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Regular and biregular functions in the sense of Fueter—some problems

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Abstract. The biregular functions in the sense of Fueter are investigated. In particular, the class of LR-biregular mappings (left regular with a right regular inverse) is introduced. Moreover, the existence of non-affine biregular mappings is established via examples. Some applications to the quaternionic manifolds are given.

Introduction. A basic question in the development of quaternionic analysis is the proper generalization of the notion of holomorphicity. At the outset it may not be clear which of several conditions, equivalent for holomorphic mappings of complex numbers, can best be generalized to the quaternionic skew field $\mathbb{H}$. It is well known (see [12]) that the notions of quaternionic differentiability and of arbitrary quaternionic power series do not lead to interesting classes of functions.

In the 1930's, R. Fueter [3] and others investigated a notion of "regularity" for quaternionic functions defined via an analogue of the Cauchy–Riemann equations. The class of Fueter regular functions expresses in many ways the spirit of complex analysis in the quaternionic context, as many classical results (e.g., Cauchy's integral formula, Morera's theorem, the Laurent expansion) carry over in a more or less natural way [11], [12]. This theory is still being developed, both over the quaternionic field and over Clifford algebras in general.

In this paper we begin an investigation of biregular functions, that is, invertible regular functions with regular inverse. One conclusion is that the composition of two biregular mappings need not be biregular. Section 1 contains various formulations of the Fueter derivative while Section 2 develops families of Chain Rules. Section 3 begins the study of biregular mappings; a pointwise criterion is given for local biregularity in terms of the Jacobian differential, and the existence of non-affine biregular mappings is established

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via examples. The class of LR-biregular mappings (left regular with a right regular inverse) is introduced, and its pointwise criterion is found to be considerably simpler. In Section 4 we apply the above to define regular and LR-biregular functions on affine quaternionic manifolds, and to justify the assertion that a quaternionic manifold admitting a notion of regular quaternion-valued function is necessarily affine.

M. Shapiro and N. Vasilevski [9] have investigated an extended family of differential operators which are "equally good" as the one studied by Fueter. Since the basic theory for each of these operators is essentially equivalent, we will not work in such generality here.

The manifold point of view suggests that the notion of LR-biregularity is more natural than biregularity. It is hoped that this work may serve as a starting point for investigation of properties of such mappings. We warmly thank E. Ramírez de Arellano for numerous discussions regarding quaternionic functions, and to the referee who made valuable suggestions for improving a previous version of this paper.

1. Fueter derivative and regular functions. Well known facts about quaternions and regular functions which are not cited specifically in this article may be found, for example, in the general references [11], [12].

Let

$$q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} \tag{1}$$

be a quaternion, where $x_\alpha \in \mathbb{R}$, $\alpha = 0,1,2,3$, and where the standard quaternionic units satisfy $i^2 = j^2 = k^2 = ijk = -1$. Write $\partial_\alpha = \partial/\partial x_\alpha$ and form the following symbolic differential expressions:

$$D^+ = \tfrac{1}{4}(\partial_0 + i\partial_1 + j\partial_2 + k\partial_3)\,,$$
$$D = \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)\,.$$

Definition. A function $f = f^0 + if^1 + jf^2 + kf^3 : \Omega \to \mathbb{H}$ is said to be (*left*) *regular* in the domain $\Omega \subseteq \mathbb{H}$ if f is differentiable in the usual sense as a mapping of an open set in $\mathbb{R}^4$ to $\mathbb{R}^4$, and $D^+ \cdot f = 0$ in Ω; it is (*left*) *antiregular* if $D \cdot f = 0$. Similarly, f is *right regular* (resp. *right antiregular*) if $f \cdot D^+ = 0$ (resp. $f \cdot D = 0$). (By notational convention, $g\partial_\alpha = \partial_\alpha g$ for any real function g.) If f is invertible and both f and f^{-1} are regular, then f is said to be *biregular*.

We will occasionally use the " $\cdot$ " as above to stress that quaternionic multiplication is performed. For clarity, some statements about the left operators will be accompanied by the symmetric results for the corresponding right sided operators. From the definition it follows that the linear mapping with matrix $A = (a_{\alpha\beta}) \in \mathbb{R}^{4\times 4} = \{4\times 4 \text{ real matrices}\}$ is regular if and only if the four *real Fueter equations*

$$\begin{aligned}
&\varepsilon_{00}a_{00} + \varepsilon_{11}a_{11} + \varepsilon_{22}a_{22} + \varepsilon_{33}a_{33} = 0\,,\\
&\varepsilon_{01}a_{01} + \varepsilon_{10}a_{10} + \varepsilon_{23}a_{23} + \varepsilon_{32}a_{32} = 0\,,\\
&\varepsilon_{02}a_{02} + \varepsilon_{13}a_{13} + \varepsilon_{20}a_{20} + \varepsilon_{31}a_{31} = 0\,,\\
&\varepsilon_{03}a_{03} + \varepsilon_{12}a_{12} + \varepsilon_{21}a_{21} + \varepsilon_{30}a_{30} = 0
\end{aligned} \tag{2}$$

are satisfied, where $\varepsilon_{\alpha\beta} = \pm 1$, the signs being given by the first of the following matrices:

$$\begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{pmatrix}, \quad \begin{pmatrix} + & - & - & - \\ + & + & + & - \\ + & - & + & + \\ + & + & - & + \end{pmatrix},$$

$$\begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{pmatrix}, \quad \begin{pmatrix} + & - & - & - \\ + & + & - & + \\ + & + & + & - \\ + & - & + & + \end{pmatrix}.$$

The remaining three sets of signs correspond to $Df = 0$, $fD^+ = 0$, $fD = 0$ respectively.

Write $\mathcal{F} = \mathcal{F}_{\mathbb{R}} = \{A \in \mathbb{R}^{4\times 4} : A \text{ is regular}\}$. Observe that $A_1, A_2 \in \mathcal{F}$ does not imply $A_1A_2 \in \mathcal{F}$, and that the identity matrix I is not in $\mathcal{F}$. A smooth function f is regular if its real Jacobian differential *df* lies in $\mathcal{F}$ at every point of its domain.

Complex notation. As is well known, the quaternion q can be expressed uniquely as $q = u + vj$ where $u = x_0 + ix_1$, $v = x_2 + ix_3 \in \mathbb{R} + i\mathbb{R}$. Note that $jv = \overline{v}j$, so

$$(u_1 + v_1 j)(u_2 + v_2 j) = (u_1u_2 - v_1\overline{v}_2) + (u_1v_2 + v_1\overline{u}_2)j\,. \tag{3}$$

Similarly we can write

$$\begin{aligned}
2D^+ &= \frac{1}{2}(\partial_0 + i\partial_1) + \frac{1}{2}(\partial_2 + i\partial_3)j = \frac{\partial}{\partial\overline{u}} + \frac{\partial}{\partial\overline{v}}j\,,\\
2D &= \frac{\partial}{\partial u} - \frac{\partial}{\partial\overline{v}}j \quad \text{and} \quad f(q) = \phi(q) + \psi(q)j\,.
\end{aligned}$$

Then a formal calculation analogous to (3) yields

$$D^+f = \tfrac{1}{2}(\phi_{\bar u} - \overline{\psi}_{\bar v}) + \tfrac{1}{2}(\psi_{\bar u} + \overline{\phi}_{\bar v})j, \quad Df = \tfrac{1}{2}(\phi_u + \overline{\psi}_{\bar v}) + \tfrac{1}{2}(\psi_u - \overline{\phi}_{\bar v})j\,,$$

an expression of the quaternionic operators in complex coordinates, where the subscripts indicate formal complex derivatives. The following consequence is immediate.

Proposition 1.1. *Let $f : \Omega \to \mathbb{H}$ be differentiable. Then $D^+ f = 0$ if and only if*

$$\phi_v = -\overline{\psi}_u \,, \qquad \psi_v = \overline{\phi}_u \,. \tag{4}$$

Similarly, $Df = 0$ if and only if $\phi_v = \overline{\psi}_{\bar u}$, $\psi_v = -\overline{\phi}_{\bar u}$.

We will refer to (4) as the "complex Fueter equations".

The following 4×4 matrix notation will prove convenient for our purposes. For f as above define

$$J_{\mathbb{C}}(f) \;=\; \begin{pmatrix} \phi_u & \phi_{\bar u} & \phi_v & \phi_{\bar v} \\ \overline{\phi}_u & \overline{\phi}_{\bar u} & \overline{\phi}_v & \overline{\phi}_{\bar v} \\ \psi_u & \psi_{\bar u} & \psi_v & \psi_{\bar v} \\ \overline{\psi}_u & \overline{\psi}_{\bar u} & \overline{\psi}_v & \overline{\psi}_{\bar v} \end{pmatrix} \in \mathcal{M} \,,$$

where $\mathcal{M} \subseteq \mathbb{C}^{4\times 4}$ is the collection of matrices with block form $B = (B_{00}, B_{01};\ B_{10}, B_{11})$, with each $B_{\alpha\beta}$ of the form $(a, \overline{b};\ b, \overline{a})$, $a, b \in \mathbb{C}$. The second and fourth columns of matrices in $\mathcal{M}$ are redundant; this notation permits expressing the Chain Rule as $J_{\mathbb{C}}(f \circ g) = J_{\mathbb{C}}(f) \circ J_{\mathbb{C}}(g)$, $J_{\mathbb{C}}(f^{-1}) = (J_{\mathbb{C}} f)^{-1}$. The complex Fueter equations (4) say that $D^+ f = 0$ is equivalent to $J_{\mathbb{C}}(f) \in \mathcal{F}_{\mathbb{C}}$, where

$$\mathcal{F}_{\mathbb{C}} = \{(b_{\alpha\beta}) \in \mathcal{M} :\ b_{02} = -b_{30},\ b_{22} = b_{10}\} \,. \tag{5}$$

Similarly, $Df = 0$ corresponds to the conditions $b_{02} = \overline{b}_{20}$, $b_{22} = -\overline{b}_{00}$, while $fD^+ = 0$ corresponds to $b_{12} = -\overline{b}_{20}$, $b_{22} = \overline{b}_{10}$ and $fD = 0$ corresponds to $b_{12} = b_{30}$, $b_{22} = -b_{00}$.

2. Quaternionic partial derivatives and chain rules. We describe one further way of expressing the Fueter equations. Introduce the following "conjugations":

$$\begin{aligned} \overline{q} &= x_0 - ix_1 - jx_2 - kx_3 \,, \\ \overline{q}^{(1)} &= x_0 - ix_1 + jx_2 + kx_3 \,, \\ \overline{q}^{(2)} &= x_0 + ix_1 - jx_2 + kx_3 \,, \\ \overline{q}^{(3)} &= x_0 + ix_1 + jx_2 - kx_3 \,, \end{aligned} \tag{6}$$

in terms of which we may express the real coordinates,

$$\begin{aligned} x_0 &= \tfrac{1}{4}(\overline{q} + \overline{q}^{(1)} + \overline{q}^{(2)} + \overline{q}^{(3)}) \,, \\ x_1 &= \tfrac{1}{4}(\overline{q} + \overline{q}^{(1)} - \overline{q}^{(2)} - \overline{q}^{(3)})i \,, \\ x_2 &= \tfrac{1}{4}(\overline{q} - \overline{q}^{(1)} + \overline{q}^{(2)} - \overline{q}^{(3)})j \,, \\ x_3 &= \tfrac{1}{4}(\overline{q} - \overline{q}^{(1)} - \overline{q}^{(2)} + \overline{q}^{(3)})k \,. \end{aligned}$$

To define symbolic partial derivatives with respect to the $\overline{q}^{(\alpha)}$, consider more generally any four $\mathbb{R}$-linear isomorphisms $\sigma_0, \sigma_1, \sigma_2, \sigma_3 : \mathbb{H} \to \mathbb{H}$ with the property that one may solve for the x_α uniquely in terms of the $\sigma_\alpha(q)$; that is, the expressions $\partial x_\alpha/\partial\sigma_\beta(q)$ are defined uniquely by the condition

$$x_\alpha = \sum_\beta \sigma_\beta(q)\cdot\frac{\partial x_\alpha}{\partial\sigma_\beta(q)} \tag{7}$$

for $0 \le \alpha \le 3$. For any quaternionic function f define

$$\frac{\partial f}{\partial\sigma_\alpha(q)} := \sum_\beta \frac{\partial x_\beta}{\partial\sigma_\alpha(q)}\cdot\frac{\partial f}{\partial x_\beta}\,. \tag{8}$$

LEMMA 2.1. *Assume f is linear. Then*

$$f(q) = \sum_\alpha \sigma_\alpha(q)\cdot\frac{\partial f}{\partial\sigma_\alpha(q)}\,.$$

Proof. By (8) and then (7),

$$\sum_\alpha \sigma_\alpha(q)\frac{\partial f}{\partial\sigma_\alpha(q)} = \sum_\alpha\sum_\beta \sigma_\alpha(q)\frac{\partial x_\beta}{\partial\sigma_\alpha(q)}\frac{\partial f}{\partial x_\beta} = \sum_\beta x_\beta\frac{\partial f}{\partial x_\beta} = f(q)\,.$$

PROPOSITION 2.2. *Let $p\mapsto q\mapsto f$ be smooth mappings. Then*

$$\frac{\partial f}{\partial\sigma_\alpha(p)} = \sum_\beta \frac{\partial\sigma_\beta(q)}{\partial\sigma_\alpha(p)}\cdot\frac{\partial f}{\partial\sigma_\beta(q)}\,.$$

Proof. It suffices to prove the statement assuming both maps are linear. Then by two applications of Lemma 2.1,

$$f(q(p)) = \sum_\beta \sigma_\beta(q)\frac{\partial f}{\partial\sigma_\beta(q)} = \sum_\beta\Big(\sum_\alpha \sigma_\alpha(p)\frac{\partial\sigma_\beta(q)}{\partial\sigma_\alpha(p)}\Big)\frac{\partial f}{\partial\sigma_\beta(q)}$$

while by Lemma 2.1 again,

$$f(q(p)) = \sum_\alpha \sigma_\alpha(p)\cdot\frac{\partial f}{\partial\sigma_\alpha(p)}\,.$$

Comparison of the coefficients of $\sigma_\alpha(q)$ gives the desired result.

Returning to the specific conjugations (6), we see that $\partial x_\alpha/\partial\overline{q} = \partial x_\alpha/\partial\sigma_0(q)$ takes the values $1/4,\ i/4,\ j/4,\ k/4$ for $\alpha = 0,1,2,3$. Therefore according to (8) we have defined $\partial f/\partial\overline{q} = \sum_\alpha(\partial x_a/\partial\overline{q})\partial_\alpha f = D^+ f$.

If we take instead $\sigma_0(q)$, $\sigma_1(q)$, $\sigma_2(q)$, $\sigma_3(q)$ equal to

$$\begin{aligned} q &= x_0 + ix_1 + jx_2 + kx_3\,,\\ q^{(1)} &= x_0 + ix_1 - jx_2 - kx_3\,,\\ q^{(2)} &= x_0 - ix_1 + jx_2 - kx_3\,,\\ q^{(3)} &= x_0 - ix_1 - jx_2 + kx_3\,, \end{aligned} \tag{9}$$

respectively (thus $q^{(1)} = iqi^{-1}$, etc.), then it will be found easily that $\partial f/\partial q = D\cdot f$.

It must be stressed that $\partial/\partial\overline{q}$, $\partial/\partial q$ are not intrinsically determined by the definitions of $\overline{q}$, q but depend on the complete system of four conjugations used. It is not difficult to find systems other than (6) for which $\partial/\partial q$ is still equal to the left operator D, but with the remaining symbolic partial derivatives $\partial/\partial\overline{q}^{(1)}$, $\partial/\partial\overline{q}^{(2)}$, $\partial/\partial\overline{q}^{(3)}$ turning out quite different.

Further, it should be noted that if the order of multiplication in definitions (7), (8) is reversed, giving

$$x_\alpha = \sum_\beta \frac{\partial x_\alpha}{\partial\sigma_\beta(q)}\sigma_\beta(q)\,, \qquad \frac{\partial f(q)}{\partial\sigma_\alpha(q)} = \sum_\beta \frac{\partial f}{\partial x_\beta}\frac{\partial x_\beta}{\partial\sigma_\alpha(q)}\,,$$

then Proposition 2.2 will need to be adjusted in the obvious way, and the two sets of conjugations (6), (9) will cause $\partial f/\partial\overline{q}$, $\partial f/\partial q$ to evaluate the right-sided operators $f\cdot D^+$, $f\cdot D$ respectively. From Proposition 2.2 one may derive various "Chain Rules" such as the following.

Proposition 2.3. *Let $f:\Omega_2\to\mathbb{H}$, $g:\Omega_1\to\Omega_2$ be differentiable mappings of domains in $\mathbb{H}$. Let $\partial/\partial\overline{q}^{(\alpha)}$, $\partial/\partial q^{(\alpha)}$ be determined by the systems* (6), (9) *respectively. Then*

$$D^+(f\circ g) = (D^+\overline{g})\cdot((D^+f)\circ g) + \sum_{\alpha=1}^{3}(D^+\overline{g}^{(\alpha)})\cdot\left(\frac{\partial f}{\partial\overline{q}^{(\alpha)}}\circ g\right),$$

$$D(f\circ g) = (Dg)\cdot((Df)\circ g) + \sum_{\alpha=1}^{3}(Dg^{(\alpha)})\cdot\left(\frac{\partial f}{\partial q^{(\alpha)}}\circ g\right),$$

$$D^+(f\circ g) = \sum_{\alpha=0}^{3}(D^+g^\alpha)\cdot(\partial_\alpha f)\circ g\,,$$

$$D(f\circ g) = \sum_{\alpha=0}^{3}(Dg^\alpha)\cdot(\partial_\alpha f)\circ g\,,$$

where in the latter two formulas, $g = g^0 + ig^1 + jg^2 + kg^3$.

Proof. The first two rules are an application of Proposition 2.2 with $\partial f/\partial\overline{q} = D^+f$, $\partial f/\partial q = Df$ respectively. To verify the other two, use $\sigma_\alpha(q) = x_\alpha$ and deduce from Proposition 2.2 that $\partial_\beta(f\circ g) = \sum_\alpha(\partial_\beta g^\alpha)((\partial_\alpha f)\circ g)$.

As an application of the Chain Rules we have the following elementary formulas.

Corollary 2.4. *Let $g(q) = aq+b$, $h(q) = qc+d$ be left and right affine quaternionic mappings*, $a,b,c,d\in\mathbb{H}$. *Then*

$$D^+(f\circ g) = \overline{a}(D^+f)\circ g\,,$$

$$
\begin{aligned}
D(f\circ g) &= \left(a_0 Df + i a_1 \frac{\partial f}{\partial q^{(1)}} + j a_2 \frac{\partial f}{\partial q^{(2)}} + k a_3 \frac{\partial f}{\partial q^{(3)}}\right)\circ g\,,\\
D^+(f\circ h) &= \left(c_0 D^+ f - i c_1 \frac{\partial f}{\partial \overline{q}^{(1)}} - j c_2 \frac{\partial f}{\partial \overline{q}^{(2)}} - k c_3 \frac{\partial f}{\partial \overline{q}^{(3)}}\right)\circ h\,,\\
D(f\circ h) &= \ c(Df)\circ h\,,
\end{aligned}
$$

where $a = a_0 + ia_1 + ja_2 + ka_3$, $c = c_0 + ic_1 + jc_2 + kc_3$. *Further*, $D^+(h\circ f) = (D^+ f)c$, $D(h\circ f) = (Df)c$.

Proof. Observe that $D^+\overline{g} = \overline{a}$, $Dh = c$, $D^+\overline{g}^{(\alpha)} = 0 = Dh^{(\alpha)}$ for $\alpha = 1,2,3$. Thus the formulas for $D^+(f\circ g)$, $D(f\circ h)$ follow from the first two Chain Rules of Proposition 2.3. The formulas for $D(f\circ g)$, $D^+(f\circ h)$ are obtained similarly. Finally, since $\partial_0 h = c$, $\partial_1 h = ic$, etc., the formulas for $D^+(h\circ f)$, $D(h\circ f)$ follow from the last two Chain Rules of Proposition 2.3 (or directly from the definitions of D, D^+).

THEOREM 2.5. *Let* $A : \mathbb{H}\to\mathbb{H}$ *be* $\mathbb{R}$*-linear. If* $\partial A/\partial\overline{q}^{(1)} = \partial A/\partial\overline{q}^{(2)} = \partial A/\partial\overline{q}^{(3)} = 0$, *then* $\overline{A}$ *is a left multiplication mapping. If* $\partial\overline{A}/\partial q^{(1)} = \partial\overline{A}/\partial q^{(2)} = \partial\overline{A}/\partial q^{(3)} = 0$, *then* $\overline{A}$ *is a right multiplication mapping.*

Proof. The system of conjugations (6) gives $4\partial/\partial\overline{q}^{(1)} = \partial_0 + i\partial_1 - j\partial_2 - k\partial_3$, $4\partial/\partial\overline{q}^{(2)} = \partial_0 - i\partial_1 + j\partial_2 - k\partial_3$, $4\partial/\partial\overline{q}^{(3)} = \partial_0 - i\partial_1 - j\partial_2 + k\partial_3$. From this a simple calculation shows that $\partial A/\partial\overline{q}^{(\gamma)} = 0$ precisely when $A = (a_{\alpha\beta})$ satisfies (2) with the signs $\varepsilon_{\alpha\beta}$ given by

$$
\begin{pmatrix} + & + & - & - \\ + & - & + & - \\ + & + & + & + \\ + & - & - & + \end{pmatrix},\quad
\begin{pmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \\ + & + & + & + \end{pmatrix},\quad
\begin{pmatrix} + & - & - & + \\ + & + & + & + \\ + & - & + & - \\ + & + & - & - \end{pmatrix},
$$

for $\gamma = 1,2,3$ respectively. If these hold simultaneously, then it follows easily that A is of the form

$$
\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix},
$$

which is the real matrix corresponding to the left multiplication $q \mapsto (a + bi + cj + dk)\cdot q$. The statement regarding $\partial\overline{A}/\partial q^{(\gamma)}$ is proved similarly with the aid of the system (9).

The following dual statement may be verified in the same way.

THEOREM 2.6. *If* $D^+\overline{A}^{(1)} = D^+\overline{A}^{(2)} = D^+\overline{A}^{(3)} = 0$, *then* A *is a left multiplication*; *if* $DA^{(1)} = DA^{(2)} = DA^{(3)} = 0$, *then* A *is a right multiplication.*

3. Biregular mappings in the sense of Fueter

Biregular linear mappings. From the real Fueter equations (2) the set $\mathcal{F}$ of real regular matrices is seen to be a 12-dimensional $\mathbb{R}$-linear subspace of $\mathbb{R}^{4\times 4}$. The collection of biregular linear mappings is thus $\mathcal{F}^* = \mathcal{F}^*_{\mathbb{R}} = \{A \in \mathcal{F} : \det A \neq 0,\ A^{-1} \in \mathcal{F}\}$. If we write $\Phi(A) = A^{-1}$, we see that $\mathcal{F}^* = \mathcal{F} \cap \Phi(\mathcal{F} \cap GL(4,\mathbb{R}))$, which is a real algebraic variety evidently of dimension no less than 8.

Inasmuch as the equations defining $\mathcal{F}^*$ are exceedingly cumbersome, we will turn to the complex formulation. Define $\mathcal{F}^*_{\mathbb{C}} = \{B \in \mathcal{F}_{\mathbb{C}} : \det B \neq 0,\ B^{-1} \in \mathcal{F}_{\mathbb{C}}\}$. There is a natural one-to-one correspondence $\mathcal{F}^* \leftrightarrow \mathcal{F}^*_{\mathbb{C}}$. By (5), an invertible $B \in \mathcal{F}_{\mathbb{C}}$ is in $\mathcal{F}^*_{\mathbb{C}}$ precisely when $(B^{-1})_{0,2} = -(B^{-1})_{3,0}$ and $(B^{-1})_{2,2} = (B^{-1})_{1,0}$. These elements of the inverse matrix can be easily obtained via expansion by minors, and we have the following characterization of biregular matrices.

THEOREM 3.1. *Let* $B = (b_{\alpha\beta}) \in \mathcal{M}$. *Then* $B \in \mathcal{F}^*_{\mathbb{C}}$ *if and only if* $\det B \neq 0$, $b_{02} = -b_{30}$, $b_{22} = b_{10}$, *and*

$$(10)\quad \begin{aligned} \begin{vmatrix} \overline{b}_{10} & -b_{30} & \overline{b}_{12} \\ \overline{b}_{00} & b_{12} & -\overline{b}_{30} \\ \overline{b}_{20} & b_{32} & \overline{b}_{10} \end{vmatrix} &= \begin{vmatrix} b_{10} & \overline{b}_{00} & b_{12} \\ b_{20} & \overline{b}_{30} & b_{10} \\ b_{30} & \overline{b}_{20} & b_{32} \end{vmatrix}, \\ \begin{vmatrix} b_{00} & \overline{b}_{10} & \overline{b}_{12} \\ b_{10} & \overline{b}_{00} & -\overline{b}_{30} \\ b_{30} & \overline{b}_{20} & \overline{b}_{10} \end{vmatrix} &= -\begin{vmatrix} b_{10} & b_{12} & -\overline{b}_{30} \\ b_{20} & b_{10} & \overline{b}_{32} \\ b_{30} & b_{32} & \overline{b}_{10} \end{vmatrix}. \end{aligned}$$

From this formulation, some simple solutions may be obtained by inspection. For example, (10) is satisfied when $b_{10} = b_{30} = b_{20} + \overline{b}_{12} = 0$. Allowing the remaining parameters to vary freely, we obtain a few particular cases of biregular maps:

COROLLARY 3.2. *Let* $t_1, t_2, t_3 \in \mathbb{C}$ *with* $t_1 t_2 + t_3^2 \neq 0$. *Then*

$$\begin{pmatrix} t_1 & 0 & 0 & -t_3 \\ 0 & \overline{t}_1 & -\overline{t}_3 & 0 \\ t_3 & 0 & 0 & t_2 \\ 0 & \overline{t}_3 & \overline{t}_2 & 0 \end{pmatrix} \in \mathcal{F}^*_{\mathbb{C}}.$$

From this it may be seen that the composition of two biregular mappings is not, in general, biregular.

Existence of nonlinear biregular mappings. Thus far all of the biregular mappings considered have been linear. We now give examples of biregular mappings with nonconstant differential. Corollary 3.2 implies that $f = \phi +$

ψj will be locally biregular wherever it is nonsingular when its complex coefficient functions satisfy, for example,

$$\phi_{\bar{u}} = \psi_{\bar{u}} = 0\,, \quad \phi_v = \psi_v = 0\,, \quad \phi_{\bar{v}} = -\psi_u\,.$$

This is solved by $\phi(u,v) = \lambda'(u)\overline{v} + \mu(u)$, $\psi(u,v) = -\lambda(u) + \nu(u)$, where λ, μ are holomorphic and ν is antiholomorphic. Here $t_1 = \lambda''(u)\overline{v} + \mu'(u)$, $t_2 = 0$, $t_3 = -\lambda'(u)$ so $t_1 t_2 + t_3^2 = \lambda'(u)^2$ is nonzero whenever $\lambda'(u) \neq 0$. This provides many nontrivial examples of biregular mappings; we will not investigate the general solution here.

LR-biregular mappings. One may define numerous classes of mappings by requiring f, f^{-1} to have any desired properties related to regularity. We will say that the invertible mapping f is *left-right biregular* (*LR-biregular*) when f is (left) regular and f^{-1} is right regular. Let $\mathcal{F}_{\mathbb{C}}^{\mathrm{LR}} \subseteq \mathcal{M}$ denote the set of matrices corresponding to LR-biregular mappings. By the same reasoning as above, one finds that if $B \in \mathcal{F}_{\mathbb{C}}^{*}$ is invertible, then B^{-1} is right regular if and only if $(B^{-1})_{1,2} = -(\overline{B}^{-1})_{2,0}$, $(B^{-1})_{2,2} = (\overline{B}^{-1})_{1,0}$; that is,

$$(11)\qquad \begin{gathered} -\begin{vmatrix} b_{00} & -b_{30} & \overline{b}_{12} \\ b_{10} & b_{12} & -\overline{b}_{30} \\ b_{30} & b_{32} & \overline{b}_{10} \end{vmatrix} = -\begin{vmatrix} \overline{b}_{10} & b_{00} & -b_{30} \\ \overline{b}_{20} & b_{30} & b_{32} \\ \overline{b}_{30} & b_{20} & b_{10} \end{vmatrix}\,, \\ \begin{vmatrix} b_{00} & \overline{b}_{10} & \overline{b}_{12} \\ b_{10} & \overline{b}_{00} & -\overline{b}_{30} \\ b_{30} & \overline{b}_{20} & \overline{b}_{10} \end{vmatrix} = -\begin{vmatrix} \overline{b}_{10} & \overline{b}_{12} & -b_{30} \\ \overline{b}_{20} & \overline{b}_{10} & b_{32} \\ \overline{b}_{30} & \overline{b}_{32} & b_{10} \end{vmatrix}\,. \end{gathered}$$

Although it might not appear so at first sight, equations (11) are considerably simpler than (10). After expansion of these determinants and some rearranging, we have the following characterization of LR-biregular matrices.

THEOREM 3.3. *Let* $B = (b_{\alpha\beta}) \in \mathcal{M}$. *Then* $B \in \mathcal{F}_{\mathbb{C}}^{\mathrm{LR}}$ *if and only if* $\det B \neq 0$, $b_{22} = b_{10}$, $b_{02} = -b_{30}$, *and*

$$(12)\qquad \begin{aligned} &b_{12}(b_{00}\overline{b}_{10} - \overline{b}_{12}b_{30}) + b_{10}(b_{00}\overline{b}_{20} + \overline{b}_{12}b_{32}) \\ &\qquad\qquad\qquad + b_{20}(\overline{b}_{10}b_{32} + \overline{b}_{20}b_{30}) = 0\,, \\ &\overline{b}_{00}(b_{00}\overline{b}_{10} - \overline{b}_{12}b_{30}) + \overline{b}_{30}(b_{00}\overline{b}_{20} + \overline{b}_{12}b_{32}) \\ &\qquad\qquad\qquad - \overline{b}_{32}(\overline{b}_{10}b_{32} + \overline{b}_{20}b_{30}) = 0\,. \end{aligned}$$

It is difficult to simplify further the system (12) of two equations in 6 unknowns. However, we have the following simple parametrization of an open set of solutions:

Corollary 3.4. *Let $t_1, t_2, t_3, t_4 \in \mathbb{C}$ with $t_1 t_2 + t_3^2 \neq 0$. Then*

$$B = \begin{pmatrix} t_1 & t_3 t_4 & -t_3 & t_1 t_4 \\ \overline{t}_3 \overline{t}_4 & \overline{t}_1 & \overline{t}_1 \overline{t}_4 & -\overline{t}_3 \\ -\overline{t}_2 \overline{t}_4 & \overline{t}_3 & \overline{t}_3 \overline{t}_4 & \overline{t}_2 \\ t_3 & -t_2 t_4 & t_2 & t_3 t_4 \end{pmatrix} \in \mathcal{F}_{\mathbb{C}}^{\mathrm{LR}} . \tag{13}$$

Proof. Since $\det B = -|t_1 t_2 + t_3^2|^2 (|t_4|^2 + 1)^2$, B is nonsingular. It is immediate that the entries $b_{\alpha\beta}$ of B satisfy the remaining conditions of Theorem 3.3; indeed, all the terms in parentheses in (12) vanish.

It must be stressed that (13) is by no means the most general LR-biregular linear mapping. Even so, it would be difficult to find explicitly the family $\{f = \phi + \psi j : J_{\mathbb{C}} f$ is of the form (13)$\}$. We give some examples of non-affine LR-biregular mappings in this family. Suppose ϕ and $\overline{\psi}$ are holomorphic functions of u and v; that is, $\phi_{\bar{u}} = \psi_u = 0$, $\phi_{\bar{v}} = \psi_v = 0$. Suppose further that $\overline{\psi}_u = -\phi_v$. (As a simple illustration, consider $\phi(u, v) = \lambda(u) + \mu(v)$, $\psi(u, v) = -\overline{u}\overline{\mu'(v)}$ for λ, μ holomorphic.) Then for $f = \phi + \psi j$, $J_{\mathbb{C}} f$ is of the form (13). The nonsingularity condition $\phi_u \overline{\psi}_v + (\overline{\psi}_u)^2 \neq 0$ is easily achieved.

One may easily see that the class $\mathcal{F}_{\mathbb{C}}^{\mathrm{LR}}$ is also not preserved under composition.

4. Regular functions on manifolds. In this section we show that LR-biregularity is a natural notion to consider on certain quaternionic manifolds.

Let M be a real differentiable manifold of dimension 4. Let $x = (x_0, x_1, x_2, x_3) : U \to \mathbb{R}^4$ be a smooth local coordinate system in an open set U in M. Define $q \in \mathbb{H}$ by (1). Then a function $F : M \to \mathbb{H}$ is *regular* with respect to this coordinate system if $f = F \circ x^{-1}$ is a regular function of q. The following fact limits the manifolds which can admit regular functions in this sense.

Proposition 4.1. *Let $g : \Omega_2 \to \Omega_1$ be a diffeomorphism of domains in $\mathbb{H}$. Suppose $f \circ g$ is regular in Ω_2 for every regular f defined in Ω_1. Then g is a left affine map $g(q) = aq + b$, $a, b \in \mathbb{H}$. Conversely, for any left affine g, $f \circ g$ is regular whenever f is.*

Proof. According to Proposition 2.3, we have

$$D^+ \overline{g}^{(1)} \frac{\partial f}{\partial \overline{q}^{(1)}} + D^+ \overline{g}^{(2)} \frac{\partial f}{\partial \overline{q}^{(2)}} + D^+ \overline{g}^{(3)} \frac{\partial f}{\partial \overline{q}^{(3)}} = 0$$

whenever $D^+ f = 0$. For this to hold for all regular f, we must have

$$D^+ \overline{g}^{(1)} = D^+ \overline{g}^{(2)} = D^+ \overline{g}^{(3)} = 0 .$$

By Theorem 2.6 the differential $dg \in \mathbb{R}^{4 \times 4}$ has the form of a left multiplication mapping. It is well known [12] that any quaternionic function whose differential is of this form at each point is left affine. Conversely, by Corollary 2.4, pre-composition by such functions preserves regularity.

There are, of course, contexts in which mappings more general than affine can preserve regularity in a weaker sense. (For instance, let $g(q) = (aq+b)(cq+d)^{-1}$ $(a,b,c,d \in \mathbb{H},\ a^{-1}b - c^{-1}d \neq 0)$ be a quaternionic Möbius transformation. Then f is regular if and only if $\varrho \cdot (f \circ g)$ is regular [12], where $\varrho(q) = (|b - ac^{-1}d|^{-2}|cq+d|^{-2})(cq+d)^{-1}$.) However, the definition we have given for regular F on M implies that if x^α, x^β are compatible local coordinates, and if $f^\alpha = F \circ (x^\alpha)^{-1}$, $f^\beta = F \circ (x^\beta)^{-1}$, then $f^\beta = f^\alpha \circ g$ where $g = x^\alpha \circ (x^\beta)^{-1}$ must satisfy the condition of Proposition 4.1. This may be summarized as follows.

COROLLARY 4.2. *The only manifolds M modelled locally on $\mathbb{H}$ and admitting a well-defined notion of "regular function" $F : M \to \mathbb{H}$ are the left affine manifolds.*

Examples of affine quaternionic manifolds are discussed, for example, in [10]. Associated with a left affine manifold M is its *canonical bundle* ζ_M. This is the bundle with fiber $\mathbb{H}$ for which the portions $U \times \mathbb{H}$, $V \times \mathbb{H}$ over coordinate neighborhoods U, V sharing the transition function $g(q) = aq+b$ are identified via $(q,p) \approx (g(q), pa^{-1})$. Note that ζ_M is a right $\mathbb{H}$-bundle. There are three other natural $\mathbb{H}$-line bundles $\overline{\zeta}_M$, ζ_M^{-1}, $\overline{\zeta}_M^{-1}$ obtained by replacing pa^{-1} with $\overline{a}^{-1}p$, ap, $p\overline{a}$ respectively. (Due to the noncommutativity of $\mathbb{H}$, a rule such as pa does not satisfy the cocycle condition for a bundle, nor can we define higher powers of ζ_M.) The *conjugate manifold* $\overline{M}$ is the right affine manifold obtained by replacing each chart of M by its conjugate. One may identify naturally $\zeta_{\overline{M}} = \overline{\zeta}_M$. Let $C^\infty(M, \mathbb{H})$ denote the collection of smooth $\mathbb{H}$-valued functions on M.

PROPOSITION 4.3. *Let M be a left affine manifold. The left Fueter operator $f \mapsto D^+ f$ induces a linear operator $C^\infty(M, \mathbb{H}) \to \Gamma(M, \overline{\zeta}_M)$.*

Proof. Let $F \in C^\infty(M, \mathbb{H})$. For each chart x_α, write $f_\alpha = F \circ x_\alpha$ and $\sigma_\alpha = D^+ f_\alpha$. The coordinate transition functions are $g_{\alpha\beta}(q) = x_\alpha \circ x_\beta^{-1}(q) = a_{\alpha\beta}q + b_{\alpha\beta}$. By Corollary 2.4, $D^+ f_\beta = D^+(f_\alpha \circ g_{\alpha\beta}) = \overline{a}_{\alpha\beta}(D^+ f_\alpha \circ g_{\alpha\beta})$, so $\sigma_\alpha \circ g_{\alpha\beta} = \overline{a}_{\alpha\beta}^{-1}\sigma_\beta$. Therefore $D^+ F = (\sigma_\alpha)$ is a section of $\overline{\zeta}_M$.

A minor extension of Corollary 4.2 shows that in order for regular functions $F : M \to N$ to be defined between two manifolds, M must be left affine and N right affine. Since the natural functions defined on N are the right regular ones, the natural notion of invertibly regular function in this context is that of LR-biregular. Corollary 2.4 gives the following.

THEOREM 4.4. *Let M, M' be left affine quaternionic manifolds, and N, N' right affine ones. Let $G : M \to M'$ and $H : N \to N'$ be left and* *right affine mappings respectively. Let $F : M \to N$ be LR-biregular. Then $H \circ F \circ G$ is LR-biregular.*

In closing we remark that similar statements hold when "regular" is replaced by "antiregular" and "right" is exchanged with "left".

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