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Second order evolution equations with parameter

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Abstract. We give some theorems on continuity and differentiability with respect to (h,t) of the solution of a second order evolution problem with parameter $h \in \Omega \subset \mathbb{R}^m$. Our main tool is the theory of strongly continuous cosine families of linear operators in Banach spaces.

1. Introduction. We consider the second order evolution problem

$$\text{(1)}\qquad \begin{aligned} &\frac{d^2u}{dt^2} = A_h u + f(h,t), \qquad t \in (0,T],\\ &u(0) = u_h^0,\\ &u'(0) = u_h^1\,, \end{aligned}$$

with parameter $h \in \Omega$, where $(A_h)_{h\in\Omega}$ is a family of linear (possibly unbounded) operators from a real Banach space X into itself, u is a mapping $\mathbb{R} \to X$, $f : \Omega \times \mathbb{R} \to X$, Ω is an open subset of $\mathbb{R}^m$, and $u_h^0, u_h^1 \in X$ for $h \in \Omega$.

It is well known (see e.g. [1], [6]) that if A_h is the infinitesimal generator of a strongly continuous cosine family $\{C_h(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself, for $h \in \Omega$, and f satisfies some regularity conditions, then the problem (1) has exactly one solution u_h given by

$$\text{(2)}\quad u_h(t) = C_h(t)u_h^0 + S_h(t)u_h^1 + \int_0^t S_h(t-s)f(h,s)\,ds, \qquad t \in [0,T],\ h \in \Omega\,.$$

In (2), S_h, for $h \in \Omega$, is the operator sine function associated with C_h, defined by

$$\text{(3)}\qquad S_h(t)x := \int_0^t C_h(s)x\,ds, \qquad x \in X,\ t \in \mathbb{R}\,.$$

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The purpose of this paper is to present some theorems on continuity and differentiability with respect to (h,t) of the solution of problem (1). Similar questions for the first order evolution problem are considered in [7]–[9].

2. Preliminaries. Assuming that X, Y are Banach spaces we let $B(X,Y)$ be the Banach space of all bounded linear operators from X to Y. If $X=Y$, then $B(X,X)$ is denoted by $B(X)$. The space of closed linear operators from X into itself will be denoted by $C(X)$. For a given operator A, $D(A)$, $R(A)$ and $P(A)$ will denote its domain, range and resolvent set, respectively.

DEFINITION 1 (cf. [6]). Let $A_h \in C(X)$ with domain $D(A_h)=D_h$ for $h\in\Omega$. We call the family $(A_h)_{h\in\Omega}$ *R-continuous* at $h_0\in\Omega$ if there exists a Banach space Z and a family $T_h \in B(Z,X)$, $h\in\Omega$, such that

(i) $R(T_h)=D_h$ and the mapping $Z\ni z\to T_h z\in D_h$ is bijective for all $h\in\Omega$,

(ii) the mappings $\Omega\ni h\to T_h\in B(Z,X)$ and $\Omega\ni h\to V_h=A_hT_h\in B(Z,X)$ are continuous at h_0.

The continuity in Ω is defined to be the continuity at every point of Ω.

We shall use the following simple lemma (cf. [7], Corollary 1).

LEMMA 1. *Let $A_h\in C(X)$ for $h\in\Omega$ and suppose $\lambda\in P(A_h)$ for all $h\in\Omega$. Then the mapping*

$$\Omega\ni h\to A_h\in C(X)$$

is R-continuous at $h_0\in\Omega$ if and only if the mapping

$$\Omega\ni h\to(\lambda-A_h)^{-1}\in B(X)$$

is continuous at h_0.

Our main tool in this paper is the theory of strongly continuous cosine families of linear operators in Banach space. The basic ideas and results of this theory can be found for example in [6].

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A:X\supset D(A)\to X$ defined by

$$Ax:=(d^2/dt^2)C(t)x\big|_{t=0},\qquad x\in D(A),\tag{4}$$

where

$$D(A):=\{x\in X: C(t)x \text{ is twice continuously differentiable in } t\}.\tag{5}$$

Let

$$E:=\{x\in X: C(t)x \text{ is once continuously differentiable in } t\}.$$

It is known (see [6], Proposition 2.2) that $D(A)$ is dense in X and A is a closed operator in X.

If A is the generator of $C(t)$, there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq M e^{\omega|t|} \quad \text{for } t \in \mathbb{R}\,. \tag{6}$$

Moreover, let us notice (see [6], (2.17)–(2.19)) that

$$S(t)X \subset E \quad \text{and} \quad S(t)E \subset D(A) \quad \text{for } t \in \mathbb{R}\,,$$
$$(d/dt)C(t)x = AS(t)x \quad \text{for } x \in E \text{ and } t \in \mathbb{R}\,,$$
$$(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax \quad \text{for } x \in D(A) \text{ and } t \in \mathbb{R}\,.$$

The proof of the next propositions can be found in [2].

PROPOSITION 1 (see [6]). *Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X satisfying* (6), *and let A be the infinitesimal generator of $C(t)$, $t \in \mathbb{R}$. Then, for* $\operatorname{Re}\lambda > \omega$, *$\lambda^2$ is in the resolvent set of A and*

$$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} C(t)x\, dt \quad \textit{for } x \in X\,, \tag{7}$$

and

$$R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} S(t)x\, dt \quad \textit{for } x \in X\,. \tag{8}$$

PROPOSITION 2. *Under the assumptions of Proposition* 1, *for* $\operatorname{Re}\lambda > \omega$, *$\lambda^2$ is in the resolvent set of A and*

$$\|(d/d\lambda)^k \lambda(\lambda^2 - A)^{-1}\| \leq \frac{Mk!}{(\operatorname{Re}\lambda - \omega)^{k+1}} \quad \textit{for } k = 0, 1, \ldots \tag{9}$$

3. Assumptions and some helpful lemmas. Let $\{A_h\}_{h\in\Omega}$ be the family of linear operators defined in the Introduction. We make the following assumptions on $\{A_h\}_{h\in\Omega}$:

(Z_1) For each $h \in \Omega$, A_h is the infinitesimal generator of a strongly continuous cosine family $\{C_h(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself.

(Z_2) The domain $D(A_h) = D$, for $h \in \Omega$, is independent of h and the family $\{C_h(t)\}$ satisfies the inequality (6) with constants M and ω independent of $h \in \Omega$.

Under assumptions (Z_1) and (Z_2), for each $h \in \Omega$, A_h satisfies (9) with constants M and ω independent of $h \in \Omega$.

In the sequel we shall need the following assumption.

(Z_3) There exist constants $M \geq 1$ and $\omega \geq 0$ independent of $h \in \Omega$ such that for $\operatorname{Re}\lambda > \omega$, λ^2 is in the resolvent set of A_h and

$$\|\lambda(\lambda^2 - A_h)^{-1}\| \leq M(|\lambda| - \omega)^{-1}\,. \tag{10}$$

The assumption (10) is stronger than the inequality resulting from (9) for $k=0$. Assumption (Z_3) has a technical character.

LEMMA 2. *Suppose assumptions* (Z_1)–(Z_3) *are satisfied. If the mapping*

$$\Omega \ni h \to A_h \in C(X) \tag{11}$$

is R-continuous, then the mapping

$$U \ni (\lambda, h) \to (\lambda^2 - A_h)^{-1} \in B(X)\,, \tag{12}$$

where

$$U := \{(\lambda, h) \in \mathbb{C} \times \Omega : \operatorname{Re}\lambda > \omega\}\,, \tag{13}$$

is continuous.

Proof. Fix $(\lambda_0, h_0) \in U$ and let $(\lambda, h) \in U$. The assertion follows directly from the equality

$$\begin{aligned}(\lambda^2 - A_h)^{-1} &- (\lambda_0^2 - A_{h_0})^{-1}\\ &= (\lambda_0^2 - \lambda^2)(\lambda^2 - A_h)^{-1}(\lambda_0^2 - A_h)^{-1} + (\lambda_0^2 - A_h)^{-1} - (\lambda_0^2 - A_{h_0})^{-1}\,.\end{aligned}$$

THEOREM 1. *Under the assumptions of Lemma* 2, *the mapping*

$$\Omega \times \mathbb{R} \ni (h, t) \to S_h(t)x \in X \tag{14}$$

is continuous for each $x \in X$.

Proof. By assumption (Z_1), the formula (8) holds for each $h \in \Omega$ and $\operatorname{Re}\lambda > \omega$, i.e.

$$R(\lambda^2; A_h)x = \int_0^\infty e^{-\lambda t} S_h(t)x\,dt\,, \qquad h \in \Omega,\ \operatorname{Re}\lambda > \omega,\ x \in X.$$

A formal application of the inverse Laplace transform yields (cf. for example [4], p. 31)

$$S_h(t)x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} R(\lambda^2; A_h)x\,d\lambda\,, \qquad t \in \mathbb{R}\,,\ h \in \Omega,\ x \in X, \tag{15}$$

where $c > \omega$ is any constant, i.e. the line integral in (15) is taken along the straight line $\operatorname{Re}\lambda = c$. From (15) it follows that

$$S_h(t)x = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+i\sigma)t}[(c+i\sigma)^2 - A_h]^{-1}x\,d\sigma\,, \tag{16}$$

where $\lambda = c + i\sigma$, $\sigma \in (-\infty, \infty)$, is the path of integration in (15). By (10) we get

$$\|e^{(c+i\sigma)t}[(c+i\sigma)^2 - A_h]^{-1}x\| \le Me^{ct}\frac{1}{\sqrt{c^2+\sigma^2}(\sqrt{c^2+\sigma^2}-\omega)}\|x\|\,. \tag{17}$$

From (17) it follows that the improper integral in (16) is absolutely convergent uniformly in $(h,t) \in \Omega \times I$, where $I \subset \mathbb{R}$ is any bounded set.

Fix $(h_0,t_0) \in \Omega \times \mathbb{R}$, a compact neighborhood $K \subset \Omega \times \mathbb{R}$ of (h_0,t_0) and an interval $[a,b] \subset \mathbb{R}$. By Lemma 2 the integrand in (16) is uniformly continuous in $K \times [a,b]$ as a function of (h,t,σ). Therefore, using the well known theorem on the continuity of the improper integral with respect to parameters, we get the continuity of the mapping (14) at (h_0,t_0). This completes the proof.

LEMMA 3. *If*

(i) *the mapping* $\Omega \ni h \to A_h \in C(X)$ *is R-continuous*,

(ii) *the mapping* $\Omega \times \mathbb{R} \ni (h,t) \to B_h(t) \in B(X)$ *is continuous*,

then the mapping

$$\Omega \times \mathbb{R} \ni (h,t) \to B_h(t)A_h \tag{18}$$

is R-continuous.

Proof. Fix $(h_0,t_0) \in \Omega \times \mathbb{R}$. By (i) there exist a Banach space Z and operators U_h, $U_{h_0}, V_h, V_{h_0} \in B(Z,X)$ such that U_h, U_{h_0} map Z bijectively onto $D(A_h)$, $D(A_{h_0})$, respectively, $A_hU_h = V_h$, $A_{h_0}U_{h_0} = V_{h_0}$ and $\|U_h - U_{h_0}\| \to 0$ and $\|V_h - V_{h_0}\| \to 0$ as $h \to h_0$.

Define $\widetilde{U}_h(t) := U_h$ and $\widetilde{V}_h(t) := B_h(t)V_h$. We have

$$\|\widetilde{U}_h(t) - \widetilde{U}_{h_0}(t_0)\| = \|U_h - U_{h_0}\| \to 0 \tag{19}$$

and

$$\begin{aligned}\|\widetilde{V}_h(t) - \widetilde{V}_{h_0}(t_0)\| &= \|B_h(t)V_h - B_{h_0}(t_0)V_{h_0}\| \\ &\le \|B_h(t) - B_{h_0}(t_0)\|\|V_h\| \\ &\quad + \|B_{h_0}(t_0)\|\|V_h - V_{h_0}\| \to 0\end{aligned} \tag{20}$$

as $(h,t) \to (h_0,t_0)$.

On the other hand,

$$\widetilde{V}_h(t) = B_h(t)V_h = B_h(t)A_hU_h = (B_h(t)A_h)\widetilde{U}_h(t)$$

and

$$\widetilde{V}_{h_0}(t_0) = B_{h_0}(t_0)V_{h_0} = B_{h_0}(t_0)A_{h_0}U_{h_0} = (B_{h_0}(t_0)A_{h_0})\widetilde{U}_{h_0}(t_0)\,.$$

Now the R-continuity of (18) follows from (19) and (20). The proof of Lemma 3 is complete.

THEOREM 2. *Under the assumptions of Lemma* 2 *the mapping*

$$\Omega \times \mathbb{R} \ni (h,t) \to C_h(t)x \in X \tag{21}$$

is continuous for each $x \in X$.

Proof. From the known formula

$$C(t+s)-C(t-s)=2AS(t)C(s)\,,\qquad t,s\in\mathbb{R}\,,$$

(see [6], (2.23)), it follows that

$$(C_h(t)-I)x=2A_hS_h^2(t/2)x\,,\qquad t\in\mathbb{R}\,,\ h\in\Omega\,,\ x\in X. \tag{22}$$

If $x\in D(A_h)=D$ we have

$$(C_h(t)-I)x=2S_h^2(t/2)A_hx\,. \tag{23}$$

Lemma 3 with $B_h(t):=2S_h^2(t/2)$, Theorem 1, and (23) show the R-continuity of the mapping

$$\Omega\times\mathbb{R}\ni(h,t)\to C_h(t)|_D\ . \tag{24}$$

On the other hand, by (6) and (Z_2), $C_h(t):X\to X$ is a uniformly bounded operator for $h\in\Omega$ and $t\in[a,b]$, where $[a,b]\subset\mathbb{R}$ is any bounded interval. This gives the continuity of (24) in the norm of $B(X)$ (see [3], p. 206). Using the Banach–Steinhaus theorem we obtain the assertion of Theorem 2 (cf. [4], p. 9).

4. Continuity with respect to a parameter. Let $(A_h)_{h\in\Omega}$ be a family of linear operators from X into X such that assumptions (Z_1), (Z_2) are satisfied.

LEMMA 4. *Let $h_0\in\Omega$. If for any $x\in X$,*

$$\lim_{h\to h_0}C_h(t)x=C_{h_0}(t)x\qquad\textit{uniformly in } t\in[0,T] \tag{25}$$

and the family $(A_h)_{h\in\Omega}$ is R-continuous at h_0, then

$$\lim_{h\to h_0}C_h(t)x=C_{h_0}(t)x\qquad\textit{uniformly in } (t,x)\in[0,T]\times K, \tag{26}$$

where K is any compact subset of X.

The proof is the same as that of Proposition 1 in [7] with $\Phi(t,h)=C_h(t)-C_{h_0}(t)$.

As a consequence of Lemma 4 we have

COROLLARY 1. $\lim_{h\to h_0}S_h(t)x=S_{h_0}(t)x$ *uniformly in* $[0,T]\times K$.

Proof. By (3) we have

$$\|S_h(t)x-S_{h_0}(t)x\|\le\int_0^T\|C_h(s)x-C_{h_0}(s)x\|\,ds\,.$$

By (26), for any $\varepsilon>0$, there exists $\delta>0$ such that if $|h-h_0|<\delta$, then

$$\|S_h(t)x-S_{h_0}(t)x\|<\varepsilon T\qquad\text{for } t\in[0,T]\,,\ x\in K\,.$$

THEOREM 3. *If the assumptions of Lemma* 4 *are satisfied*, *the mappings*

(a) $\Omega \ni h \to u_h^0 \in X$,
(b) $\Omega \ni h \to u_h^1 \in X$,
(c) $f : \Omega \times [0,T] \to X$

are continuous and

(d) $f_h = f(h,\cdot) : [0,T] \to X$

is C^1 *for* $h \in \Omega$, *then for every* $h \in \Omega$ *there exists exactly one solution* u_h *of the problem* (1) *and*

$$\lim_{h\to h_0} u_h(t) = u_{h_0}(t)\,,$$

uniformly in $t \in [0,T]$.

Proof. By the assumptions, the solution of (1) is given by (2). Thus, by standard calculation we have

$$\begin{aligned} u_h(t) - u_{h_0}(t) &= (C_h(t) - C_{h_0}(t))u_h^0 + C_{h_0}(t)(u_h^0 - u_{h_0}^0) \\ &\quad + (S_h(t) - S_{h_0}(t))u_h^1 + S_{h_0}(t)(u_h^1 - u_{h_0}^1) \\ &\quad + \int_0^t [S_h(t-s) - S_{h_0}(t-s)]f_h(s)\,ds \\ &\quad + \int_0^t S_{h_0}(t-s)[f_h(s) - f_{h_0}(s)]\,ds\,. \end{aligned} \tag{27}$$

Let K be a compact neighborhood of h_0. Since the mappings (a), (b), (c) are continuous, the sets $K_1 = \{u_h^0 : h \in K\}$, $K_2 = \{u_h^1 : h \in K\}$ and $K_3 = \{f_h(s) : h \in K, s \in [0,T]\}$ are compact subsets of X. By Lemma 4 and Corollary 1,

$$[C_h(t) - C_{h_0}(t)]u_h^0 \xrightarrow{h\to h_0} 0\,, \qquad [S_h(t) - S_{h_0}(t)]u_h^1 \xrightarrow{h\to h_0} 0\,,$$

and

$$[S_h(t-s) - S_{h_0}(t-s)]f_h(s) \xrightarrow{h\to h_0} 0\,,$$

uniformly in $t, s \in [0,T]$. By assumption (Z_2) we have

$$\|C_{h_0}(t)(u_h^0 - u_{h_0}^0)\| \le Me^{\omega T}\|u_h^0 - u_{h_0}^0\| \xrightarrow{h\to h_0} 0\,,$$

uniformly in $t \in [0,T]$ and

$$\|S(t)x\| \le \int_0^t Me^{\omega s}\|x\|\,ds \le \frac{M}{\omega}(e^{\omega t} - 1)\|x\|\,.$$

Therefore

$$\|S_h(t)\| \le \frac{M}{\omega}(e^{\omega T} - 1) \quad \text{for } h \in \Omega\,.$$

Hence

$$\|S_{h_0}(t)(u_h^1 - u_{h_0}^1)\| \overset{h\to h_0}{\longrightarrow} 0 \quad \text{and} \quad S_{h_0}(t-s)[f_h(s) - f_{h_0}(s)] \overset{h\to h_0}{\longrightarrow} 0\,,$$

uniformly in $t, s \in [0,T]$. Thus the left hand side of (27) converges to zero, uniformly in $t \in [0,T]$.

COROLLARY 2. *If the assumptions of Theorem* 3 *are satisfied for any* $h_0 \in \Omega$, *then the mapping*

$$u : \Omega \times [0,T] \ni (h,t) \to u_h(t) \in X$$

is continuous.

5. Differentiability with respect to a parameter. Let us recall (see [7], p. 223) the definition of differentiability of $\Omega \ni h \to A_h$.

Let D be a normed vector space over $\mathbb{R}$ such that there exist a Banach space Z and a bounded, linear, bijective mapping $T : Z \to D$. Setting $sB(D,Y) = \{A : D \to Y : A$ is linear and $AT \in B(Z,Y)\}$ we see that $sB(D,Y)$ is independent of (Z,T).

DEFINITION 2. Let Ω be an open subset of $\mathbb{R}$. A function $\Omega \ni h \to A_h \in sB(D,Y)$ is said to be (continuously) *differentiable* at a point $h_0 \in \Omega$ if there exist a Banach space Z and a bounded, linear, bijective mapping $T : Z \to D$ such that the mapping $\Omega \ni h \to A_hT \in B(Z,D)$ is (continuously) differentiable in the Fréchet sense.

In this case we put

$$A'_{h_0} = \left(\frac{d}{dh} A_h T\bigg|_{h=h_0}\right) T^{-1}\,.$$

The higher differentiability classes are defined in the standard manner.

LEMMA 5. *If* $Au^0+f(0) \in D(A)$, $Au^1+(df/dt)(0) \in E$, A *is the generator of* $C(t)$ *and* $f : [0,T] \to X$ *is of class* C^3 *then the problem*

$$\begin{aligned} &\frac{d^2u}{dt^2} = Au + f\,,\\ &u(0) = u^0\,,\\ &\frac{du}{dt}(0) = u^1\,, \end{aligned} \tag{28}$$

has exactly one solution which is of class C^4 *in* $[0,T]$.

Proof. It is well known that, under our assumptions, the solution of the problem (28) has the form

$$u(t) = C(t)u^0 + S(t)u^1 + \int_0^t S(t-s)f(s)\,ds\,.$$

Hence

$$\frac{d^2u}{dt^2} = C(t)(Au^0 + f(0)) + S(t)\left(Au^1 + \frac{df}{dt}(0)\right) + \int_0^t S(s)\frac{d^2f}{dt^2}(t-s)\,ds\,.$$

Thus $w = d^2u/dt^2$ is the solution of the problem

$$\begin{aligned} &\frac{d^2w}{dt^2} = Aw + \frac{d^2f}{dt^2}\,,\\ &w(0) = Au^0 + f(0)\,,\\ &\frac{dw}{dt}(0) = Au^1 + \frac{df}{dt}(0)\,. \end{aligned} \tag{29}$$

By Proposition 2.4 of [6] we conclude that u is C^4 in $[0,T]$.

LEMMA 6. *Suppose that the assumptions of Theorem* 3 *are satisfied at every* $h_0 \in \Omega$. *If* $A_h u^0 + f_h(0) \in D$, $A_h u_h^1 + (df_h/dt)(0) \in E$ *for* $h \in \Omega$, $f_h = f(h,\cdot) : [0,T] \to X$ *is of class* C^3, $d^2f/dt^2 : \Omega \times [0,T] \to X$ *is continuous and the mappings*

$$\Omega \ni h \to A_h u_h^0 + f_h(0), \qquad \Omega \ni h \to A_h u_h^1 + \frac{df_h}{dt}$$

are continuous, *then the mapping*

$$\Omega \times [0,T] \ni (h,t) \to \frac{d^2u_h}{dt^2}(t) \in X$$

is continuous in $[0,T]$.

Lemma 6 is an immediate consequence of Lemma 5 and Theorem 3.

Now we prove

THEOREM 4. *Let* Ω *be an open subset of* $\mathbb{R}$ *and suppose that assumptions* (Z_1)–(Z_3) *hold. If*

(1) *the mappings* $\Omega \ni h \to A_h$, $\Omega \ni h \to u_h^0$ *are continuous in* Ω *and differentiable at* $h_0 \in \Omega$,

(2) $f_h : [0,T] \to f_h(t)$ *is of class* C^3 *for* $h \in \Omega$,

(3) *the mappings*

$$\begin{aligned} &\Omega \ni h \to A_h u_h^0 + f_h(0)\,,\\ &\Omega \ni h \to A_h u_h^1 + \frac{df_h}{dt}(0)\,,\\ &\Omega \times [0,T] \ni (h,t) \to \frac{\partial f}{\partial h}(h,t) \end{aligned}$$

are continuous,

(4) $A_h u_h^0 + f_h(0) \in D$ *and* $A_h u_h^1 + (df_h/dt)(0) \in E$ *for* $h \in \Omega$,

then there exists exactly one solution $u_h(t) = u(h,t)$ *of the problem* (28) *which is of class* C^2 *with respect to* t *and differentiable with respect to* h *at* h_0.

Moreover,

$$\lim_{h\to h_0} \frac{u_h(t) - u_{h_0}(t)}{h - h_0} = u'_{h_0}(t)\,,$$

uniformly in $t \in [0,T]$, *and* u'_{h_0} *is the solution of the problem*

$$\begin{aligned}
&\frac{d^2 u'_{h_0}}{dt^2} = A_{h_0} u'_{h_0} + A'_{h_0} u_{h_0} + f'_{h_0}\,,\\
&u'_{h_0}(0) = (u_{h_0}^0)'\,,\\
&\frac{du'_{h_0}}{dt}(0) = (u_{h_0}^1)'\,.
\end{aligned}$$

Proof. We proceed similarly to the proof of Theorem 2 in [6]. For $h, h_0 \in \Omega$ we have

$$\frac{d^2}{dt^2}\left(\frac{u_h - u_{h_0}}{h - h_0}\right) = A_h\left(\frac{u_h - u_{h_0}}{h - h_0}\right) + \frac{A_h - A_{h_0}}{h - h_0} u_{h_0} + \frac{f_h - f_{h_0}}{h - h_0}\,,$$

and

$$\begin{aligned}
\frac{u_h(0) - u_{h_0}(0)}{h - h_0} &= \frac{u_h^0 - u_{h_0}^0}{h - h_0}\,,\\
\frac{\frac{du_h}{dt}(0) - \frac{du_{h_0}}{dt}(0)}{h - h_0} &= \frac{u_h^1 - u_{h_0}^1}{h - h_0}\,.
\end{aligned}$$

If we take

$$F_h = \begin{cases} \dfrac{A_h - A_{h_0}}{h - h_0} u_{h_0} + \dfrac{f_h - f_{h_0}}{h - h_0} & \text{for } h \neq h_0,\\ A'_{h_0} u_{h_0} + f'_{h_0} & \text{for } h = h_0, \end{cases}$$

$$v_h^0 = \begin{cases} \dfrac{u_h^0 - u_{h_0}^0}{h - h_0} & \text{for } h \neq h_0,\\ (u_{h_0}^0)' & \text{for } h = h_0, \end{cases}$$

$$v_h^1 = \begin{cases} \dfrac{u_h^1 - u_{h_0}^1}{h - h_0} & \text{for } h \neq h_0,\\ (u_{h_0}^1)' & \text{for } h = h_0, \end{cases}$$

and

$$v_h = \frac{u_h - u_{h_0}}{h - h_0} \qquad \text{for } h \neq h_0 \, ,$$

then v_h, for $h \neq h_0$, is the solution of the problem

$$\begin{aligned} &\frac{d^2 v_h}{dt} = A_h v_h + F_h \, , \\ &v_h(0) = v_h^0 \, , \\ &\frac{dv_h}{dt}(0) = v_h^1 \, . \end{aligned}$$

Therefore Theorem 4 will be proved if we can show that Theorem 3 can be applied.

Since the family $(A_h)_{h\in\Omega}$ and the mappings $\Omega \ni h \to v_h^0 \in X$, $\Omega \ni h \to v_h^1 \in X$ satisfy all the assumptions of Theorem 3, we only have to prove that the mapping $\Omega \ni h \to F_h$ satisfies them as well. Taking $\lambda \in P(A_{h_0})$ and $T = (A_{h_0} - \lambda I)^{-1}$ we have

$$\frac{A_h - A_{h_0}}{h - h_0} u_{h_0}(t) = \left(\frac{A_h - A_{h_0}}{h - h_0} T \right) T^{-1} u_{h_0}(t) \, .$$

Then, by Lemma 5, $T^{-1}u_{h_0}$ is of class C^2 in $[0, T]$. This completes the proof.

Theorem 4 is the key to establishing theorems on higher regularity of the solution of (28) with respect to the parameter h.

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