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Generic properties of generalized hyperbolic partial differential equations

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Abstract. The existence and uniqueness of solutions and convergence of successive approximations are considered as generic properties for generalized hyperbolic partial differential equations with unbounded right-hand sides.

1. Introduction. In this note we consider the Darboux problem

$$(1)\qquad \begin{cases} u_{xy} = f(x,y,u,u_x,u_y,u_{xy})\,, \\ u(0,y) = \psi(y)\,, \quad u(x,0) = \phi(x)\,, \end{cases}$$

where $f : [0,a] \times [0,b] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, $\phi : [0,a] \to \mathbb{R}$ and $\psi : [0,b] \to \mathbb{R}$ are continuously differentiable on $[0,a]$ and $[0,b]$ respectively and satisfy $\phi(0) = \psi(0)$. The family of all pairs of functions (ϕ,ψ) satisfying the above conditions is denoted by $\mathcal{Z}$.

By a solution of the problem (1) we mean a continuous function $u : [0,a] \times [0,b] \to \mathbb{R}$ with continuous derivatives u_x, u_y, $u_{xy} = u_{yx}$ which satisfies (1) for $x \in [0,a]$ and $y \in [0,b]$.

The problem (1) was investigated by Goebel [6]. Using Darbo's fixed point theorem for α-contractions [4] and Bielecki's norms [2] he showed that under the additional assumptions that f is bounded and satisfies the Lipschitz condition

$$|f(x,y,u,p,q,s) - f(x,y,u,\overline{p},\overline{q},\overline{s})| \le M|p-\overline{p}| + N|q-\overline{q}| + k|s-\overline{s}|\,,$$

where $k < 1$, the problem (1) has at least one solution.

Much attention was paid to the quasilinear hyperbolic equation

$$(2)\qquad \begin{cases} u_{xy} = f(x,y,u,u_x,u_y)\,, \\ u(0,y) = \psi(y)\,, \quad u(x,0) = \phi(x)\,, \end{cases} \qquad x \in [0,a]\,,\ y \in [0,b]\,,$$

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where $(\phi,\psi)\in\mathcal{Z}$ (see [2] and [7]). It is well known that the continuity of f is not sufficient to guarantee the existence of solution of the problem (2). However, Costello [3] proved that the existence, uniqueness, and continuous dependence of the solution for the problem (2) is a generic property in the space $\mathcal{H}$ of all continuous bounded $f:[0,a]\times[0,b]\times\mathbb{R}^3\to\mathbb{R}$ endowed with the norm of uniform convergence. This result is a generalization of an earlier paper of Alexiewicz and Orlicz [1]. Costello's method of proof is similar to that of Lasota and Yorke [8]. His result was strengthened by De Blasi and Myjak [5] who proved that the convergence of successive approximations is a generic property in $\mathcal{H}$. In our approach, applying Bielecki's norms, we are able to study generic properties of the problem (1) in two cases: with unbounded right-hand sides under the metric of uniform convergence on bounded sets or with bounded right-hand sides under the norm of uniform convergence.

2. Results. The space of all continuous $v:[0,a]\times[0,b]\to\mathbb{R}$ is denoted by $\mathcal{C}$. The family of Bielecki's norms in $\mathcal{C}$ for $\lambda\ge 0$ is defined by

$$\|v\|_\lambda=\sup\{\exp(-\lambda(x+y))|v(x,y)|:x\in[0,a],\ y\in[0,b]\}\,.$$

Note that all these norms are equivalent.

We associate with every continuous $f:[0,a]\times[0,b]\times\mathbb{R}^4\to\mathbb{R}$ and $(\phi,\psi)\in\mathcal{Z}$ the continuous mapping $\widehat{f}^{\,1}_{\phi,\psi}:\mathcal{C}\to\mathcal{C}$ and its successive approximations given as follows:

$$\begin{aligned}\widehat{f}^{\,1}_{\phi,\psi}(v)(x,y)=f\Big(x,y,\ &\int_0^x\int_0^y v(\xi,\eta)\,d\xi\,d\eta+\phi(x)+\psi(y)-\phi(0)\,,\\ &\int_0^y v(x,\eta)\,d\eta+\phi'(x),\ \int_0^x v(\xi,y)\,d\xi+\psi'(y),v(x,y)\Big)\,,\\ &\widehat{f}^{\,i+1}_{\phi,\psi}(v)=\widehat{f}^{\,1}_{\phi,\psi}(\widehat{f}^{\,i}_{\phi,\psi}(v))\,,\qquad v\in\mathcal{C}\,.\end{aligned}$$

Putting $v(x,y)=u_{xy}(x,y)$ we can now write the problem (1) in an equivalent form

$$\widehat{f}^{\,1}_{\phi,\psi}(v)=v\,,\qquad v\in\mathcal{C}\,. \tag{3}$$

By $(\mathcal{F},d)$ we denote the complete metric space of all continuous $f:[0,a]\times[0,b]\times\mathbb{R}^4\to\mathbb{R}$ satisfying the condition

$$|f(x,y,u,p,q,s)-f(x,y,u,p,q,\overline{s})|\le|s-\overline{s}|\,,$$

for $x\in[0,a]$, $y\in[0,b]$, $u,p,q,s,\overline{s}\in\mathbb{R}$, endowed with the metric of uniform convergence on bounded sets given by

$$d(f,g)=\sum_{n=1}^{\infty}2^{-n}\sup\left\{\frac{|f(x,y,u,p,q,s)-g(x,y,u,p,q,s)|}{1+|f(x,y,u,p,q,s)-g(x,y,u,p,q,s)|}:\right.$$
$$\left. x\in[0,a],\ y\in[0,b],\ |u|,|p|,|q|,|s|\le n\right\}.$$

Recall that a subset of a metric space is said to be residual if its complement is of the first category.

THEOREM 1. *Let* $\mathcal{F}_0$ *be the subset of all* $f\in\mathcal{F}$ *such that* (i) *for every* $(\phi,\psi)\in\mathcal{Z}$ *the problem* (1) (*or* (3)) *has exactly one solution*; (ii) *for every* $v\in\mathcal{C}$ *and* $(\phi,\psi)\in\mathcal{Z}$ *the sequence* $\widehat{f}^{\,i}_{\phi,\psi}(v)$ *is convergent in* $\mathcal{C}$ *as* $i\to\infty$. *Then* $\mathcal{F}_0$ *is a residual set in the space* $(\mathcal{F},d)$.

P r o o f. Let $\mathcal{G}$ be the subset of all $g\in\mathcal{F}$ satisfying the following Lipschitz condition:

$$|g(x,y,u,p,q,s)-g(x,y,\overline{u},\overline{p},\overline{q},\overline{s})|\le L|u-\overline{u}|+M|p-\overline{p}|+N|q-\overline{q}|+k|s-\overline{s}|\,, \tag{4}$$

for $x\in[0,a]$, $y\in[0,b]$, $u,p,q,s,\overline{u},\overline{p},\overline{q},\overline{s}\in\mathbb{R}$, where L, M, N, k are some constants and $k<1$.

We will see that for $g\in\mathcal{G}$ there exist constants $K(g)<1$ and $\lambda(g)\ge 0$ such that

$$\|\widehat{g}^{\,1}_{\phi,\psi}(v)-\widehat{g}^{\,1}_{\phi,\psi}(w)\|_{\lambda(g)}\le K(g)\|v-w\|_{\lambda(g)}\,,\qquad v,w\in\mathcal{C}\,,\ (\phi,\psi)\in\mathcal{Z}\,. \tag{5}$$

In fact, from (4) we get

$$\begin{aligned}
&\exp(-\lambda(x+y))|\widehat{g}^{\,1}_{\phi,\psi}(v)(x,y)-\widehat{g}^{\,1}_{\phi,\psi}(w)(x,y)|\\
&\le \exp(-\lambda(x+y))\Big(L\int_0^x\int_0^y|v(\xi,\eta)-w(\xi,\eta)|\,d\xi\,d\eta\\
&\quad+M\int_0^y|v(x,\eta)-w(x,\eta)|\,d\eta\\
&\quad+N\int_0^x|v(\xi,y)-w(\xi,y)|\,d\xi+k|v(x,y)-w(x,y)|\Big)\\
&\le \exp(-\lambda(x+y))\Big(L\int_0^x\int_0^y\exp(\lambda(\xi+\eta))\,d\xi\,d\eta+M\int_0^y\exp(\lambda(x+\eta))\,d\eta\\
&\quad+N\int_0^x\exp(\lambda(\xi+y))\,d\xi+k\exp(\lambda(x+y))\Big)\|v-w\|_\lambda\\
&\le(2L\lambda^{-2}+(M+N)\lambda^{-1}+k)\|v-w\|_\lambda\,,
\end{aligned}$$

so it suffices to set $\lambda(g)=(1-k)^{-1}(2L+M+N)+1$ and $K(g)=\lambda(g)^{-1}(2L+M+N)+k$.

Observe that $\mathcal{G}$ is dense in $\mathcal{F}$. To show this suppose that $f\in\mathcal{F}$. For $n=1,2,\ldots$ let continuously differentiable functions $\gamma_n:\mathbb{R}^4\to[0,\infty)$ be such that $\int_{\mathbb{R}^4}\gamma_n\,d\mu=1$ and $\sup\{|z|:\gamma_n(z)>0\}\to 0$ as $n\to\infty$. For $n=1,2,\ldots$ consider the nonexpansive retractions $r_n:\mathbb{R}\to[-n,n]$ given by

$$r_n(u)=\begin{cases} u & \text{if } |u|\le n,\\ nu|u|^{-1} & \text{if } |u|>n.\end{cases}$$

Define $g_n:[0,a]\times[0,b]\times\mathbb{R}^4\to\mathbb{R}$ by

$$g_n(x,y,u,p,q,s)$$
$$=\frac{n}{n+1}\int_{\mathbb{R}^4} f(x,y,r_n(\overline{u}),r_n(\overline{p}),r_n(\overline{q}),r_n(\overline{s}))\gamma_n(u-\overline{u},p-\overline{p},q-\overline{q},s-\overline{s})\,d\mu\,.$$

One checks that $d(f,g_n)\to 0$ as $n\to\infty$ and g_n satisfies (4) with some constants L, M, N, and $k=n/(n+1)$.

We denote by $\mathcal{Z}_n$ $(n=1,2,\ldots)$ the subset of all pairs $(\phi,\psi)\in\mathcal{Z}$ such that $|\phi(x)|,|\phi'(x)|\le n$ for $x\in[0,a]$ and $|\psi(y)|,|\psi'(y)|\le n$ for $y\in[0,b]$. Note that $\bigcup_{n=1}^{\infty}\mathcal{Z}_n=\mathcal{Z}$. Put

$$\overline{B}_\lambda(0,R)=\{v\in\mathcal{C}:\|v\|_\lambda\le R\}\,.$$

For every $g\in\mathcal{G}$ and $n=1,2,\ldots$ define

$$R(g,n)=(1-K(g))^{-1}(nK(g)+G_n)\,,\tag{6}$$

where

$$G_n=\sup\{|g(x,y,u,p,q,s)|:\\ x\in[0,a],\ y\in[0,b],\ |u|\le 3n\,,\ |p|,|q|\le n\,,\ s=0\}\,.$$

We claim that for $(\phi,\psi)\in\mathcal{Z}_n$,

$$\widehat{g}^{\,1}_{\phi,\psi}(\overline{B}_{\lambda(g)}(0,R(g,n)+n))\subset\overline{B}_{\lambda(g)}(0,R(g,n))\,.\tag{7}$$

In fact, if $\|v\|_{\lambda(g)}\le R(g,n)+n$ and $(\phi,\psi)\in\mathcal{Z}_n$, then from (5) and (6) we obtain

$$\begin{aligned}\|\widehat{g}^{\,1}_{\phi,\psi}(v)\|_{\lambda(g)}&\le\|\widehat{g}^{\,1}_{\phi,\psi}(v)-\widehat{g}^{\,1}_{\phi,\psi}(0)\|_{\lambda(g)}+\|\widehat{g}^{\,1}_{\phi,\psi}(0)\|_{\lambda(g)}\\&\le K(g)\|v\|_{\lambda(g)}+\|\widehat{g}^{\,1}_{\phi,\psi}(0)\|_0\le K(g)(R(g,n)+n)+G_n\\&=R(g,n)\,.\end{aligned}$$

Since d is the metric of uniform convergence on bounded sets, it is possible to find for every $g\in\mathcal{G}$ and $n=1,2,\ldots$ a positive number $\varepsilon(g,n)$ such that for $f\in\mathcal{F}$, $d(f,g)<\varepsilon(g,n)$ and for $x\in[0,a]$, $y\in[0,b]$,

$$|u|\le ab\exp(\lambda(g)(a+b))(R(g,n)+n)+3n\,,$$

$$|p| \le b\exp(\lambda(g)(a+b))(R(g,n)+n)+n\,,$$
$$|q| \le a\exp(\lambda(g)(a+b))(R(g,n)+n)+n\,,$$
$$|s| \le \exp(\lambda(g)(a+b))(R(g,n)+n)\,,$$

we have

$$|f(x,y,u,p,q,s)-g(x,y,u,p,q,s)| \le (3n)^{-1}(1-K(g))\exp(-\lambda(g)(a+b))\,.$$

This means that for $g\in\mathcal{G}$, $d(f,g)<\varepsilon(g,n)$ we have

$$\begin{aligned} &v\in \overline{B}_{\lambda(g)}(0,R(g,n)+n)\,,\ (\phi,\psi)\in\mathcal{Z}_n \\ &\quad \Rightarrow \|\widehat{f}^{\,1}_{\phi,\psi}(v)-\widehat{g}^{\,1}_{\phi,\psi}(v)\|_0 \le (3n)^{-1}(1-K(g))\exp(-\lambda(g)(a+b))\,. \end{aligned} \tag{8}$$

Now we show that

$$\bigcap_{n=1}^{\infty}\bigcup_{g\in\mathcal{G}} B(g,\varepsilon(g,n))\subset\mathcal{F}_0\,, \tag{9}$$

where $B(g,\varepsilon(g,n))$ denotes the open ball in $\mathcal{F}$ with center g and radius $\varepsilon(g,n)$.

Suppose that f belongs to the left side of the inclusion (9). This implies that for every $n=1,2,\ldots$ there exists $g_n\in\mathcal{G}$ satisfying $d(g_n,f)<\varepsilon(g_n,n)$.

First, we prove that the equation (3) for $(\phi,\psi)\in\mathcal{Z}$ has at most one solution in $\mathcal{C}$. To this end it suffices to check that $I-\widehat{f}^{\,1}_{\phi,\psi}$ is injective (I denotes the identity map of $\mathcal{C}$). For $v,w\in\mathcal{C}$ such that $\|v-w\|_0\ge 1/n$, $\|v\|_0,\|w\|_0\le n$ and $(\phi,\psi)\in\mathcal{Z}_n$ from (5) and (8) we obtain

$$\begin{aligned} &\|(I-\widehat{f}^{\,1}_{\phi,\psi})(v)-(I-\widehat{f}^{\,1}_{\phi,\psi})(w)\|_{\lambda(g_n)} \\ &\quad\ge \|v-w\|_{\lambda(g_n)}-\|\widehat{g}^{\,1}_{n,\phi,\psi}(v)-\widehat{g}^{\,1}_{n,\phi,\psi}(w)\|_{\lambda(g_n)} \\ &\qquad -\|\widehat{g}^{\,1}_{n,\phi,\psi}(v)-\widehat{f}^{\,1}_{\phi,\psi}(v)\|_{\lambda(g_n)}-\|\widehat{g}^{\,1}_{n,\phi,\psi}(w)-\widehat{f}^{\,1}_{\phi,\psi}(w)\|_{\lambda(g_n)} \\ &\quad\ge (1-K(g_n))\|v-w\|_{\lambda(g_n)}-2(3n)^{-1}(1-K(g_n))\exp(-\lambda(g_n)(a+b)) \\ &\quad\ge (3n)^{-1}(1-K(g_n))\exp(-\lambda(g_n)(a+b))>0\,. \end{aligned}$$

Since n can be chosen arbitrarily large, for $\|v-w\|_0>0$ and $(\phi,\psi)\in\mathcal{Z}$ we get

$$(I-\widehat{f}^{\,1}_{\phi,\psi})(v)\neq(I-\widehat{f}^{\,1}_{\phi,\psi})(w)\,,$$

so $I-\widehat{f}^{\,1}_{\phi,\psi}$ is injective.

Now we prove that the sequence of successive approximations $\widehat{f}^{\,i}_{\phi,\psi}(v)$ is convergent in $\mathcal{C}$ as $i\to\infty$ for all $v\in\mathcal{C}$ and $(\phi,\psi)\in\mathcal{Z}$. To this end fix ϕ, ψ and v and consider any positive integer n satisfying $(\phi,\psi)\in\mathcal{Z}_n$ and $\|v\|_0\le n$. Observe that

$$\widehat{f}^{\,1}_{\phi,\psi}(\overline{B}_{\lambda(g_n)}(0,R(g_n,n)+n)\subset\overline{B}_{\lambda(g_n)}(0,R(g_n,n)+n)\,. \tag{10}$$

In fact, for $\|w\|_{\lambda(g_n)} \le R(g_n,n)+n$ from (7) and (8) we obtain

$$\begin{aligned}\|\widehat{f}^1_{\phi,\psi}(w)\|_{\lambda(g_n)} &\le \|\widehat{f}^1_{\phi,\psi}(w) - \widehat{g}^1_{n,\phi,\psi}(w)\|_{\lambda(g_n)} + \|\widehat{g}^1_{n,\phi,\psi}(w)\|_{\lambda(g_n)} \\ &\le (3n)^{-1}(1-K(g_n))\exp(-\lambda(g_n)(a+b)) + R(g_n,n) < R(g_n,n)+n\,.\end{aligned}$$

Since $v \in \overline{B}_0(0,n) \subset \overline{B}_{\lambda(g_n)}(0,R(g_n,n)+n)$, by (10) we see that for every $i=1,2,\ldots$,

$$\widehat{f}^i_{\phi,\psi}(v) \in \overline{B}_{\lambda(g_n)}(0,R(g_n,n)+n)\,. \tag{11}$$

Now we show by induction that for every positive integer i we have

$$\|\widehat{f}^i_{\phi,\psi}(v) - \widehat{g}^i_{n,\phi,\psi}(v)\|_{\lambda(g_n)} \le (3n)^{-1}(1-K(g_n)^i)\exp(-\lambda(g_n)(a+b))\,. \tag{12}$$

For $i=1$, (12) follows from (8). If (12) is true for a fixed i, then by (5), (8) and (11) we have

$$\begin{aligned}&\|\widehat{f}^{i+1}_{\phi,\psi}(v) - \widehat{g}^{i+1}_{n,\phi,\psi}(v)\|_{\lambda(g_n)} \\ &\quad\le \|\widehat{f}^1_{\phi,\psi}\widehat{f}^i_{\phi,\psi}(v) - \widehat{g}^1_{n,\phi,\psi}\widehat{f}^i_{\phi,\psi}(v)\|_{\lambda(g_n)} \\ &\qquad + \|\widehat{g}^1_{n,\phi,\psi}\widehat{f}^i_{\phi,\psi}(v) - \widehat{g}^1_{n,\phi,\psi}\widehat{g}^i_{n,\phi,\psi}(v)\|_{\lambda(g_n)} \\ &\quad\le ((3n)^{-1}(1-K(g_n)) + (3n)^{-1}K(g_n)(1-K(g_n)^i))\exp(-\lambda(g_n)(a+b)) \\ &\quad= (3n)^{-1}(1-K(g_n)^{i+1})\exp(-\lambda(g_n)(a+b))\,,\end{aligned}$$

so (12) is true for $i+1$.

From (5) and (12) it follows that

$$\begin{aligned}&\|\widehat{f}^i_{\phi,\psi}(v) - \widehat{f}^{i+j}_{\phi,\psi}(v)\|_0 \\ &\quad\le \exp(\lambda(g_n)(a+b))\|\widehat{f}^i_{\phi,\psi}(v) - \widehat{f}^{i+j}_{\phi,\psi}(v)\|_{\lambda(g_n)} \\ &\quad\le \exp(\lambda(g_n)(a+b))(\|\widehat{f}^i_{\phi,\psi}(v) - \widehat{g}^i_{n,\phi,\psi}(v)\|_{\lambda(g_n)} \\ &\qquad + \|\widehat{f}^{i+j}_{\phi,\psi}(v) - \widehat{g}^{i+j}_{n,\phi,\psi}(v)\|_{\lambda(g_n)} + \|\widehat{g}^i_{n,\phi,\psi}(v) - \widehat{g}^{i+j}_{n,\phi,\psi}(v)\|_{\lambda(g_n)}) \\ &\quad\le 2(3n)^{-1} + K(g_n)^i(1-K(g_n))^{-1}\|v - \widehat{g}^1_{n,\phi,\psi}(v)\|_{\lambda(g_n)}\,.\end{aligned}$$

Therefore, for sufficiently large i and every $j \in \mathbb{N}$ we have

$$\|\widehat{f}^i_{\phi,\psi}(v) - \widehat{f}^{i+j}_{\phi,\psi}(v)\|_0 < 1/n\,.$$

Since one can choose n arbitrarily large, the sequence $\{\widehat{f}^i_{\phi,\psi}(v)\}$ is Cauchy in the Banach space $\mathcal{C}$, so it is convergent. As $\widehat{f}^1_{\phi,\psi} : \mathcal{C} \to \mathcal{C}$ is continuous it has at least one fixed point $\lim_{i\to\infty}\widehat{f}^i_{\phi,\psi}(v)$.

From (9) it follows that $\mathcal{F}_0$ contains a dense G_δ subset of $\mathcal{F}$, so $\mathcal{F}_0$ is a residual subset of $\mathcal{F}$. This completes the proof.

Remark. Theorem 1 is not true when the metric d is replaced by the metric $\widetilde{d}$ of uniform convergence given as follows:

$$\widetilde{d}(f,g)=\sup\left\{\frac{|f(x,y,u,p,q,s)-g(x,y,u,p,q,s)|}{1+|f(x,y,u,p,q,s)-g(x,y,u,p,q,s)|}:\ x\in[0,a],\ y\in[0,b],\ u,p,q,s\in\mathbb{R}\right\}.$$

In fact, consider the open ball $B(f_0,1/2)$ in $(\mathcal{F},\widetilde{d})$, where $f_0(x,y,u,p,q,s)=s+1$. One can check that for every $f\in B(f_0,1/2)$ there is no solution of the equation (3).

Now consider the Banach space $(\widetilde{\mathcal{F}},\|\cdot\|_\infty)$ of all bounded $f\in\mathcal{F}$ with the norm of uniform convergence

$$\|f\|_\infty=\sup\{|f(x,y,u,p,q,s)|:x\in[0,a],\ y\in[0,b],\ u,p,q,s\in\mathbb{R}\}.$$

THEOREM 2. *Let $\widetilde{\mathcal{F}}_0$ be the subset of all $f\in\widetilde{\mathcal{F}}$ such that* (i) *for every $(\phi,\psi)\in\mathcal{Z}$ the problem* (1) (*or* (3)) *has exactly one solution*; (ii) *for every $v\in\mathcal{C}$ and $(\phi,\psi)\in\mathcal{Z}$ the sequence $\widehat{f}^{\,i}_{\phi,\psi}(v)$ is convergent in $\mathcal{C}$ as $i\to\infty$. Then $\widetilde{\mathcal{F}}_0$ is a residual set in the Banach space $(\widetilde{\mathcal{F}},\|\cdot\|_\infty)$.*

Proof. Let $\widetilde{\mathcal{G}}$ be the subset of all $g\in\widetilde{\mathcal{F}}$ satisfying, on every bounded subset of $[0,a]\times[0,b]\times\mathbb{R}^4$, the Lipschitz condition (4) with some constants $k<1$, L, M, N. We show that $\widetilde{\mathcal{G}}$ is dense in $\widetilde{\mathcal{F}}$. Suppose that $f\in\widetilde{\mathcal{F}}$ and $\varepsilon>0$. As we have seen in the proof of Theorem 1, for every positive integer n there exists $f_n\in\mathcal{G}$ such that

$$\sup\{|f_n(x,y,u,p,q,s)-f(x,y,u,p,q,s)|:\ x\in[0,a],\ y\in[0,b],\ |u|,|p|,|q|,|s|\le n\}<2^{-n}\varepsilon.$$

Let us find continuously differentiable $\beta_m:\mathbb{R}^3\to[0,\infty)$ such that $\mathrm{supp}(\beta_1)\subset B(0,1)$, $\mathrm{supp}(\beta_m)\subset B(0,m)-\overline{B}(0,m-2)$ for $m\ge 2$ and $\sum_{m=1}^{\infty}\beta_m(z)=1$ for $z\in\mathbb{R}^3$. Define $h_n:[0,a]\times[0,b]\times\mathbb{R}^4\to\mathbb{R}$ $(n=1,2,\ldots)$ by

$$h_n(x,y,u,p,q,s)=\sum_{m=1}^{\infty}\beta_m(u,p,q)f_{n+m}(x,y,u,p,q,s).$$

Note that for $n=1,2,\ldots,$

$$\sup\{|h_n(x,y,u,p,q,s)-f(x,y,u,p,q,s)|:\ x\in[0,a],\ y\in[0,b],\ u,p,q\in\mathbb{R},\ |s|\le n\}\le 2^{-n-1}\varepsilon.$$

A function $g\in\widetilde{\mathcal{G}}$ such that $\|f-g\|_\infty<\varepsilon$ may be given by

$$g(x,y,u,p,q,s)=h_{[|s|]+1}(x,y,u,p,q,s)+\sum_{i=1}^{[|s|]}(h_i(x,y,u,p,q,i\operatorname{sgn}(s)) - h_{i+1}(x,y,u,p,q,i\operatorname{sgn}(s)))$$

(we assume that the sum $\sum_{i=1}^{[|s|]}\ldots$ vanishes for $s\in(-1,1)$).

For every $g\in\widetilde{\mathcal{G}}$ define

$$R(g)=\sup\{|g(x,y,u,p,q,s)|: x\in[0,a],\ y\in[0,b],\ u,p,q,s\in\mathbb{R}\}.$$

Notice that for $(\phi,\psi)\in\mathcal{Z}$ we have

$$\widehat{g}^1_{\phi,\psi}(\widetilde{\mathcal{F}})\subset\overline{B}_0(0,R(g)).$$

Let $k<1$, L, M, N be such that (4) is satisfied for every $x\in[0,a]$, $y\in[0,b]$, and $|u|,|\overline{u}|\le ab(R(g)+n)+3n$, $|p|,|\overline{p}|\le b(R(g)+n)+n$, $|q|,|\overline{q}|\le a(R(g)+n)+n$, $|s|,|\overline{s}|\le R(g)+n$. This means that there exist constants $\widetilde{K}(g)<1$ and $\widetilde{\lambda}(g)\ge 0$ such that for $(\phi,\psi)\in\mathcal{Z}_n$ and $v,w\in\overline{B}_0(0,R(g)+n)$ we have

$$\|\widehat{g}^1_{\phi,\psi}(v)-\widehat{g}^1_{\phi,\psi}(w)\|_{\widetilde{\lambda}(g)}\le\widetilde{K}(g)\|v-w\|_{\widetilde{\lambda}(g)}.$$

We set

$$\widetilde{\varepsilon}(g,n)=(3n)^{-1}(1-\widetilde{K}(g))\exp(-\widetilde{\lambda}(g)(a+b)).$$

Now as in the proof of Theorem 1 one can show that

$$\bigcap_{n=1}^{\infty}\bigcup_{g\in\widetilde{\mathcal{G}}}B(g,\widetilde{\varepsilon}(g,n))\subset\widetilde{\mathcal{F}}_0,$$

where $B(g,\widetilde{\varepsilon}(g,n))=\{f\in\widetilde{\mathcal{F}}:\|f\|_\infty\le\widetilde{\varepsilon}(g,n)\}$. This completes the proof.

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