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A note on generic chaos

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Abstract. We consider dynamical systems on a separable metric space containing at least two points. It is proved that weak topological mixing implies generic chaos, but the converse is false. As an application, some results of Piórek are simply reproved.

1. Definitions and results. By a semigroup of times we mean one of the following semigroups: $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{R}^+$ (the nonnegative reals) and $\mathbb{R}$ (the reals). Let X be a metric space and let G be a semigroup of times. By a *dynamical system* on X relative to G we mean a continuous map $S : X \times G \to X$ satisfying:

(1) $S(x, 0) = x$, $\forall x \in X$,
(2) $S(S(x, t), s) = S(x, t + s)$, $\forall x \in X$, $t, s \in G$.

We call $S|_{Y\times G}$ a *subsystem* of S if $Y \subset X$ and $S|_{Y\times G} : Y \times G \to Y$ is a dynamical system.

Let $S : X \times G \to X$ be a dynamical system and let $x \in X$. The set

$$O^+(x) = \{S_x(t) = S(x, t) : t \geq 0,\ t \in G\}$$

is said to be the *positive semiorbit* of x under S. The positive semiorbit $O^+(x)$ is *periodic* if there is a $T > 0$ such that $S_x(T) = S_x(0) = x$. It is easy to see that for each $t \in G$, S induces a continuous map $S^t : X \to X$ by $S^t(x) = S(x, t)$ (for simplicity, sometimes $S^t(x)$ is written $S^t x$). Conversely, if $f : X \to X$ is a continuous map then it induces naturally a dynamical system f_* on X relative to the semigroup $\mathbb{Z}^+$ in the following sense:

(1) $f_*(x, 0) = x$, $\forall x \in X$,
(2) $f_*(x, 1) = f(x)$ and inductively $f_*(x, n + 1) = f \circ f_*(x, n)$ for any $n \in \mathbb{Z}^+$.

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In addition, if we define $S \times S : X \times X \times G \to X \times X$ by

$$S \times S(x, y, t) = (S(x, t), S(y, t)), \quad \forall (x, y) \in X \times X,\ t \in G,$$

then $S \times S$ is a dynamical system on $X \times X$ relative to G.

Let S be a dynamical system on X relative to G, where X is a metric space with metric d and G a semigroup of times. A point $(x, y) \in X \times X$ is said to be *chaotic* with respect to S if

$$\liminf_{t \to +\infty} d(S^t x, S^t y) = 0 \quad \text{and} \quad \limsup_{t \to +\infty} d(S^t x, S^t y) > 0.$$

For a continuous map f of an interval I, Li and Yorke [4] proved that if f has a periodic point of period 3 then there is an uncountable set E such that all $x, y \in E$ with $x \neq y$ form a chaotic point of $I \times I$ with respect to the dynamical system f_* induced by f. This result was improved by many authors (e.g., see [1]–[3] and [5]–[6]).

Piórek [8]–[9] developed these ideas and made the following

DEFINITION. Let S be a dynamical system on X relative to G. S is said to be *generically chaotic* if there is a residual set E in $X \times X$ (i.e., E contains a countable intersection of everywhere dense sets) such that each point in E is chaotic with respect to S.

In the present paper, we shall discuss the relations between generic chaos and mixing in the topological sense.

Let S be a dynamical system on X relative to G. S is said to be *transitive* if for any pair of nonempty open sets U and V in X, there is a $T > 0$ such that

$$S^T(U) \cap V \neq \emptyset.$$

S is said to be *weakly topologically mixing* if $S \times S : X \times X \times G \to X \times X$ is transitive.

S is said to be *topologically mixing* if for any pair of nonempty open sets U and V in X, there is a $T > 0$ such that $S^t(U) \cap V \neq \emptyset$ for all $t \geq T$.

It is clear that a topologically mixing system is weakly topologically mixing, and a weakly topologically mixing system is transitive.

We shall prove the following

THEOREM. *Let S be a dynamical system on X relative to G, where X is a separable metric space containing at least two points and G is a semigroup of times. If S is weakly topologically mixing, then it is generically chaotic.*

From the Theorem, we get immediately

COROLLARY. *Let S be a dynamical system on X relative to G, where X is a separable metric space containing at least two points and G is a semigroup of times. If S is topologically mixing, then it is generically chaotic.*

2. Proof of Theorem

LEMMA. *Let S be a dynamical system on X relative to G, where X is a metric space and G a semigroup of times. Let $S|_{Y\times G}$ be a transitive subsystem of S and let $x \in X$. If $O^+(x)$ is dense in Y, then for each open set U in Y, the set*

$$\{t : S_x(t) \in U,\ t \in G\}$$

is not bounded above.

Proof. For $z \in X$, $T > 0$, we write

$$A_z(T) = \{S_z(t) : 0 \le t \le T,\ t \in G\}\,.$$

Suppose U is an open set in Y and suppose $T_0 > 0$ is given. We must show that there is a $t > T_0$ such that $S_x(t) \in U$.

For this we first assume $Y \not\subset A_x(T_0)$. Let

$$V = Y - A_x(T_0)\,.$$

Thus $V \neq \emptyset$. Since $A_x(T_0)$ is closed in X, V is open in Y. So by transitivity of $S|_{Y\times G}$ there exist $v \in V$ and $t_v > 0$ such that $S(v, t_v) \in U$. If $v \in O^+(x)$, then there exists $t_0 > T_0$ such that $v = S(x, t_0)$. Let $t = t_0 + t_v$. Then $t > T_0$ and we have

$$S_x(t) = S(x, t_0 + t_v) = S(S(x, t_0), t_v) = S(v, t_v) \in U\,.$$

If $v \notin O^+(x)$, then $S_x(t_i) \to v$ for some $t_i \to +\infty$, which implies that

$$S^{t_v}(S_x(t_i)) \to S^{t_v}(v)\,,$$

i.e., for $S(v, t_v) \in U$ and $t_v + t_i \to +\infty$,

$$S_x(t_v + t_i) \to S(v, t_v)\,.$$

Hence we also have $S_x(t) \in U$ for some $t > T_0$.

Assume now $Y \subset A_x(T_0)$. Let $y \in U$. Then $y \in O^+(x)$, i.e., there is a $t_y > 0$ such that $y = S_x(t_y)$. Since S maps $Y \times G$ into Y, $S_y(t) \in Y$ for each $t > T_0$. It follows from $Y \subset A_x(T_0)$ that there exists a periodic positive semiorbit P such that $S_y(t) \in P$ for some $t \ge 0$. We write

$$\overline{t} = \min\{t \ge 0 : S_y(t) \in P,\ t \in G\}$$

and define

$$B_y(\overline{t}/2) = \{S_y(t) : t \ge \overline{t}/2,\ t \in G\}\,.$$

Let

$$V_1 = Y - A_x(\overline{t}/2 + t_y)\,, \qquad V_2 = Y - B_y(\overline{t}/2)\,.$$

If $\bar{t} > 0$, then V_1, V_2 are both nonempty open sets of Y. For any $q \in V_1$, there exists $t_q > \bar{t}/2$ such that $q = S_y(t_q)$. Then for any $t > 0$,

$$S^t(q) = S(q,t) = S(S(y,t_q),t) = S(y,t_q+t) = S_y(t_q+t) \in B_y(\bar{t}/2)\,,$$

since $t_q + t > \bar{t}/2$. By the definition of V_2, we have $S^t(q) \notin V_2$. So for each $t > 0$, $S^t(V_1) \cap V_2 = \emptyset$, which contradicts the transitivity of $S|_{Y\times G}$. Thus the only possibility is $\bar{t} = 0$; that is to say, $S_y(0) = y \in P$. And clearly there is a $t > T_0$ such that $S_x(t) = y \in U$.

The proof of the Lemma is complete.

Proof of Theorem. Put

$$E = \{(x,y) \in X \times X : \{(S^t x, S^t y) : t \ge 0,\ t \in G\} \text{ is dense}\}\,.$$

Since X being a separable metric space implies the same for $X \times X$, we know that $X \times X$ has a countable base $\{U_1, U_2, \ldots\}$. It is easy to check that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{t\ge 0} (S^{-t} \times S^{-t})(U_n)\,.$$

For each $n > 0$, $\bigcup_{t\ge 0}(S^{-t} \times S^{-t})(U_n)$ is clearly open in $X \times X$ and by the transitivity of $S \times S$, it is dense in $X \times X$. So E is a residual set in $X \times X$.

Select $x_0, y_0 \in X$ with $x_0 \ne y_0$. Let $(x,y) \in E$. By the Lemma, there are increasing sequences $t_i \to +\infty$ and $t_j \to +\infty$ in G such that

$$\lim_{i\to+\infty} (S^{t_i}x, S^{t_i}y) = (x_0,x_0) \quad\text{and}\quad \lim_{j\to+\infty} (S^{t_j}x, S^{t_j}y) = (x_0,y_0)\,.$$

Clearly, (x,y) is chaotic with respect to S, and hence the proof is complete.

3. Examples. In this section, we give three examples. The first two examples are applications of our results. The last one shows that the converse of the Theorem is false.

(1) Let k be an integer greater than one. We denote by Y_k the set of symbols $\{1,2,\ldots,k\}$ with the discrete topology, and by Σ_k the product space $(Y_k)^{\mathbb{Z}}$ equipped with the product topology. The shift $\sigma : \Sigma_k \to \Sigma_k$ is defined by $(\sigma(x))_i = x_{i+1}$, where $x = (\ldots x_{-2}x_{-1}x_0x_1x_2\ldots)$. The product topology on Σ_k is induced by the metric

$$d(x,y) = \sum_{n=-\infty}^{+\infty} 2^{-(2|n|+1)}\delta_n(x,y)\,,$$

where $\delta_n(x,y)$ is 0 if $x_n = y_n$, and 1 otherwise. Since Σ_k is compact, it follows that (Σ_k, d) is a separable metric space. As is well known, $\sigma : \Sigma_k \to \Sigma_k$ is continuous and the dynamical system σ_* induced by σ is topologically mixing (see [7]). Therefore σ_* is generically chaotic by the Corollary.

(2) We use $\mathbb{R}$ to denote the space of all reals, $\mathbb{R}^m$ to denote the m-dimensional Euclidean space. Let $M = C^r(\mathbb{R},\mathbb{R}^m)$; for $r > 0$, this is the set of all r times continuously differentiable maps from $\mathbb{R}$ into $\mathbb{R}^m$, and for $r = 0$, the set of all continuous maps from $\mathbb{R}$ into $\mathbb{R}^m$.

For $f, g \in M$, we put

$$\varrho_n^r(f,g) = \sum_{j=1}^{m}\sum_{i=0}^{r} \max_{x\in[-n,n]} |f_j^{(i)}(x) - g_j^{(i)}(x)|\,,$$

$$\varrho^r(f,g) = \sum_{n=1}^{+\infty} 2^{-n} h(\varrho_n^r(f,g))\,,$$

where $f = (f_1,\ldots,f_m)$, $g = (g_1,\ldots,g_m)$ and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = t/(t+1)$. Then (M,ϱ^r) is a separable metric space and the convergence in ϱ^r is the uniform convergence with derivatives up to order r on compact subsets of $\mathbb{R}$.

For $f \in M$ and $t \in \mathbb{R}$, let $S(f,t)x = f(x-t)$ for $x \in \mathbb{R}$. It is not difficult to check that $S : M \times \mathbb{R} \to M$ is continuous and therefore it is a dynamical system on M relative to $\mathbb{R}$. We prove S is topologically mixing.

Suppose U, V are open sets, $f \in U$ and $g \in V$. Then there is an $\varepsilon > 0$ such that $N^r(f,\varepsilon) \subset U$ and $N^r(g,\varepsilon) \subset V$, where

$$N^r(f,\varepsilon) = \{f' \in M : \varrho^r(f,f') < \varepsilon\}\,.$$

Furthermore, for some $N > 0$,

$$\sum_{n=N}^{+\infty} 2^{-n} < \varepsilon\,.$$

We define $\bar f \in M$ by

$$\bar f(x) = \begin{cases} f(x), & |x| \le N,\\ g(x), & |x| \ge N+1.\end{cases}$$

Such a map exists by using a partition of unity. Clearly, $\bar f \in N^r(f,\varepsilon)$. Let $T = 2N+1$. If $t \ge T$, then

$$\varrho^r(S^t\bar f, g) = \sum_{n=N}^{+\infty} 2^{-n} h(\varrho_n^r(S^t\bar f, g)) \le \sum_{n=N}^{+\infty} 2^{-n} < \varepsilon\,.$$

Thus $S^t(U)\cap V \ne \emptyset$ and so S is topologically mixing. By the Corollary, S is generically chaotic.

Indeed, we have thus given a new simple proof to some results of Piórek in [9].

(3) Let $p : \mathbb{R} \to S^1$ be defined by $x \mapsto p(x) = e^{2\pi i x}$, and define f :

$S^1 \to S^1$ by

$$f(e^{2\pi ix}) = e^{4\pi ix} \quad \text{for } e^{2\pi ix} \in S^1 .$$

It is easy to verify that f is a continuous open map and the dynamical system f_* induced by f is topologically mixing. So by the Corollary, f_* is generically chaotic.

Now let C be another circle in the complex plane so that $C \cap S^1 = \emptyset$. Let X denote the topological sum of S^1 and C. Define $F : X \to X$ by

$$F(x) = \begin{cases} f(x), & x \in S^1, \\ g(x), & x \in C, \end{cases}$$

where $g : C \to S^1$ is a homeomorphism. Then F is a continuous open map. Since C is open in X and for any positive integer n, $F_*^n(C) \cap C = \emptyset$, it follows that the dynamical system F_* induced by F is not weakly topologically mixing.

However, F_* is generically chaotic. Indeed, let $\{V_1, V_2, \ldots\}$ be a countable base of $S^1 \times S^1$. If W is a nonempty open subset of $S^1 \times S^1$, then for any $V \in \{V_1, V_2, \ldots\}$, there is an $n > 1$ such that

$$(F_* \times F_*)^n(W) \cap V \neq \emptyset , \tag{$*$}$$

since $F_*|_{S^1 \times \mathbb{Z}^+}$ being topologically mixing implies the same for $(F_* \times F_*)|_{S^1 \times S^1 \times \mathbb{Z}^+}$. Now suppose U is open in $X \times X$. Then $(F \times F)(U)$ is open in $S^1 \times S^1$, since $F : X \to S^1$ being open implies the same for $F \times F : X \times X \to S^1 \times S^1$. Therefore for each $n \geq 1$, by $(*)$, the set

$$\bigcup_{m=1}^{+\infty} (F_* \times F_*)^{-m}(V_n)$$

is dense in $X \times X$. Put

$$E = \bigcap_{n=1}^{+\infty} \bigcup_{m=1}^{+\infty} (F_* \times F_*)^{-m}(V_n) .$$

Then E is residual in $X \times X$ and equals

$$\{(x, y) \in X \times X : \{(F_*^n x, F_*^n y) : n \in \mathbb{Z}^+\} \text{ is dense in } S^1 \times S^1\} .$$

By repeating the argument used in the proof of the Theorem, we see that each point of E is chaotic with respect to F_*. So F_* is generically chaotic.

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