ε). Let

$$M=\min\{(\operatorname{ord}_\varepsilon a_{i,j}(\varepsilon))/(3k-i-kj):0<i+kj<3k\}\,.$$

We shall base on the following estimate:

Lemma 1. *Let $(x(\varepsilon),y(\varepsilon))$ be a critical point of F_ε tending to the origin as $\varepsilon\to 0$. Then*

$$\operatorname{ord}_\varepsilon x(\varepsilon)\ge M,\qquad \operatorname{ord}_\varepsilon y(\varepsilon)\ge kM\,.$$

Proof. Let the coordinates of the critical point have the following Puiseux expansions:

$$x(\varepsilon)=x_0\varepsilon^m+o(\varepsilon^m),\qquad y(\varepsilon)=y_0\varepsilon^km+o(\varepsilon^{km})\,,$$

where $0<m<M$ and $(x_0,y_0)\ne(0,0)$ (we denote by $o(\cdot)$ the terms of higher order). After substitution we obtain

$$\frac{\partial F_\varepsilon}{\partial x}(x(\varepsilon),y(\varepsilon))=(3kx_0^{3k-1}+k\lambda y_0^2x_0^{k-1})\varepsilon^{(3k-1)m}+o(\varepsilon^{(3k-1)m})\,,$$
$$\frac{\partial F_\varepsilon}{\partial y}(x(\varepsilon),y(\varepsilon))=(3y_0^2+2\lambda y_0x_0^k)\varepsilon^{2km}+o(\varepsilon^{2km})\,.$$

Since $4\lambda^3+27\ne 0$, the above implies that $x_0=y_0=0$, a contradiction.

To prove Theorem 1 we apply the method similar to [5], Section 3. We make a substitution $\varepsilon=\alpha^{4k}$ and apply the quasihomogeneous transformation for $t=1/\alpha$:

$$\begin{aligned}\widetilde F_\alpha(x,y)&=\alpha^{-3k}F_{\alpha^{4k}}(\alpha x,\alpha^ky)\\&=\sum\alpha^{i+jk-3k}a_{i,j}(\alpha^{4k})x^iy^j=\sum\widetilde a_{i,j}(\alpha)x^iy^j\,.\end{aligned}$$

Now

(i) if $i+kj<3k$ then $a_{i,j}(0)=0$, hence

$$\operatorname{ord}_\alpha a_{i,j}(\alpha^{4k})=4k\operatorname{ord}_\varepsilon a_{i,j}(\varepsilon)\ge 4k\,,$$

thus

$$\operatorname{ord}\widetilde a_{i,j}(\alpha)=i+jk-3k+\operatorname{ord}_\alpha a_{i,j}(\alpha^{4k})>0\,;$$

(ii) if $i+kj=3k$ then $\widetilde a_{i,j}(\alpha)=a_{i,j}(\alpha^{4k})$,

(iii) if $i+kj>3k$ then

$$\operatorname{ord}_\alpha\widetilde a_{i,j}(\alpha)\ge i+jk-3k>0\,.$$

Therefore

$$\widetilde{F}_0(x,y) = y^3 + \lambda y^2 x^k + x^{3k}\,.$$

Corollary. *Any one-parameter deformation of a semi-quasihomogeneous singularity $J_{k,c}(\lambda,c)$ may be transformed by a quasihomogeneous transformation to a deformation of the quasihomogeneous singularity $J_{k,0}(\lambda,0)$. Moreover, both deformed curves*

$$F_\varepsilon^{-1}(0) \quad \text{and} \quad \widetilde{F}_\alpha^{-1}(0) \quad \text{for } \alpha,\varepsilon \neq 0,\ \varepsilon = \alpha^{4k}\,,$$

have the same singular points (up to analytic equivalence) which tend to the origin as ε respectively α tends to 0.

Indeed, the singular point $(x(\varepsilon), y(\varepsilon))$ of $F_\varepsilon = 0$ is transformed to the singular point $(\widetilde{x}(\alpha), \widetilde{y}(\alpha))$ of $\widetilde{F}_\alpha$, where

$$\widetilde{x}(\alpha) = \alpha^{-1} x(\alpha^{4k}), \qquad \widetilde{y}(\alpha) = \alpha^{-k} y(\alpha^{4k})\,.$$

Hence

$$\operatorname{ord}_\alpha \widetilde{x}(\alpha) = 4k \operatorname{ord}_\varepsilon x(\varepsilon) - 1, \qquad \operatorname{ord}_\alpha \widetilde{y}(\alpha) = 4k \operatorname{ord}_\varepsilon y(\varepsilon) - k\,.$$

The deformation F_ε is analytic, thus M is greater than $1/(3k)$. Therefore, by the above lemma, if $(x(\varepsilon), y(\varepsilon))$ tends to the origin then so does $(\widetilde{x}(\alpha), \widetilde{y}(\alpha))$.

This finishes the proof of Theorem 1.

5. Reduction to an underdiagonal deformation. Repeating the procedure of the previous section for other α and t we may reduce our problem to the investigation of polynomials with constant coefficients.

Let F_ε be a nontrivial deformation. Without loss of generality we may assume that $F_\varepsilon(0,0) = 0$ for all ε, i.e. the constant term $a_{0,0}$ is zero. Let p and q be positive integers such that

$$p/q = M = \min\{(\operatorname{ord}_\varepsilon a_{i,j}(\varepsilon))/(3k - i - kj) : 0 < i + kj < 3k\}\,.$$

We make the substitution $\varepsilon = \alpha^q$, and apply the quasihomogeneous transformation for $t = 1/\alpha^p$:

$$\begin{aligned}\widetilde{F}_\alpha(x,y) &= \alpha^{-3kp} F_{\alpha^q}(\alpha^p x, \alpha^{kp} y)\\ &= \sum \alpha^{(i+kj-3k)p} a_{i,j}(\alpha^q) x^i y^j = \sum \widetilde{a}_{i,j}(\alpha) x^i y^j\,.\end{aligned}$$

Obviously all $\widetilde{a}_{i,j}(\alpha)$ are analytic germs of α. But now in contrast to the previous section there is at least one pair of indices $(i,j), 0 < i + kj < 3k$, such that $\widetilde{a}_{i,j}(0)$ is not zero, namely those for which $q(\operatorname{ord}_\varepsilon a_{i,j}(\varepsilon)) = p(3k - i - jk)$. Hence

$$\widetilde{F}_0(x,y) = y^3 + \lambda y^2 x^k + x^{3k} + \sum b_{i,j} x^i y^j, \qquad i + jk < 3k\,.$$

Moreover, $b_{i,j} = a_{i,j}(0)$, thus not all $b_{i,j}$ are zeros.

Remark. The singular points of $F_\varepsilon^{-1}(0)$, $\varepsilon \neq 0$, which tend to the origin as $\varepsilon \to 0$, correspond to the singular points of $\widetilde{F}_\alpha^{-1}(0)$, $\alpha \neq 0$, which tend to the singular points of $\widetilde{F}_0^{-1}(0)$ as $\alpha \to 0$. Moreover, the corresponding points have the same analytic type (for $\varepsilon = \alpha^q$).

6. The shift transformation of an underdiagonal deformation. In this section we shall consider the orbit of the polynomial $\widetilde{F}_0$ from the previous section under the shift transformation. By the change of coordinates

$$y \to y + d_0 + d_1 x + \ldots + d_k x^k, \qquad x \to x\,,$$

we transform the polynomial $\widetilde{F}_0$ to the form

$$G(x,y) = y^3 + y\sum_{i=0}^{2k} A_i x^i + \sum_{i=0}^{3k} B_i x^i\,.$$

The j-invariant of the leading part of G,

$$y^3 + A_{2k}x^{2k}y + B_{3k}x^{3k}\,,$$

equals

$$j = \frac{4A_{2k}^3}{4A_{2k}^3 + 27B_{3k}^2}\,.$$

We apply to G the shift transformation:

$$\begin{aligned}
\widetilde{G}(x,y) &= (1+tx)^{3k} G\left(\frac{x}{1+tx}, \frac{y}{(1+tx)^k}\right)\\
&= (1+tx)^{3k}\left(\frac{y^3}{(1+tx)^{3k}} + y\sum_{i=0}^{2k}\frac{A_i x^i}{(1+tx)^{k+i}} + \sum_{i=0}^{3k}\frac{B_i x^i}{(1+tx)^i}\right)\\
&= y^3 + y\sum_{i=0}^{2k} A_i x^i (1+tx)^{2k-i} + \sum_{i=0}^{3k} B_i x^i (1+tx)^{3k-i}\\
&= y^3 + y\sum_{i=0}^{2k} A_i x^i \left(\frac{1}{x} + t\right)^{2k-i} + \sum_{i=0}^{3k} B_i x^i \left(\frac{1}{x}+t\right)^{3k-i}\,.
\end{aligned}$$

The leading part of $\widetilde{G}$ is $y^3 + yA(t)x^{2k} + B(t)x^{3k}$, where $A(t) = \sum A_i t^{2k-i}$ and $B(t) = \sum B_i t^{3k-i}$. The new j-invariant is

$$J(t) = \frac{4A(t)^3}{4A(t)^3 + 27B(t)^2}\,.$$

Lemma 2. (i) *$J(t)$ is constant only in one of the following cases*:

(a) $A \equiv 0$; *then* $J \equiv 0$;
(b) $B \equiv 0$; *then* $J \equiv 1$;
(c) $A(t)^3 = cB(t)^2, c \neq 0$; *then* $J \equiv 4c/(4c+27)$.

(ii) *Otherwise the image of* $J(t)$ *is the whole complex line except possibly a finite number of points.*

Proof. Obviously $J(t)$ is constant only in one of cases (a)–(c) of (i). Otherwise $J(t)$ is a nonconstant function defined on the complex line except the zeros of the discriminant $\Delta = 4A^3 + 27B^2$. On the other hand, it may be extended to a rational function $\widetilde{J} : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$\widetilde{J}(t) = \begin{cases} J(t) & \text{if } \Delta(t) \neq 0\,, \\ \infty & \text{if } \Delta(t) = 0,\ A(t) \neq 0\,, \\ \lim_{\tau \to t} J(\tau) & \text{if } A(t) = B(t) = 0, \text{ or } t = \infty. \end{cases}$$

Since $\widetilde{J}$ is onto, the image of J contains the complement of the set of $\widetilde{J}(t)$ where t is a common zero of $A(t)$ and $B(t)$ or $t = \infty$, i.e. the image of J contains the whole complex line except possibly a finite number of points.

Next we shall investigate the exceptional cases.

(a) $A \equiv 0$. Then $G = y^3 + b(x)$. The multiple factors of $b(x)$ give rise to singular points of $G^{-1}(0)$. Let

$$b(x) = b \prod (x - x_i)^{\alpha_i}\,,$$

where all x_i are different. In the local coordinate system $\{\widetilde{x}, \widetilde{y}\}$ in a neighbourhood of the point $(x_i, 0)$,

$$G = \widetilde{y}^3 + \widetilde{x}^{\alpha_i}\,,$$

where

$$\widetilde{x} = (x - x_i)\Big(b \prod_{j \neq i} (x - x_j)^{\alpha_j}\Big)^{1/\alpha_i}, \qquad \widetilde{y} = y\,.$$

Therefore the point $(x_i, 0)$ has the following singularity type:

$$\begin{array}{ll} \alpha_i = 2 & \Rightarrow A_2; \\ \alpha_i = 3 & \Rightarrow D_4; \\ \alpha_i = 3m+1, m \geq 1 & \Rightarrow E_{6m}; \\ \alpha_i = 3m+2, m \geq 1 & \Rightarrow E_{6m+2}; \\ \alpha_i = 3m, m \geq 2 & \Rightarrow J_{m,0},\ j = 0,\ \text{quasihomogeneous.} \end{array}$$

(b) $B \equiv 0$. Then

$$G = y^3 + a(x)y = y(y^2 + a(x))\,.$$

The factors of $a(x)$ give rise to singular points of $G^{-1}(0)$. Let

$$a(x) = a \prod (x - x_i)^{\alpha_i}\,,$$

where all x_i are different. In the local coordinate system $\{\widetilde{x},\widetilde{y}\}$ in a neighbourhood of $(x_i,0)$,

$$G=\widetilde{y}(\widetilde{y}^2+\widetilde{x}^{\alpha_i})\,,$$

where

$$\widetilde{x}=(x-x_i)\Big(a\prod_{j\neq i}(x-x_j)^{\alpha_j}\Big)^{1/\alpha_i},\qquad \widetilde{y}=y\,.$$

Therefore we have the following types of singularity at $(x_i,0)$:

$$\begin{array}{ll}\alpha_i=1 & \Rightarrow A_1;\\ \alpha_i=2 & \Rightarrow D_4;\\ \alpha_i=2m+1, m\geq 1 & \Rightarrow E_{6m+1};\\ \alpha_i=2m, m\geq 2 & \Rightarrow J_{m,0},\ j=1, \text{quasihomogeneous.}\end{array}$$

(c) $A(t)^3=cB(t)^2$, $c\neq 0$. Then

$$G=y^3+a(x)y+b(x),\qquad a(x)^3=cb(x)^2,\ c\neq 0\,.$$

In this case $a(x)$ and $b(x)$ have the same roots. Thus there is a polynomial $d(x)$ such that $b(x)=cd(x)^3$ and $a(x)=cd(x)^2$. Hence

$$G=y^3+cd(x)^2y+cd(x)^3\,.$$

The factors of $d(x)$ give rise to singular points of $G^{-1}(0)$. Let

$$d(x)=d\prod(x-x_i)^{\alpha_i}\,,$$

where all x_i are different. In the local coordinate system $\{\widetilde{x},\widetilde{y}\}$ in a neighbourhood of $(x_i,0)$,

$$G=\widetilde{y}^3+c\widetilde{x}^{2\alpha_i}\widetilde{y}+c\widetilde{x}^{3\alpha_i}\,,$$

where

$$\widetilde{x}=(x-x_i)\Big(d\prod_{j\neq i}(x-x_j)^{\alpha_j}\Big)^{1/\alpha_i},\qquad \widetilde{y}=y\,.$$

This gives the following singularity types:

$$\begin{array}{ll}\alpha_i=1 & \Rightarrow D_4;\\ \alpha_i=m,\ m\geq 2 & \Rightarrow J_{m,0},\ j=4c/(4c+27),\ \text{quasihomogeneous.}\end{array}$$

7. Proofs of Theorems 2 and 4. Theorems 2 and 4 follow directly from the following proposition.

Proposition 1. *If there exists a decomposition*

$$J_{k,0}(\lambda,0)\to S_1,\ldots,S_m,\qquad \textit{for some } \lambda \textit{ with } j(\lambda)\neq 0,1,\infty\,,$$

then there exists a decomposition

$$J_{k,0}(\kappa,0)\to S_1,\ldots,S_m,\qquad \textit{for any } \kappa \textit{ with } j(\kappa)\neq\infty\,.$$

P r o o f. We note that there are two obvious cases:

(i) $m = 0$: the curve $F_\varepsilon = 0$ is smooth for small nonzero ε;

(ii) $m = 1$, $S_1 = J_{k,0}$: the trivial deformation (all $J_{k,0}$ singularities have the same topological type).

Hence we may restrict ourselves to nontrivial deformations with at least one singular point tending to the origin as $\varepsilon \to 0$. Moreover, we may shift this point to the origin (if necessary we substitute $\varepsilon := \varepsilon^\nu$). Let F_ε be such a deformation of the singularity $J_{k,0}(\lambda, 0)$, $j(\lambda) \neq 0, 1, \infty$. We shall show that, for any κ, there exists a deformation H_α of $J_{k,0}(\kappa, 0)$ which has the same (up to topological type) singular points as F_ε .

We apply the quasihomogeneous transformation to F_ε, as in Section 5. We obtain

$$\widetilde{F}_\alpha(x, y) = \alpha^{-3kp} F_{\alpha^q}(\alpha^p x, \alpha^{kp} y) ,$$

where

$$\widetilde{F}_0(x, y) = y^3 + \lambda y^2 x^k + x^{3k} + \sum b_{i,j} x^i y^j , \qquad i + jk < 3k ,$$

where not all $b_{i,j}$ vanish.

Let $X_1, \ldots, X_v$ be the analytic types of the singular points of $\widetilde{F}_0^{-1}(0)$. Thus the decomposition induced by F_ε has the form

$$J_{k,0}(\lambda, 0) \to S_{1,1}, \ldots, S_{1,m_1}, \ldots, S_{v,1}, \ldots, S_{v,m_v} ,$$

where

$$X_i \to S_{i,1}, \ldots, S_{i,m_i}, \qquad i = 1, \ldots, v .$$

To prove the proposition it is enough to show that there exists a deformation $\widetilde{H}_\alpha$ of $J_{k,0}(\kappa, 0)$ which has, for $\alpha \neq 0$, singular points of analytic types $X_1', \ldots, X_v'$, where X_i' is topologically equivalent to X_i and all the decompositions of X_i occur for X_i'. Indeed, we may deform $\widetilde{H}_\alpha$ and obtain a deformation H_α of $J_{k,0}(\kappa, 0)$ which has singular points of types

$$S_{1,1}, \ldots, S_{1,m_1}, S_{2,1}, \ldots, S_{v,m_v} ,$$

where

$$X_i' \to S_{i,1}, \ldots, S_{i,m_i}, \qquad i = 1, \ldots, v$$

(cf. [7]).

We now construct the $\widetilde{H}_\alpha$.

As in Section 6 we transform $\widetilde{F}_0$ to

$$G(x, y) = y^3 + a(x)y + b(x) .$$

Next we apply the shift transformation:

$$G_t(x, y) = (1 + tx)^{3k} G\left(\frac{x}{1 + tx}, \frac{y}{(1 + tx)^k}\right) .$$

There are three cases to be considered (in terms of Section 6):

(i) $j(\kappa)$ belongs to the image of $J(t)$;
(ii) $j(\kappa)$ belongs to the closure of $J(t)$;
(iii) $J(t)$ is constant and not equal to $j(\kappa)$.

Remark. If $J(t)$ is constant and equal to $j(\kappa)$ then $j(\kappa) = J(0) = j(\lambda)$, hence the singularity $J_{k,0}(\kappa, 0)$ is analytically equivalent to $J_{k,0}(\lambda, 0)$. Thus they have the same decompositions.

Case (i): $j(\kappa) = J(\tau)$ for some $\tau \in \mathbb{C}$. We apply the quasihomogeneous transformation:

$$\widetilde{H}_\alpha(x, y) = \alpha^{3k} G_\tau\left(\frac{x}{\alpha}, \frac{y}{\alpha^k}\right).$$

Obviously $\widetilde{H}_\alpha^{-1}(0)$, $\alpha \neq 0$, has the same singular points as $\widetilde{F}_0^{-1}(0)$. Moreover, $\widetilde{H}_0$ is a quasihomogeneous germ of type $J_{k,0}$ and its j-invariant equals $j(\kappa)$.

Case (ii): $j(\kappa) = \lim_{t\to\tau} J(t)$ for some $\tau \in \mathbb{C}$ or $\tau = \infty$. We consider the leading part of the family of polynomials G_t,

$$\Gamma G_t(x, y) = y^3 + A(t) y x^{2k} + B(t) x^{3k}.$$

If τ is finite then both $A(\tau)$ and $B(\tau)$ vanish. Let

$$p = \min\{\tfrac{1}{2} \operatorname{ord}_t A(\tau + t), \tfrac{1}{3} \operatorname{ord}_t B(\tau + t)\}.$$

In the case $\tau = \infty$ we put

$$p = \max\{\tfrac{1}{2} \deg_t A(t), \tfrac{1}{3} \deg_t B(t)\}.$$

Next we shall deal with the meromorphic family of polynomials

$$\widetilde{G}_\alpha(x, y) = \begin{cases} \alpha^{-3p} G_{\tau+\alpha}(x, \alpha^p y) & \text{if } \tau \in \mathbb{C}, \\ \alpha^{3p} G_{1/\alpha}(x, y/\alpha^p) & \text{if } \tau = \infty. \end{cases}$$

For $\alpha \neq 0$ the family $\widetilde{G}_\alpha(x, y)$ is locally analytic in α. The coefficients are Puiseux series of α and may have poles at the origin but the coefficients of the leading part are finite. Indeed,

$$\begin{aligned} \Gamma \widetilde{G}_\alpha(x, y) &= y^3 + \widetilde{A}(\alpha) y x^{2k} + \widetilde{B}(\alpha) x^{3k} \\ &= \begin{cases} y^3 + A(\tau + \alpha)\alpha^{-2p} y x^{2k} + B(\tau + \alpha)\alpha^{-3p} x^{3k} & \text{if } \tau \in \mathbb{C}, \\ y^3 + A(1/\alpha)\alpha^{2p} y x^{2k} + B(1/\alpha)\alpha^{3p} x^{2k} & \text{if } \tau = \infty. \end{cases} \end{aligned}$$

Moreover, $\widetilde{A}(\alpha)$ and $\widetilde{B}(\alpha)$ do not vanish simultaneously at $\alpha = 0$ and the j-invariant of $\Gamma \widetilde{G}_0$ equals $\lim_{t\to\tau} J(t)$:

$$\begin{aligned}
j &= \frac{4\widetilde{A}(0)^3}{4\widetilde{A}(0)^3+27\widetilde{B}(0)^2} = \lim_{\alpha\to 0}\frac{4\widetilde{A}(\alpha)^3}{4\widetilde{A}(\alpha)^3+27\widetilde{B}(\alpha)^2} \\
&= \begin{cases} \lim\limits_{\alpha\to 0}\dfrac{4A(\tau+\alpha)^3\alpha^{-6p}}{4A(\tau+\alpha)^3\alpha^{-6p}+27B(\tau+\alpha)^2\alpha^{-6p}} & \text{if } \tau\in\mathbb{C} \\ \lim\limits_{\alpha\to 0}\dfrac{4A(1/\alpha)^3\alpha^{6p}}{4A(1/\alpha)^3\alpha^{6p}+27B(1/\alpha)^2\alpha^{6p}} & \text{if } \tau=\infty \end{cases} \\
&= \lim_{t\to\tau}\frac{4A(t)^3}{4A(t)^3+27B(t)^2} = \lim_{t\to\tau} J(t)\,.
\end{aligned}$$

Next we apply the quasihomogeneous transformation:

$$\widetilde{H}_\alpha(x,y)=\alpha^{3kq}\widetilde{G}_\alpha\left(\frac{x}{\alpha^q},\frac{y}{\alpha^{kq}}\right),$$

where we choose q to be large enough not only to eliminate poles but also to make $\widetilde{H}_0(x,y)$ quasihomogeneous:

$$\widetilde{H}_0(x,y)=y^3+\widetilde{A}(0)yx^{2k}+\widetilde{B}(0)x^{3k}\,.$$

Obviously for $\alpha\neq 0$, $\widetilde{H}_\alpha^{-1}(0)$ has the same singularities as $\widetilde{F}_0^{-1}(0)$.

Case (iii): $J(t)$ is constant. This case will be proved by induction on k. We assume that the proposition is valid for $J_{2,0},\ldots,J_{k-1,0}$. We have (see Lemma 1)

$$G(x,y)=y^3+cd(x)^2+ycd(x)^3\,.$$

Moreover, the singular points of $G^{-1}(0)$ are analytically equivalent to D_4 or to quasihomogeneous $J_{m,0}, m<k$, with j-invariant equal to the j-invariant of the leading part of G, i.e. $j=j(\lambda)\neq 0,1,\infty$.

We choose c' such that

$$\frac{4c'}{4c'+27}=j(\kappa).$$

We put

$$G'(x,y)=y^3+c'd(x)^2y+c'd(x)^3\,.$$

Obviously, the singular points of $G^{-1}(0)$ and $G'^{-1}(0)$ are pairwise topologically equivalent, they can only have different j-invariants. But $j(\lambda)\neq 0,1,\infty$, hence the decompositions of singular points of $G^{-1}(0)$ occur for singular points of $G'^{-1}(0)$ (up to topological type). Therefore we may put

$$\widetilde{H}_\alpha(x,y)=\alpha^{3k}G'\left(\frac{x}{\alpha},\frac{y}{\alpha^k}\right).$$

Remark. One may prove in the same way the extension of the proposition for nonisolated singularities, i.e. for $j(\kappa)=\infty$.

8. Proof of Theorem 3

8.1. *Existence.* First we show that the decompositions

$$J_{k,0}(\lambda,0) \to E_6, E_{6k-10} \quad \text{and} \quad J_{k,0}(\lambda,0) \to E_7, E_{6k-9}$$

exist for $k \geq 3$ and respectively $j(\lambda) = 0$ and $j(\lambda) = 1$. Indeed, they are given by the deformations

$$F_\alpha(x,y) = y^3 + x^{3k-4}(x-\alpha)^4, \qquad F_0(x,y) = y^3 + x^{3k}\,,$$

and

$$F_\alpha(x,y) = y^3 + yx^{2k-3}(x-\alpha)^3, \quad F_0(x,y) = y^3 + yx^{2k}$$

(cf. Section 6, cases (a) and (b)).

8.2. *Uniqueness. The case* $k = 3$. It is well known that for $k = 3$ the decompositions

$$J_{3,0}(\lambda,0) \to E_6, E_8 \quad \text{and} \quad J_{3,0}(\lambda,0) \to E_7, E_7$$

occur only if respectively $j(\lambda) = 0$ or 1 (see [3, 2, 8]). We give a short proof of this fact.

We consider the dimension of the stratum in the base of the (right) miniversal deformation of the germ $J_{3,0}(\lambda,0)$ corresponding to a given decomposition. The dimension of the base is

$$\mu(J_{3,0}) = 16$$

(cf. [1], §8). The codimensions of the strata (E_6, E_8) and (E_7, E_7) are 14. Moreover, the decompositions of $J_{3,0}$ do not depend on the upper diagonal modulus (c_1 in our notation) (see [9]). So the above decompositions may exist only for distinct values of the j-invariant. But if one of them exists for respectively $j(\lambda) \neq 0$ or $j(\lambda) \neq 1$ then the shift technique from Sections 6 and 7 gives us that it exists for any j: a contradiction (cf. the proof of Proposition 1, cases (i) and (ii).

8.3. *Uniqueness. The case* $k > 3$. Let F_α be any deformation of

$$F_0 = y^3 + \lambda y^2 + x^k + x^{3k}$$

such that $F_\alpha^{-1}(0)$ has two singular points of types E_{6k-10} and E_6 or respectively E_{6k-11} and E_7. We shift the singular point E_{6k-10} or respectively E_{6k-11} to the origin and apply the blowing-up transformation

$$\widetilde{F}_\alpha(x,y) = x^{-3(k-3)}F_\alpha(x, x^{k-3}y)\,.$$

Then $\widetilde{F}_\alpha(x,y)$ is a deformation of

$$\widetilde{F}_0(x,y) = y^3 + \lambda y^2x^3 + x^9\,,$$

i.e. of $J_{3,0}$; moreover, the j-invariant is left unaltered. For $\alpha \neq 0$, $\widetilde{F}_\alpha^{-1}(0)$ has two singular points: E_8 or E_7 at the origin and E_6 or respectively E_7

elsewhere. From Subsection 8.2 we know that the j-invariant must be 0 or 1 respectively. Hence $j(\lambda) = 0$ or 1 respectively.

This finishes the proof of Theorem 3.

9. Applications. The Legendrian singularities. The theory of Legendrian maps is closely connected with the theory of hypersurface singularities (see [1], §20). Our results concerning deformations of $J_{k,0}$ hypersurface singularities may be restated in terms of $J_{k,0}$ Legendrian singularities.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singular point at the origin of $J_{k,0}(\lambda, c)$ type.

Let $F(x, q), x \in \mathbb{C}^n, q \in \mathbb{C}^{\mu-1}$, be a (right-) miniversal deformation of germ $f(x)$ without the constant term.

Then F is a generating family of the germ of a Legendrian submanifold

$$L(\lambda, c) = \{(p, q, z) : \exists x\ \partial F/\partial x = 0,\ p = \partial F/\partial q,\ z = F(x, q)\}$$

in the space $\mathbb{C}^{\mu-1} \times \mathbb{C}^{\mu-1} \times \mathbb{C}$ with contact structure $dz - pdq$.

The image of the submanifold L by the projection

$$\pi : (p, q, z) \to (q, z)$$

is called a front. The restriction of the projection π to the Legendrian submanifold L is a Legendrian map. In the notation of [1], §21, the germ of $\pi|_{L(\lambda,c)}$ at the origin has type $J_{k,0}(\lambda, c)$.

Let $L'(\lambda, c)$ be a subset of $L(\lambda, c)$ consisting only of simple points, i.e. of such points (p, q, z) that the hypersurface $F^{-1}(z) \subset \mathbb{C}^n$ has only simple singularities. From Theorems 1 and 4 we deduce that the Legendrian singularities $J_{k,0}(\kappa, 0), j(\kappa) = 1, 0$, are universal.

THEOREM 5. *Let $c \in \mathbb{C}^{k-2}$, $j(\lambda) \neq 0, 1, \infty$. There exists an open ball $B_1 = B(0, r_1) \subset \mathbb{C}^{2\mu-1}$ such that for any open ball $B_2 = B(0, r_2) \subset \mathbb{C}^{2\mu-1}$ and for any point $P_1 \in L'(\lambda, c) \cap B_1$ there exists a point $P_2 \in L'(\kappa, 0) \cap B_2$ with $j(\kappa) = 1$ (respectively $j(\kappa) = 0$) such that the germ of the Legendrian map $\pi|_{L(\lambda,c)}$ at P_1 is equivalent to the germ of $\pi|_{L(\kappa,0)}$ with $j(\kappa) = 1$ (respectively $j(\kappa) = 0$) at P_2.*

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