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Nonlinear orthogonal projection

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Abstract. We discuss some properties of an orthogonal projection onto a subset of a Euclidean space. The special stress is laid on projection's regularity and characterization of the interior of its domain.

0. Introduction. Let M be a non-empty subset of a metric space Z. We define a relation $\mathcal{P} \subset Z \times M$, which we call the *orthogonal projection* onto M. Its domain is

$$\operatorname{dom}\mathcal{P} := \{z \in Z : \text{ there exists a unique point } z' \in M \text{ such that } d(z, z') = \varrho(z, M)\},$$

where d denotes the metric of Z and $\varrho(z, M) := \inf_{x\in M} d(z, x)$. Obviously, $M \subset \operatorname{dom}\mathcal{P}$. The orthogonal projection of $z \in \operatorname{dom}\mathcal{P}$ is defined to be the unique point $(z' =)\ \mathcal{P}(z) \in M$ which realizes the distance of z to M. If M is a closed linear subspace of a Hilbert space Z, then $\mathcal{P}$ is the well-known linear orthogonal projection: $Z = \operatorname{dom}\mathcal{P} \to M$.

The need of considering orthogonal projections onto non-linear sets has been noticed since a long time. For example, if $Z = \mathbb{R}^n$ and M is a smooth (or analytic) submanifold, then the composition $f \circ \mathcal{P}|_{\operatorname{int}\operatorname{dom}\mathcal{P}}$ is the most natural smooth (analytic) extension of a given smooth (analytic) function $f : M \to \mathbb{R}$ on an open neighbourhood of M (because in this case $M \subset \operatorname{int}\operatorname{dom}\mathcal{P}$ (see the generalization (3.8) of the classical result of Federer [5] and (4.1))). Of course, there are other methods of extending such functions, e.g. in the non-analytic case by local straightening of M or by applying Whitney's theory. However, in numerous problems the extension $f \circ \mathcal{P}$ is most useful, since it is simple and effective. The set $\operatorname{int}\operatorname{dom}\mathcal{P}$ is in some sense a star-shaped neighbourhood of M (see (1.5), (3.13)), so the retraction $\mathcal{P}|_{\operatorname{int}\operatorname{dom}\mathcal{P}}$ is helpful in studies on differentiable homotopy, e.g. for a given solenoidal vector field $v : G \to \mathbb{R}^n$ vanishing on the boundary of a

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domain $G \subset \mathbb{R}^n$ one can construct—with the aid of $\mathcal{P}$—a sequence $(\Phi_\nu)_{\nu=1}^\infty$ of solenoidal vector fields on G equal to zero in a neighbourhood of $M = \partial G$ such that $\Phi_\nu \to v$ together with derivatives as $\nu \to \infty$.

The notion of the nonlinear orthogonal projection enables us to formulate a new, curvilinear version of the theorem on the existence of the Fréchet differential (see (4′.13)).

Furthermore, the differential properties of the distance function $z \mapsto |z - \mathcal{P}(z)| = \varrho(z, M)$ are useful (see e.g. the lemma of Hopf in Lions [13] (Part 1, Lemma 7.2) or Hopf [8] or applications of ϱ in Serrin [16] and Gilbarg–Trudinger [6]).

Literature we know contains only studies on restrictions of orthogonal projections onto submanifolds (of a Riemannian manifold Z) to small neighbourhoods of M (e.g. the tubular neighbourhood theorem in Hirsch [7]). In this paper we present various properties of the mapping $\mathcal{P}$ without assuming that M is a submanifold. These are topological properties; for example, we formulate a criterion for z ($\in \operatorname{dom}\mathcal{P}$) to be an interior point of $\operatorname{dom}\mathcal{P}$ (see (2.8)). In so general a situation one cannot expect the orthogonal projection to be differentiable; G. Jasiński [9] proved that the class M of values of a C^1-retraction of a domain $U \in \operatorname{top}\mathbb{R}^n$ is a differentiable submanifold (of $\mathbb{R}^n$) whenever $M \in \operatorname{cotop} U$ (i.e. M is closed in U). In the present work we wish to investigate the orthogonal projection globally; in particular, almost all theorems we formulate refer also to arguments z ($\in \operatorname{dom}\mathcal{P}$) which may lie at a large distance from M.

The theorems presented here can, for the most part, be modified for the case of a Riemannian manifold Z. Nevertheless, we restrict our attention to the basic situation $Z = \mathbb{R}^n$.

1. Projection onto an arbitrary subset of a Euclidean space. Let Z denote a Euclidean space, i.e. a real finite-dimensional Hilbert space (e.g. $Z = \mathbb{R}^n$) with a scalar product $(\cdot \mid \cdot)$, which defines the norm $|\cdot|$. From now on Ω stands for the interior of the domain of the projection $\mathcal{P}$.

(1.1) Example. If M is the unit sphere of Z, then

$$\operatorname{dom}\mathcal{P} = Z \setminus \{0\} \quad \text{and} \quad \forall z \in \operatorname{dom}\mathcal{P} : \ \mathcal{P}(z) = z/|z|\,.$$

Generally, if M is the unit sphere of a linear subspace Y, then

$$\Omega = \operatorname{dom}\mathcal{P} = Z \setminus \ker P_Y = Z \setminus Y^\perp \quad \text{and} \quad \mathcal{P}(z) = P_Y(z)/|P_Y(z)|$$

for any $z \in \operatorname{dom}\mathcal{P}$, where P_Y is the usual projection $Z \to Y$.

In general, neither has $\operatorname{dom}\mathcal{P}$ to be open, nor $\mathcal{P}$ to be a continuous mapping, even if M is an analytic submanifold:

(1.2) Example. If $Z = \mathbb{R}^2$ and $M = \{(x, y) : |y| = 1\} \setminus \{(0, 1)\}$, then $(0, 0) \in (\operatorname{dom} \mathcal{P}) \setminus \Omega$ and $\mathcal{P}$ is not continuous at $(0, 0)$.

(1.3) Theorem. *The restriction* $\mathcal{P}|_\Omega$ *is continuous.*

Theorems (lemmas, examples etc.) from Section k are proved or discussed in Section k', $k = 1, \ldots, 6$.

(1.4) Corollary. $M \cap \Omega$ *is closed in* Ω.

(1.5) Theorem. *If* $a \in Z$, $a' \in M$ *and* $|a - a'| = \varrho(a, M)$, *then*:

(i) $]a, a'] \subset \operatorname{dom} \mathcal{P}$ *and* $\forall z \in\,]a, a'] : \mathcal{P}(z) = a'$,
(ii) $[a, \mathcal{P}(a)[\, \subset \Omega$ *for* $a \in \Omega$.

In general, $\mathcal{P}(a) \notin \Omega$, for example when $Z = \mathbb{R}^2$, $M = \{(x, y) : y = |x|\}$ and $a = (0, -1)$.

(1.6) Corollary. *Suppose that* $f : \mathcal{O} \to M$ *is a continuous mapping of an open set* $\mathcal{O} \subset Z$. *Furthermore, assume that* $\forall z \in \mathcal{O} : |z - f(z)| = \varrho(z, M)$. *Then* $f \subset \mathcal{P}$; *in particular*, $\mathcal{O} \subset \Omega$.

Finally, let us mention that the projection is invariant with respect to isometries:

(1.7) Remark. *Let* $I : Z \to Z$ *be an isometry. Then* $I \circ \mathcal{P} \circ I^{-1}$ *is the orthogonal projection onto* $I(M)$.

2. Closedness of a set near a point. We say that a set M is *closed near* $z \in Z$ iff

$$\exists r > \varrho(z, M) : \quad M \cap \overline{B}(z, r) \in \operatorname{cotop} Z, \tag{2.1}$$

where $B(z, r) := \{x \in Z : |x - z| < r\}$ and $\overline{B}(z, r) = \overline{B(z, r)}$ (Fig. 1).

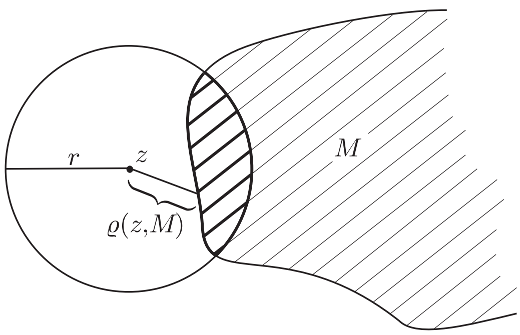

Fig. 1

(2.2) Fact. *If M is closed near z, then there exists at least one element of M which realizes the distance $\varrho(z, M)$.*

(2.3) Fact. *The set $D := \{z \in Z : M$ is closed near $z\}$ is open in Z.*

It is obvious that $D = Z$ whenever $M \in \operatorname{cotop} Z$. Then $\operatorname{dom} \mathcal{P}$ is dense in Z. This results from the following general

(2.4) Remark. $D \subset \overline{\operatorname{dom} \mathcal{P}}$.

And here is a kind of completing of Theorem (1.3):

(2.5) Remark. *The restriction $\mathcal{P}|_{D \cap \operatorname{dom} \mathcal{P}}$ is continuous. In particular, $\mathcal{P}$ is continuous whenever M is closed.*

(2.6) Proposition. *If M is locally compact, then $\Omega \subset D$.*

In general, $D \cap \operatorname{dom} \mathcal{P} \not\subset \Omega$, even if M is an analytic submanifold. For $Z = \mathbb{R}^2$ and $M = \{(x, y) : y = x^2\}$ the point $(0, 1/2)$ is in $(D \cap \operatorname{dom} \mathcal{P}) \setminus \Omega$ (see (6.1)).

Now we formulate a counterpart of Theorem (1.5):

(2.7) Theorem. *Suppose that M is locally compact, $a \in Z$, $a' \in M$ and $|a - a'| = \varrho(a, M)$. Then $]a, a'] \subset D$.*

The next theorem is a criterion for being an interior point of the projection's domain.

(2.8) Theorem. *Let M be closed near $a \in \operatorname{dom} \mathcal{P}$. Fix $t \in]0, 1]$. Then the following conditions are equivalent:*

(i) $a \in \Omega$;
(ii) $\mathcal{P}(a) + t(a - \mathcal{P}(a)) \in \Omega$ *and the mapping*

$$w \mapsto w - (1 - t)\mathcal{P}(w)$$

is an injection on a neighbourhood of $\mathcal{P}(a) + t(a - \mathcal{P}(a))$.

3. Projection onto a submanifold. We start with recalling a well-known property of the tangent space to a submanifold:

(3.1) Remark. *If M is a C^1-submanifold of Z and $a \in Z$ and $a' \in M$ satisfy $|a - a'| = \varrho(a, M)$, then*

$$a - a' \perp T_{a'} M,$$

where $T_{a'} M$ denotes the tangent space to M at a'. In particular, $\forall a \in \operatorname{dom} \mathcal{P} :\ a \in \mathcal{P}(a) + (T_{\mathcal{P}(a)} M)^{\perp}$.

For $z \in M$ and $r > 0$ we define the disc of radius r with center at z orthogonal to M by

$$K_z(r) := z + \{\zeta \in (T_z M)^{\perp} : |\zeta| < r\}.$$

(3.2) THEOREM. *Let M be a C^1-submanifold such that*

(3.3) *the mapping*: $M \ni z \mapsto T_z M \in \mathcal{E}$ *satisfies locally the Lipschitz condition in the Hausdorff metric* (*in the set* $\mathcal{E}$ *of all non-zero linear subspaces of* Z).

Then for any compact subset $F \subset M$ *there is* $r > 0$ *such that*:

(i) $\forall y, z \in F: \ \{y \neq z \Rightarrow K_y(r) \cap K_z(r) = \emptyset\}$;
(ii) $\forall z \in F: \ K_z(r) \subset \operatorname{dom} \mathcal{P}, \ \mathcal{P}|_{K_z(r)} \equiv z$.

The Hausdorff distance of two non-zero linear subspaces A, B is, by definition, the number

$$d(A, B) := d(A \cap S, B \cap S)\,, \tag{3.4}$$

where $S := \{x \in Z : |x| = 1\}$ and $d(A \cap S, B \cap S)$ is the usual Hausdorff distance of the compact sets $A \cap S$, $B \cap S$. Note that the metric (3.4) defines the usual topology on the Grassmann manifold $\mathcal{G}_k(Z)$ of all k-dimensional linear subspaces of Z; moreover,

(3.5) $\mathcal{G}_k(Z) \ni A \mapsto A^\perp \in \mathcal{G}_{N-k}(Z)$ $(1 \le k \le N-1, N = \dim Z)$ *is an isometry in this metric.*

One can also prove that

(3.6) *all C^2-submanifolds satisfy* (3.3).

Submanifolds of class C^1 generally do not:

(3.7) EXAMPLE. The curve $M := \{(t, \frac{2}{3}|t|^{3/2}) : t \in \mathbb{R}\}$ is a C^1-submanifold of $Z = \mathbb{R}^2$. One can check that

$$\forall r > 0\ \exists z \in M \setminus \{0\} : \quad K_z(r) \cap K_0(r) \neq \emptyset\,.$$

In particular, (3.3) is not satisfied.

The following theorem is important in the local analysis of a nonlinear orthogonal projection:

(3.8) THEOREM. *Let M be a C^1-submanifold satisfying* (3.3). *Then for $a \in M$ there exists an open $\mathcal{O} \subset Z$ such that $a \in \mathcal{O} \subset \operatorname{dom} \mathcal{P}$ and*

$$\forall z \in \mathcal{O}\ \forall x \in \mathcal{O} \cap M : \quad \{z - x \perp T_x M \Rightarrow \mathcal{P}(z) = x\}\,.$$

In particular, $M \subset \Omega$.

This is a generalization of Federer's theorem from [5] which states that $M \subset \Omega$ whenever M is a hypersurface of class C^2.

(3.9) COROLLARY. *If M is a C^1-submanifold satisfying* (3.3), *then*

$$\forall a \in \Omega\ \exists H \in \operatorname{top}(T_{\mathcal{P}(a)} M)^\perp : \quad 0 \in H \text{ and } \mathcal{P}|_{a+H} \equiv \mathcal{P}(a)\,.$$

(3.10) Corollary. *Suppose that M is a C^1-submanifold satisfying* (3.3) *and closed near $a \in \operatorname{dom}\mathcal{P}$. Let $\phi : \mathcal{O} \to M$ be a continuous map on a neighbourhood $\mathcal{O}$ of a such that*

$$\phi(a) = \mathcal{P}(a) \quad \text{and} \quad \forall z \in \mathcal{O}: \quad z - \phi(z) \perp T_{\phi(z)}M\,.$$

Then $a \in \Omega$ and $\phi = \mathcal{P}$ in a neighbourhood of a.

This result is a differential counterpart of Corollary (1.6). And here is a differential counterpart of the criterion (2.8):

(3.11) Theorem. *Let M be a C^2-submanifold. Fix $z_0 \in \operatorname{dom}\mathcal{P}$ and an inverse chart $f \subset \mathbb{R}^n \times M$ of M which takes on the value $\mathcal{P}(z_0)$. Consider the matrix*

$$\text{(3.12)} \quad A_f(z_0) := \left[\left(\frac{\partial f}{\partial x_i}(x_0) \,\middle|\, \frac{\partial f}{\partial x_j}(x_0)\right) + \left(\mathcal{P}(z_0) - z_0 \,\middle|\, \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right)\right],$$

where $x_0 = f^{-1}(\mathcal{P}(z_0))$. Then the following conditions are equivalent:

(i) $z_0 \in \Omega$;

(ii) *M is closed near z_0 and* $\det A_f(z_0) \neq 0$.

The additional assumption "$M \in C^2$" lets us strengthen the conclusion of Theorem (1.5):

(3.13) Theorem. *Let M be a C^2-submanifold. Then*

(a) *$]a, a'] \subset \Omega$ for all $(a, a') \in Z \times M$ such that $|a - a'| = \varrho(a, M)$*;

(b) *for every $(a, a') \in (\operatorname{dom}\mathcal{P}) \times M$ the following conditions are equivalent*:

(i) $\mathcal{P}(a) = a'$;

(ii) *$]a, a'] \subset \Omega$ and $a - a' \perp T_{a'}M$.*

(3.14) Corollary. *If M is a C^2-submanifold, then Ω is dense in $\operatorname{dom}\mathcal{P}$. If M is also closed in Z, then $\overline{\Omega} = Z$ (compare with* (2.4)*).*

4. Differentiability of an orthogonal projection

(4.1) Theorem. *Let M be a C^k-submanifold of a Euclidean space Z, where $k \in \{2, 3, \ldots, \infty, \omega\}$ and C^ω denotes $\mathbb{R}$-analyticity. Then $\mathcal{P}|_\Omega$ is of class C^{k-1} (as usual $\infty - 1 := \infty$, $\omega - 1 := \omega$) and*

(i) *for all $z \in M$, the Fréchet differential $d_z\mathcal{P}$ is the linear orthogonal projection onto T_zM*;

(ii) $\forall z \in \Omega: \ d_z\mathcal{P}(Z) = T_{\mathcal{P}(z)}M$, $\ker d_z\mathcal{P} = (T_{\mathcal{P}(z)}M)^\perp$.

Regularity of the projection $\mathcal{P}$ is higher along M itself; for example, for a C^2-submanifold the following improvement of regularity may be achieved:

(4.2) THEOREM. *If M is a C^2-submanifold, then*

(i) *for all $z \in M$, the mapping $\mathcal{P}$ is twice differentiable at z;*

(ii) *the mapping $M \ni z \mapsto d_z^2\mathcal{P} \in S_2(Z,Z)$ is continuous.* (*The symbol $S_2(Z,Z)$ stands for the Banach space of all symmetric bilinear operators $Z^2 \to Z$.*)

Despite the fact that $\mathcal{P}|_M = \mathrm{id}_M$, the restriction to M of the second derivative of $\mathcal{P}$ may have a non-trivial structure. For example, if M is the unit sphere in Z, then

$$(d_z^2\mathcal{P})(\eta,\xi) = 3(\eta \mid z)(\xi \mid z)z - (\eta \mid \xi)z - (\eta \mid z)\xi - (\xi \mid z)\eta$$

for $z \in M$ and $\eta, \xi \in Z$.

Theorem (4.2) cannot be generalized to any neighbourhood of M:

(4.3) EXAMPLE. The curve $M := \{(t,0) : t < 0\} \cup \{(t, \frac{1}{3}t^3) : t \geq 0\}$ is a C^2-submanifold of $Z = \mathbb{R}^2$. The projection $\mathcal{P}$ is not twice differentiable in any neighbourhood of $(0,0)$.

(4.4) PROPOSITION. *Let*

$$\varrho(x) := \varrho(x, M) \qquad (= |x - \mathcal{P}(x)| \text{ for } x \in \operatorname{dom}\mathcal{P}).$$

If M is of class C^2, then

$$\nabla_x \varrho = \frac{x - \mathcal{P}(x)}{|x - \mathcal{P}(x)|} \qquad \text{for any } x \in \Omega \setminus M .$$

This fact and Theorem (4.1) immediately give the following

(4.5) COROLLARY. *The function ϱ is of class C^k in $\Omega \setminus M$ for M being a submanifold of class C^k ($k \in \{2, 3, \ldots, \infty, \omega\}$).*

One of the first formulations of differential properties of ϱ was presented in Serrin [16] (Chapter I, Lemma 3.1) in the following form:

If a hypersurface M (i.e. a submanifold in Z of codimension 1) is of class C^3, then there is an open neighbourhood $\mathcal{O}$ of M such that $\varrho|_{\mathcal{O}\setminus M}$ is of class C^2.

Later on Gilbarg and Trudinger proved the following local version of Corollary (4.5) in the Appendix to [6]:

The restriction $\varrho|_{\mathcal{O}\setminus M}$ is of class C^k for some neighbourhood $\mathcal{O}$ of a C^k-hypersurface M for $k = 2, 3, \ldots, \infty, \omega$.

Obviously, the continuous function ϱ is not differentiable at any point of M. However, in the case of M being a hypersurface one can smooth ϱ by one-sided change of its sign:

(4.6) THEOREM (see Krantz–Parks [12]). *Assume that the boundary M of some set $G \in \operatorname{top} Z$ is a compact C^k-submanifold* ($k \in \{1, 2, \ldots, \infty, \omega\}$). *Suppose also that $M \subset \Omega$ (this assumption is relevant only for $k = 1$). Consider the signed distance function to M*:

$$\delta(x) := \begin{cases} \varrho(x) & \text{if } x \in G, \\ -\varrho(x) & \text{if } x \in Z \setminus G. \end{cases}$$

Then δ is of class C^k in some neighbourhood of M.

5. When is the whole space the domain of an orthogonal projection? If $\operatorname{dom} \mathcal{P} = Z$ then M is called a *Chebyshev set*.

(5.1) THEOREM. *Every non-empty closed and convex subset of Z is a Chebyshev set.*

The above theorem also holds for an infinite-dimensional Hilbert space (see Rudin [15], Theorem 4.10). Moreover, in finite-dimensional spaces also the converse theorem holds:

(5.2) THEOREM (see Bunt [2], Motzkin [14]). *If* $\operatorname{dom} \mathcal{P} = Z$, *then M is non-empty, closed and convex.*

A bounded Chebyshev set cannot be a submanifold of Z:

(5.3) THEOREM. *Assume that M is a C^2-submanifold of a Euclidean space Z and* $\operatorname{dom} \mathcal{P} = Z$. *Then M is an affine subspace of Z.*

Under some additional assumptions the implication (5.2) is valid also for infinite-dimensional Hilbert spaces or even Banach spaces (see Efimov–Stechkin [4], Klee [10], Asplund [1]). Klee in [11] conjectures that in some (possibly non-separable) Hilbert spaces there exist non-convex Chebyshev sets. The question of existence of "Klee caverns" is also considered in Asplund [1].

6. Domains of projections onto graphs of some elementary functions. Now we illustrate the above theory with examples of orthogonal projections onto graphs of some numerical functions. In this section $Z = \mathbb{R}^2$.

(6.1) EXAMPLE. The set $M := \{(x, x^2) : x \in \mathbb{R}\}$ is a closed analytic submanifold of Z. The domain of the orthogonal projection onto M is $\operatorname{dom} \mathcal{P} = \mathbb{R}^2 \setminus \{(0, t) : t > 1/2\}$ (see (3.14) and Fig. 2).

(6.2) EXAMPLE. If we take $M := \{(x, x^2) : x \geq 0\}$ (clearly, M is not a submanifold but it is closed), then $\operatorname{dom} \mathcal{P} = \mathbb{R}^2 \setminus \{(x, y) : x < 0,\ y = \frac{3}{2}(-x)^{2/3} + \frac{1}{2}\}$ (see (2.4) and Fig. 3).

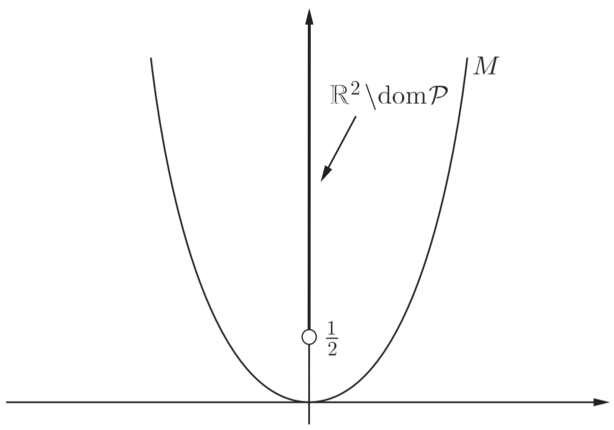

Fig. 2

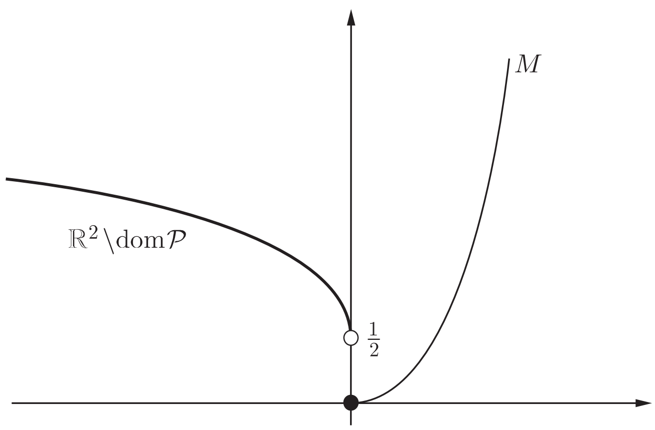

Fig. 3

(6.3) EXAMPLE. Removing the point $(0,0)$ from the set M of the preceding example we obtain a non-closed submanifold M' with

$$\operatorname{dom} \mathcal{P} = \mathbb{R}^2 \Bigg\backslash \left(\left\{ \left(0, \frac{1}{2}\right) \right\} \cup \left\{ (x,y) : x \le 0, y < \frac{3}{2}(-x)^{2/3} + \frac{1}{2} \right\} \right)$$

(see (3.14) and Fig. 4).

(6.4) EXAMPLE. Let M be the graph of $\mathbb{R} \ni x \mapsto \cos x$. Then $\operatorname{dom} \mathcal{P} = \mathbb{R}^2 \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, y) : (-1)^{k+1} y > 0\}$ (see (3.14) and Fig. 5).

(6.5) EXAMPLE. We did not manage to find the exact shape of the domain of the orthogonal projection onto $M := \{(x, e^x) : x \in \mathbb{R}\}$ (naturally,

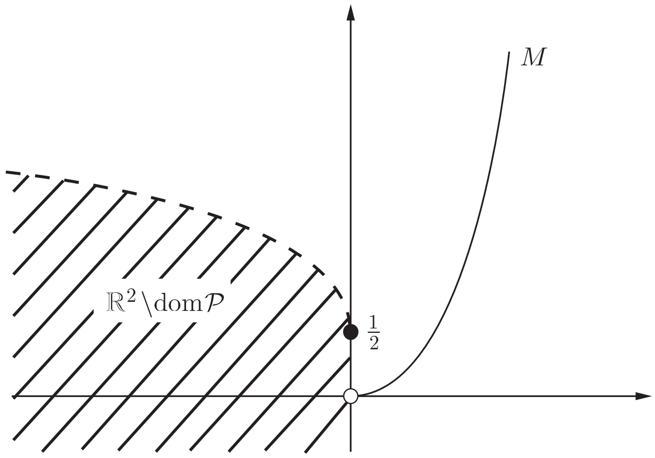

Fig. 4

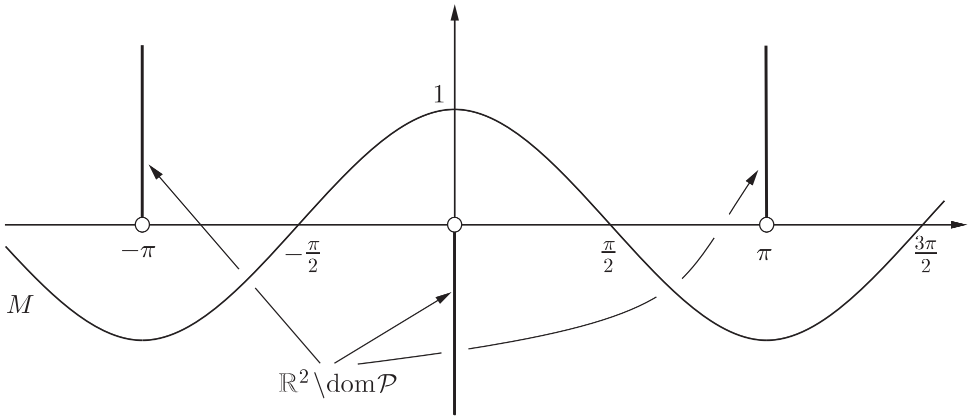

Fig. 5

M is a closed analytic submanifold). The relation

$$\mathbb{R}^2 \setminus \operatorname{dom} \mathcal{P} \subset \{(x,y) : y > 2\sqrt{2}, -\ln\sqrt{2e} - u_2(y/2\sqrt{2}) < x < -\ln\sqrt{2e} - u_1(y/2\sqrt{2})\}\,,$$

where

$$u_1(z) := z(z - \sqrt{z^2-1}) + \ln(z + \sqrt{z^2-1})\,,$$
$$u_2(z) := z(z + \sqrt{z^2-1}) - \ln(z + \sqrt{z^2-1}) \quad \text{for all } z > 1\,,$$

is all we know about $\operatorname{dom} \mathcal{P}$ (see (3.14) and (5.3); Fig. 6).

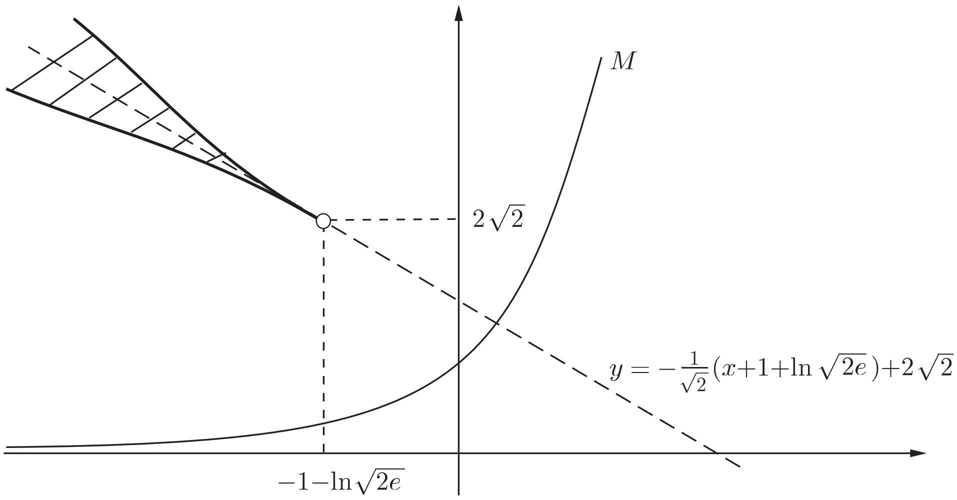

Fig. 6

1′. Proofs

(1′.1) LEMMA. *If $a \in \Omega$ and V is a neighbourhood of $\mathcal{P}(a)$ in M, then $\varrho(a, M \setminus V) > \varrho(a, M)$.*

P r o o f. If the inequality were inverse, then $\varrho(a, M\setminus V) = \varrho(a, M)$. There is a sequence $(x_\nu)_{\nu=1}^{\infty} \in (M\setminus V)^{\mathbb{N}}$ for which $|a-x_\nu| \to \varrho(a, M\setminus V) = \varrho(a, M)$ as $\nu \to \infty$. Since $|x_\nu| \le |x_\nu - a| + |a|$ $(\forall \nu)$, there are also an infinite set $A \subset \mathbb{N}$ and $b \in Z$ such that $x_\nu \to b$ as $A \ni \nu \to \infty$. Clearly, $|a-b| = \varrho(a, M)$, $b \ne \mathcal{P}(a)$, $b \notin M$ and $b \ne a$. Choosing $z \in\,]a, b] \cap \operatorname{dom} \mathcal{P}$ $(\ne \emptyset)$ we obtain

$$b - z = \frac{|b-z|}{|z-a|}(z-a)$$

and $|a-b| \le |a-\mathcal{P}(a)| \le |a-z| + |z-\mathcal{P}(z)| \le |a-z| + |z-b| = |b-a|$. This means that for $u := z-a$ and $v := \mathcal{P}(z) - z$ we have $|u|+|v| = |u+v|$ and $|v| = |z-b|$, i.e. $v = \frac{|v|}{|u|}u$. Therefore, $b - z = \frac{|v|}{|u|}u = \mathcal{P}(z) - z$, contrary to $b \notin M$. ∎

P r o o f o f T h e o r e m (1.3). Suppose that $\mathcal{P}$ is not continuous at $a \in \Omega$. There is a neighbourhood of $\mathcal{P}(a)$ on M and a sequence $(a_\nu) \in (\operatorname{dom}\mathcal{P})^{\mathbb{N}}$ with $a_\nu \to a$ as $\nu \to \infty$ and $\mathcal{P}(a_\nu) \notin V$ for all ν. Then

$$\begin{aligned} \forall \nu \in \mathbb{N}: \quad \varrho(a, M\setminus V) &\le |a - \mathcal{P}(a_\nu)| \le |a-a_\nu| + |a_\nu - \mathcal{P}(a_\nu)| \\ &= |a-a_\nu| + \varrho(a_\nu, M)\,. \end{aligned}$$

As a consequence, $\varrho(a, M\setminus V) \le \varrho(a, M)$, which contradicts Lemma (1′.1). ∎

(1′.2) Lemma. *For* $t \in]0,1]$ *define* $f_t : \operatorname{dom}\mathcal{P} \to Z$ *by* $f_t(z) := \mathcal{P}(z) + t(z - \mathcal{P}(z))$. *Then* f_t *is an injection.*

Proof. The case $t = 1$ is easy. For $t \in]0,1[$ suppose $a, b \in \operatorname{dom}\mathcal{P}$ satisfy $f_t(a) = f_t(b)$. By Remark (1.7) we can assume $f_t(a) = f_t(b) = 0$ without loss of generality. So, for $s := -t/(1-t)$ we have

$$a' := \mathcal{P}(a) = sa\,, \qquad b' := \mathcal{P}(b) = sb\,.$$

Assume that $|a| \ge |b|$ and $|a| > 0$. If a, b were linearly dependent, then we would get $b \in [0,a]$ (if not, there is $\xi \in]-1,0[$ for which $b = \xi a$; this yields $b' = s\xi a \in]0,\infty[\cdot a$ and $\varrho(a,M) \le |a - b'| = |1 - s\xi| \cdot |a| < (1+|s|\cdot|\xi|)|a| < (1-s)|a| = |a-a'| = \varrho(a,M)$). Therefore $b \in [0,a] \subset [a',a]$. By Theorem (1.5)(i), $b' = \mathcal{P}(b) = a'$. Thus $a = b$.

So it suffices to establish linear dependence of a and b. Suppose not, i.e. $(a \mid b) < |a| \cdot |b| \le |a|^2$. It follows that

$$|a|^2 - |b|^2 \ge 0 > 2\left(a + \frac{(a \mid a - b)}{t|a-b|^2}(b-a)\ \middle|\ a - b\right)$$

and for

$$x := s\left(a + \frac{(a \mid a - b)}{t|a-b|^2}(b-a)\right)$$

we get $|x-a'|^2 > |x-b'|^2$. We have $((1-t)(a-x) \mid a-b) = 0$, so $a-x \perp b-a$. Moreover, $a-x \perp x-a'$ and $a-x \perp x-b'$, because $x-a', x-b' \in \mathbb{R}\cdot(b-a)$. By Pythagoras' Theorem,

$$\begin{aligned}\varrho(a,M)^2 \le |a-b'|^2 = |a-x|^2 + |x-b'|^2 < |a-x|^2 + |x-a'|^2 \\ = |a-a'|^2 = \varrho(a,M)^2\,.\end{aligned}$$

This contradiction completes the proof. ■

Proof of Theorem (1.5). The indirect proof of the fact that $|z-a'| < |z-x|$ for all $x \in M \setminus \{a'\}$ uses the method of proof of Lemma (1′.1).

To show (ii), fix $a \in \Omega$ and a point $([a, \mathcal{P}(a)[\ni)\ z_0 := \mathcal{P}(a) + t(a - \mathcal{P}(a))$ (for some $t \in]0,1]$). The Brouwer theorem on the invariance of domain applied to the continuous injection $f_t : \Omega \ni z \mapsto \mathcal{P}(z) + t(z - \mathcal{P}(z)) \in [z, \mathcal{P}(z)] \subset \operatorname{dom}\mathcal{P}$ completes the proof. ■

Proof of Corollary (1.6). Fix $a \in \mathcal{O}$ and suppose that $|b-a| = \varrho(a,M)$ for some $b \in M \setminus \{f(a)\}$. It is clear that $a \ne b$. In view of Theorem (1.5), $]a,b] \subset \operatorname{dom}\mathcal{P}$ and $\mathcal{P}(z) = b$ for all $z \in]a,b]$. So, there is a sequence $(z_\nu) \in (]a,b] \cap (\operatorname{dom}\mathcal{P}) \cap \mathcal{O})^{\mathbb{N}}$ convergent to a. Of course, $b = \mathcal{P}(z_\nu) = f(z_\nu)$ for all ν. Since f is continuous, we obtain the equality $b = f(a)$ contradicting the choice of b. ■

2′. Proofs

P r o o f o f F a c t (2.2) is easy. ■

P r o o f o f F a c t (2.3). Fix $a \in D$. Then there is $r > \varrho(a, M) = \vartheta r$ (for some $\vartheta \in [0,1[$) such that $M \cap \overline{B}(a,r) \in \operatorname{cotop} Z$. The inclusion $B(a,(1-\vartheta)r/2) \subset D$ holds, because for $\widetilde{a} \in B(a,(1-\vartheta)r/2)$ we have $\varrho(\widetilde{a}, M) < (1+\vartheta)r/2$, $\overline{B}(\widetilde{a},(1+\vartheta)r/2) \subset \overline{B}(a,r)$ and $\overline{B}(\widetilde{a},(1+\vartheta)r/2) \cap M \in \operatorname{cotop} Z$. ■

P r o o f o f R e m a r k (2.4) rests on the simple

(2′.1) Remark. *If $\emptyset \neq \mathcal{O} \subset D$ is open, then $\mathcal{O} \cap \operatorname{dom} \mathcal{P}$ is dense in $\mathcal{O}$.*

P r o o f. Consider $G \in (\operatorname{top} \mathcal{O}) \setminus \{\emptyset\}$ and $a \in G$ $(\subset D)$. There is $a' \in M$ for which $|a - a'| = \varrho(a, M)$. Thus $]a, a'] \subset \operatorname{dom} \mathcal{P}$. Clearly, $G \cap]a, a'] \neq \emptyset$, and consequently $G \cap \mathcal{O} \cap \operatorname{dom} \mathcal{P} \neq \emptyset$. ■

(2′.2) Lemma. *If $a \in D \cap \operatorname{dom} \mathcal{P}$ and V is a neighbourhood of $\mathcal{P}(a)$ in M, then $\varrho(a, M \setminus V) > \varrho(a, M)$.*

P r o o f. For an indirect proof suppose that $\varrho(a, M \setminus V) = \varrho(a, M)$. There is a sequence $(x_\nu) \in (M \setminus V)^{\mathbb{N}}$ for which $|a - x_\nu| \to \varrho(a, M)$ as $\nu \to \infty$. It is bounded, so there is an infinite set $A \subset \mathbb{N}$ and $b \in Z \setminus \{\mathcal{P}(a)\}$ such that $x_\nu \to b$ as $A \ni \nu \to \infty$. Obviously, $|a - b| = \varrho(a, M)$. There is a radius $r > \varrho(a, M)$ such that $M \cap \overline{B}(a, r) \in \operatorname{cotop} Z$, so we have $x_\nu \in \overline{B}(a, r) \cap M$ (for almost all $\nu \in \mathbb{N}$). Hence, $b \in \overline{B}(a, r) \cap M \subset M$ and $b \neq \mathcal{P}(a)$, which contradicts $|a - b| = \varrho(a, M)$. ■

P r o o f o f R e m a r k (2.5) is analogous to the one of Theorem (1.3). ■

P r o o f o f P r o p o s i t i o n (2.6). For a given point $a \in \Omega$ there is a closed neighbourhood F of $\mathcal{P}(a)$ in M. By Lemma (1′.1) there exists r with $\varrho(a, M) < r < \varrho(a, M \setminus F)$. Clearly, $M \cap \overline{B}(a, r) = F \cap \overline{B}(a, r)$. ■

P r o o f o f T h e o r e m (2.7). Suppose $a \neq a'$. Without loss of generality we can assume $a = 0$. According to Theorem (1.5), $]0, a'] \subset \operatorname{dom} \mathcal{P}$ and $\mathcal{P}(z) = a'$ for all $z \in]0, a']$. Fix $z \in]0, a']$ and set $d := \varrho(z, M) = |z - a'| < |a'| =: r$. There is $0 < \delta < \sqrt{r(r+d)}$ for which $M \cap \overline{B}(a', \delta) \in \operatorname{cotop} Z$. We have

$$\varrho(z, M) < \sqrt{d^2 + \delta^2 \frac{r-d}{r}} =: r(z) \ (< r)\,.$$

To show that $\overline{B}(z, r(z)) \cap M \in \operatorname{cotop} Z$ (which will complete the proof) it suffices to prove that $\overline{B}(z, r(z)) \subset B(0, r) \cup \overline{B}(a', \delta)$. Indeed, let $x \in \overline{B}(z, r(z))$ with $|x| \geq r$. Then $(x \mid z) > 0$ and consequently $(x \mid a') > 0$. If

$$x' := \frac{(x \mid a')}{|a'|^2} a'$$

denotes the orthogonal projection of x onto $\mathbb{R}a'$, then

$$|x'| = \frac{|x|^2 + r^2 - |x-a'|^2}{2r}\,.$$

Further, $|x'-z| = ||x'| - (r-d)|$, since $x', z \in \mathbb{R}_+ a'$. Hence

$$\begin{aligned} r(z)^2 &\ge |x-z|^2 = |x-x'|^2 + |x'-z|^2 \\ &= (|x|^2 - |x'|^2) + |x'-z|^2 \ge \frac{r-d}{r}|x-a'|^2 + d^2\,, \end{aligned}$$

which means that indeed $\delta \ge |x-a'|$. ∎

Proof of Theorem (2.8). Define

$$f_t : \operatorname{dom}\mathcal{P} \ni z \mapsto \mathcal{P}(z) + t(z - \mathcal{P}(z)) \in Z\,.$$

It is an injection with values in $\operatorname{dom}\mathcal{P}$ (by Lemma (1′.2) and Theorem (1.5)). Moreover, $\mathcal{P}\circ f_t = \mathcal{P}$, $f_t(\Omega) \subset \Omega$ and $f_t^{-1}(w) = \frac{1}{t}(w - (1-t)\mathcal{P}(w))$.

(i)⇒(ii). In view of the Brouwer theorem on invariance of domain, $f_t(\Omega)$ is a neighbourhood of $f_t(a) = \mathcal{P}(a) + t(a - \mathcal{P}(a))$. Clearly, tf_t^{-1} is an injection on it.

(ii)⇒(i). According to the assumptions $a \in D \cap \operatorname{dom}\mathcal{P}$. There is an open neighbourhood $U \subset \Omega$ of $\widetilde{a} := f_t(a)$ such that the mapping $Q : U \ni w \mapsto \frac{1}{t}(w - (1-t)\mathcal{P}(w)) \in Z$ is a continuous injection (see Theorem (1.3)). By the Brouwer theorem, $Q(U) \in \operatorname{top} Z$ and $Q : U \to Q(U)$ is a homeomorphism. Also, $Q^{-1}(z) = f_t(z)$ and $(\mathcal{P}\circ Q^{-1})(z) = \mathcal{P}(z)$ for any $z \in f_t^{-1}(U)$ $(\subset Q(U))$. The map $f_t|_D$ is continuous by Remark (2.5), hence there is $\mathcal{O} \in \operatorname{top} Z$ for which $(f_t|_D)^{-1}(U) = \mathcal{O} \cap D \cap \operatorname{dom}\mathcal{P}$. By (2.3), $D_0 := \mathcal{O} \cap D \cap Q(U)$ is an open neighbourhood of a. It remains to show that $D_0 \subset \operatorname{dom}\mathcal{P}$. For $z \in D_0$, Remark (2′.1) enables us to choose a sequence $(z_\nu) \in (D_0 \cap \operatorname{dom}\mathcal{P})^{\mathbb{N}}$ convergent to z. Then

$$|z - (\mathcal{P}\circ Q^{-1})(z)| \underset{\nu\to\infty}{\longleftarrow} |z_\nu - (\mathcal{P}\circ Q^{-1})(z_\nu)| = |z_\nu - \mathcal{P}(z_\nu)| \underset{\nu\to\infty}{\longrightarrow} \varrho(z, M)\,.$$

Consequently, $\forall z \in D_0 : |z - (\mathcal{P}\circ Q^{-1})(z)| = \varrho(z, M)$. From Corollary (1.6) it follows directly that $D_0 \subset \operatorname{dom}\mathcal{P}$, which completes the proof. ∎

3′. Proofs

Proof of Remark (3.1) is based on the necessary condition for a local minimum of a differentiable function. ∎

(3′.1) Lemma. *If $M \subset Z$ is a submanifold and $F \subset M$ a compact set, then*

$$\sup\left\{\varrho\left(\frac{z-y}{|z-y|}, T_y M\right) : z \in M, y \in F, 0 < |z-y| < \delta\right\} \to 0 \quad \text{as } \delta \to 0\,.$$

P r o o f. It is sufficient to show that this condition holds locally on M, i.e. $\forall a \in M\ \exists \mathcal{O} \in \operatorname{top} M : \mathcal{O}$ is a neighbourhood of a and

$$\sup\left\{\varrho\left(\frac{z-y}{|z-y|}, T_y M\right) : y, z \in \mathcal{O},\ 0 < |z-y| < \delta\right\} \to 0 \quad \text{as } \delta \to 0 .$$

Fix $a \in M$ and an inverse chart $f : H \nrightarrow M$ for which $f(a_0) = a$ (where H is a linear subspace of a Euclidean space X, with $\dim X = \dim Z$; $\nrightarrow$ denotes a partial mapping). We can assume, diminishing the domain if necessary, that f^{-1} satisfies the Lipschitz condition on its domain. For a fixed convex compact neighbourhood $V \subset \operatorname{dom} f$ of a_0, the set $\mathcal{O} := f(V)$ will prove to be the suitable neighbourhood of a. The mapping

$$\alpha :]0,\infty[\ \ni \delta \mapsto \sup\{|f'(u) - f'(v)| : u, v \in V,\ |u-v| \le \delta\}$$

is increasing and $\lim_{\delta\to 0} \alpha(\delta) = 0$. Define $\vartheta := \inf_{h\in V} \min\{|d_h f(\xi)| : |\xi| = 1\}$ $(= |d_{h_1} f(\xi_0)| > 0$ for some $h_1 \in V$ and $\xi_0 \in \{z : |z| = 1\})$. Let L be the Lipschitz constant for f^{-1}. Fix $\varepsilon \in]0,1]$ and $r > 0$ for which $\alpha(r) \le \varepsilon\vartheta/2$. For $\delta \in]0, r/L[$ and $z, z_0 \in f(V)$ with $0 < |z - z_0| \le \delta$, set $h_0 := f^{-1}(z_0)$, $h := f^{-1}(z)$. We have $|h - h_0| \le r$ and, by the Lagrange theorem,

$$|f(h) - f(h_0) - d_{h_0} f(h-h_0)| \le \alpha(r)|h-h_0| \le |h-h_0| \frac{\varepsilon\vartheta}{2} .$$

In particular, for $\varepsilon = 1$ we obtain

$$\frac{\vartheta}{2} \ge \left|\frac{f(h)-f(h_0)}{|h-h_0|} - d_{h_0} f\left(\frac{h-h_0}{|h-h_0|}\right)\right| \ge \vartheta - \frac{|f(h)-f(h_0)|}{|h-h_0|}$$

and consequently

$$\frac{|h-h_0|}{|f(h)-f(h_0)|} \le \frac{2}{\vartheta} .$$

Going back to the general case of $\varepsilon > 0$ and noticing that

$$\begin{aligned}\frac{z-z_0}{|z-z_0|} &= \frac{|h-h_0|}{|f(h)-f(h_0)|} d_{h_0} f\left(\frac{h-h_0}{|f(h)-f(h_0)|}\right) \\ &\quad + \frac{|h-h_0|}{|f(h)-f(h_0)|}\left(\frac{f(h)-f(h_0)}{|h-h_0|} - d_{h_0} f\left(\frac{h-h_0}{|f(h)-f(h_0)|}\right)\right)\end{aligned}$$

(the first component is in $T_{z_0} M$) we come to the desired conclusion:

$$\begin{aligned}&\varrho\left(\frac{z-z_0}{|z-z_0|}, T_{z_0} M\right) \\ &\qquad \le \frac{|h-h_0|}{|f(h)-f(h_0)|}\left(\frac{f(h)-f(h_0)}{|h-h_0|} - d_{h_0} f\left(\frac{h-h_0}{|f(h)-f(h_0)|}\right)\right) \\ &\qquad \le \frac{2}{\vartheta} \cdot \frac{\varepsilon \cdot \vartheta}{2} = \varepsilon . \ \blacksquare\end{aligned}$$

Reasoning by reductio ad absurdum it is easy to derive from Lemma (3′.1) the following

(3′.2) Lemma. *If* $M \subset Z$ *is a submanifold and* $F \subset M$ *is compact*, *then* $\exists r > 0\ \forall z \in F : M \cap (\overline{K}_z(r) \setminus \{z\}) = \emptyset$.

(3′.3) Lemma. *Let* $S := \{z \in Z : |z| = 1\}$. *Fix subspaces* X, Y *in* Z. *Denote by* $d(X,Y)$ *their distance in the Hausdorff metric* (see (3.4)). *For* $\emptyset \neq A, B \subset Z$ *put* $\varrho(A,B) := \inf\{|a-b| : a \in A, b \in B\}$. *Consider* $R, \eta > 0$ *for which* $Rd(X,Y) < \eta$ *and a point* $a \in B_R \cap X$, *where* $B_R := \{z \in Z : |z| \le R\}$. *Assume that* $\varrho([-Ry, Ry], a + [-Rx, Rx]) \ge \eta$ *whenever* $y \in S \cap Y$, $x \in S \cap X$ *and* $|x - y| \le d(X,Y)$. *Then* $(B_R \cap Y) \cap (a + (B_R \cap X)) = \emptyset$.

Proof. Let $\Pi : Z \to Y$ stand for the linear orthogonal projection. There is $y \in S \cap Y$ such that $\Pi(a) \in [-Ry, Ry]$, and $x \in S \cap X$ for which $|x - y| = \varrho(y, S \cap X)$ $(\le d(X,Y))$. Hence $|a - \Pi(a)| \ge \eta$. Therefore $(B_R \cap Y) \cap (a + (B_R \cap X)) = \emptyset$, because if $a + tx \in Y \cap (a + (B_R \cap X))$ (for some $|t| \le R$ and $x \in S \cap X$) then $\Pi(a + tx) = a + tx$ implies

$$\eta \le |\Pi(a) - a| \le R|x - \Pi(x)| \le Rd(X,Y)\,,$$

which contradicts our assumptions. ∎

Proof of Theorem (3.2). To prove (i) we only need to consider the case $\dim M < \dim Z$. Let $L > 0$ be the Lipschitz constant for T on F. Fix $\vartheta, c > 0$ so that $\vartheta < c$ and $\vartheta + c < 1$. From Lemma (3′.1) it follows that there is $\delta \in \,]0, \vartheta/L]$ for which

$$\sup\left\{\varrho\left(\frac{z-y}{|z-y|}, T_y M\right) : y, z \in F,\ 0 < |z - y| < \delta\right\} \le \frac{1 + (c+\vartheta)^2}{2(1 + c + \vartheta)}\,.$$

If we show that $K_{z_1}(\vartheta/L) \cap K_{z_2}(\vartheta/L) = \emptyset$ whenever $z_1, z_2 \in F$ and $0 < |z_1 - z_2| \le \delta$, then $r := \delta/2$ will suit the assertion.

First, we put $S := \{z \in Z : |z| = 1\}$ and prove the auxiliary fact:

(3′.4) $\varrho\left(z_1 + \frac{\vartheta}{L}[-\zeta_1, \zeta_1], z_2 + \frac{\vartheta}{L}[-\zeta_2, \zeta_2]\right) \ge c|z_1 - z_2|$ whenever $z_1, z_2 \in F$, $0 < |z_1 - z_2| < \delta$, $\zeta_i \in S \cap T_{z_i}^{\perp}$ $(i = 1, 2)$ and $|\zeta_1 - \zeta_2| \le d(T_{z_1}^{\perp}, T_{z_2}^{\perp})$ (of course, $T_z^{\perp} := (T_z M)^{\perp}$).

If (3′.4) were false we could find $t_1, t_2 \in [-\vartheta/L, \vartheta/L]$ for which $|z_1 + t_1\zeta_1 - z_2 - t_2\zeta_2| < c|z_1 - z_2|$ and hence $c|z_1 - z_2| > |(z_1 - z_2) + (t_1 - t_2)\zeta_2| - t_1|\zeta_1 - \zeta_2|$. The following inequalities hold:

$$|t_i(\zeta_1 - \zeta_2)| \le \frac{\vartheta}{L} d(T_{z_1}^{\perp}, T_{z_2}^{\perp}) = \frac{\vartheta}{L} d(T_{z_1}, T_{z_2}) \le \vartheta|z_1 - z_2|$$

(see (3.5)). Putting $s := (t_2 - t_1)/|z_2 - z_1|$ we obtain

$$\left|\frac{z_2 - z_1}{|z_1 - z_2|} + s\zeta_2\right| < c + \vartheta \quad \text{and} \quad |s| \le c + \vartheta + 1\,.$$

Combining these two inequalities we get

$$(c+\vartheta)^2 > 1 + s^2 + 2s\left(\frac{z_2 - z_1}{|z_1 - z_2|}\;\middle|\;\zeta_2\right) \geq 1 - 2|s|\cdot\left|\left(\frac{z_2 - z_1}{|z_1 - z_2|}\;\middle|\;\zeta_2\right)\right|$$
$$\geq 1 - 2(c+\vartheta+1)\varrho\left(\frac{z_2 - z_1}{|z_1 - z_2|}, T_{z_2}\right)$$
$$\geq 1 - 2(c+\vartheta+1)\frac{1-(c+\vartheta)^2}{2(c+\vartheta+1)} = (c+\vartheta)^2\,.$$

This contradiction proves (3′.4).

Now, to show (i) fix $z_1, z_2 \in F$ such that $0 < |z_1 - z_2| \leq \delta$. We can apply Lemma (3′.3) to $X := T_{z_1}^{\perp}$, $Y := T_{z_2}^{\perp}$, $R := \vartheta/L$, $\eta := c|z_1 - z_2|$ and $a := z_2 - z_1$, for we have

$$d(T_{z_1}, T_{z_2}) \leq L|z_1 - z_2| = \frac{c|z_1 - z_2|}{\vartheta/L}\cdot\frac{\vartheta}{c} < \frac{\eta}{R}\,,$$

and in view of (3′.4) all the assumptions of Lemma (3′.3) hold. Thus indeed $K_{z_1}(\vartheta/L) \cap K_{z_2}(\vartheta/L) = \emptyset$.

To prove (ii) consider a compact set $\widetilde{F} \subset M$ for which $F \subset \mathrm{int}_M \widetilde{F}$. Fix $z_0 \in F$ and a number $0 < r < \varrho(z_0, M\backslash \mathrm{int}_M \widetilde{F})$ satisfying (i). For $z \in K_{z_0}(r)$ and $\delta := |z - z_0|$ $(< r)$ we have $z_0 \in E := \overline{B}(z,\delta) \cap M = \overline{B}(z,\delta) \cap F$. Since E is compact, there is $z_1 \in E$ $(\subset M)$ for which $|z_1 - z| = \varrho(z, E) = \varrho(z, M)$. But $|z - z_1| \leq \delta < r$, thus, by Remark (3.1), $z \in K_{z_1}(r)$ $(\cap K_{z_0}(r))$ and consequently $z_1 = z_0$. ■

Example 3.7 is easy to analyze. ■

(3′.5) Lemma. *Consider a submanifold $M \subset Z$ and a continuous function $r : M \to\,]0,\infty[$ such that for all $z \in M$, $K_z(r(z)) \subset \mathrm{dom}\,\mathcal{P}$ and $\mathcal{P}|_{K_z(r(z))} \equiv z$. Then $M \subset \bigcup_{z\in M} K_z(r(z)) \in \mathrm{top}\, Z$.*

Proof. Fix $z_0 \in M$, $x_0 \in K_{z_0}(r(z_0))$ and $\vartheta \in \mathbb{R}$ such that $|z_0 - x_0| < \vartheta < r(z_0)$. For a fixed compact neighbourhood $F \subset M$ of z_0 we put $V := F \cap \{z \in M : r(z) \geq \vartheta\}$. Compactness of ∂V enables us to choose $y_0 \in \partial V$ such that $\varrho(x_0, \partial V) = |x_0 - y_0|$ $(> |x_0 - z_0|)$. We claim that $B(x_0, s) \subset \bigcup_{z\in M} K_z(r(z))$, where $s := \min\{\delta/2, \vartheta - |z_0 - x_0|\}$ and $\delta := \varrho(x_0, \partial V) - |z_0 - x_0|$. Indeed, fix $x \in B(x_0, s)$ and $z \in V$ for which $|x - z| = \varrho(x, V)$. Then $z \in M' := \mathrm{int}_M V$, for otherwise, i.e. if $z \in \partial V$, we would have $|x - z_0| < -\delta/2 + \varrho(x_0, \partial V) \leq |x_0 - z| - \delta/2 < |x - z|$, contrary to the choice of z. Hence, by Remark (3.1), $x - z \perp T_z M' = T_z M$. Moreover, $|x - z| \leq |x - x_0| + |x_0 - z_0| < \vartheta - |z_0 - x_0| + |z_0 - x_0| \leq r(z)$, so $x \in K_z(r(z))$. ■

(3′.6) Theorem. *Consider a C^k-submanifold $M \subset Z$ $(k \in \{1, 2, \ldots, \infty\})$ satisfying condition* (3.3). *Then there exists a C^k-function $r : M \to\,]0,\infty[$ such that for all $z \in M$, $K_z(r(z)) \subset \mathrm{dom}\,\mathcal{P}$ and $\mathcal{P}|_{K_z(r(z))} \equiv z$. Moreover,*

$\bigcup_{z\in M} K_z(r(z)) \in \operatorname{top} Z$ *and*

$$\forall z,y\in M:\quad (z\neq y \Rightarrow K_z(r(z))\cap K_y(r(y))=\emptyset)\,.$$

Proof. There exists a family $\{F_i\}_{i=1}^{\infty}$ of compact subsets of M such that $M=\bigcup_{i=1}^{\infty} F_i$ and $F_i\subset \operatorname{int}_M F_{i+1}$ $(i=1,2,\ldots)$. Also there are C^k-functions $\lambda_i: M\to[0,1]$ $(i=1,2,\ldots)$ with $\lambda_i|_{F_i}\equiv 1$ and $\operatorname{supp}\lambda_i\subset \operatorname{int}_M F_{i+1}$ $(i=1,2,\ldots)$. Put $\lambda_0\equiv 0$. By Theorem (3.2), for any $i\in\{1,2,\ldots\}$ there is $r_i>0$ such that for all $z\in F_{i+1}$, $K_z(r_i)\subset \operatorname{dom}\mathcal{P}$ and $\mathcal{P}|_{K_z(r_i)}\equiv z$. Clearly, we can assume that $r_1\geq r_2\geq\ldots$ For a fixed $i\in\{1,2,\ldots\}$ define $h_i:(\operatorname{int}_M F_{i+1})\setminus\operatorname{supp}\lambda_{i-1}\to\mathbb{R}$ by $h_i(z):=r_i\lambda_i(z)+r_{i+1}(1-\lambda_i(z))$. Obviously, this is a C^k-function. If $i\neq j$ and $z\in(\operatorname{dom}h_i)\cap(\operatorname{dom}h_j)$, then $h_i(z)=h_j(z)$, so $r:=\bigcup_{i=1}^{\infty}h_i\subset M\times\mathbb{R}$ is a C^k-function on M. Moreover, for any $z\in\operatorname{dom}h_i$ we have $0<r(z)\leq r_i$, so $r:M\to\mathbb{R}$ satisfies the desired condition. ∎

Proof of Theorem (3.8). Theorem (3′.6) ensures the existence of a continuous function $r: M\to\,]0,\infty[$ such that for all $x\in M$, $K_x(r(x))\subset \operatorname{dom}\mathcal{P}$ and $\mathcal{P}|_{K_x(r(x))}\equiv x$. Fix $\vartheta\in\,]0,r(a)[$. There is $s>0$ for which $B(a,s)\cap M\subset\{x\in M: r(x)>\vartheta\}$. If $r_0:=\min\{s,\vartheta/2\}$, then

$$\mathcal{O}:=\bigcup_{x\in M\cap B(a,r_0)} K_x(r_0)=\bigcup_{x\in M\cap\mathcal{O}} K_x(r_0)$$

is an open neighbourhood of a in Z (see (3′.5)). For any $z\in\mathcal{O}$ there exists $x\in\mathcal{O}\cap M$ such that $z\in K_x(r_0)$, thus $\mathcal{O}\subset\operatorname{dom}\mathcal{P}$. The remaining assertion results from the fact that any two distinct elements of the family $\{K_x(r_0)\}_{x\in\mathcal{O}\cap M}$ are disjoint. ∎

Proof of Corollary (3.9). Theorem (3.8) implies the existence of $V\in\operatorname{top}Z$ for which $\mathcal{P}(a)\in V\subset\operatorname{dom}\mathcal{P}$ and $\forall z\in V\ \forall x\in V\cap M: (z-x\perp T_xM\Rightarrow \mathcal{P}(z)=x)$. For some $\lambda\in\,]0,1[$ we have $\widetilde{a}:=\mathcal{P}(a)+\lambda(a-\mathcal{P}(a))\in V$. The injection $f_\lambda:\operatorname{dom}\mathcal{P}\ni z\mapsto \mathcal{P}(z)+\lambda(z-\mathcal{P}(z))$ is continuous (see (1.3) and (1′.2)); moreover, $f_\lambda(a)\in V\cap f_\lambda(\Omega)\in\operatorname{top}Z$. Hence, there exists $\mathcal{O}\in\operatorname{top}T^{\perp}_{\mathcal{P}(a)}M$ such that $0\in\mathcal{O}$ and $\widetilde{a}+\lambda\mathcal{O}\subset V\cap f_\lambda(\Omega)$. Applying Theorem (3.8) to a fixed $u\in\mathcal{O}$ we obtain $\mathcal{P}(\widetilde{a}+\lambda u)=\mathcal{P}(a)$. Since $a+u=f_\lambda^{-1}(\widetilde{a}+\lambda u)\in\Omega$, it follows that $\mathcal{P}(a+u)=\mathcal{P}(a)$, completing the proof. ∎

Proof of Corollary (3.10). Theorem (3.8) lets us choose a set $V\in\operatorname{top}Z$ for which $\mathcal{P}(a)\in V\subset\operatorname{dom}\mathcal{P}$ and $\forall z\in V\ \forall x\in V\cap M: (z-x\perp T_xM\Rightarrow\mathcal{P}(z)=x)$. Assume that $\mathcal{P}(a)\neq a$ and fix $t\in\,]0,1]$ such that $\widetilde{a}:=\mathcal{P}(a)+t(a-\mathcal{P}(a))\in V$. Define $\phi_t:\mathcal{O}\ni z\mapsto\phi(z)+t(z-\phi(z))\in Z$ and $G:=\phi_t^{-1}(V)\cap\phi^{-1}(V)$; obviously, $a\in G$. By the choice of V we have $\mathcal{P}\circ\phi_t(z)=\phi(z)$ for any $z\in G$. In order to show injectivity of $\phi_t|_G$ consider the map

$$Q : \operatorname{dom} \mathcal{P} \ni v \mapsto \frac{1}{t}(v - (1-t)\mathcal{P}(v)).$$

For any $z \in G$ we have $Q(\phi_t(z)) = z$, so $\phi_t|_G$ (and surely $Q|_{\phi_t(G)}$) is an injection. Hence, by the Brouwer theorem, $\phi_t(G)$ is open. Therefore, $\widetilde{a} \in \phi_t(G) \subset \Omega$, which together with injectivity of $Q|_{\phi_t(G)}$ and Theorem (2.8) means that $a \in \Omega$.

To show that $\phi = \mathcal{P}$ in some neighbourhood of a—and end the proof in this way—it suffices to prove that $f_t = \phi_t$ in this neighbourhood, where $f_t : \operatorname{dom} \mathcal{P} \ni z \mapsto \mathcal{P}(z) + t(z - \mathcal{P}(z))$. Indeed, for every z from the open set $G_0 := G \cap (f_t|_\Omega)^{-1}(\phi_t(G))$ we have $Q \circ \phi_t(z) = Q \circ f_t(z)$, which, by injectivity of Q on the set $\phi_t(G)$ ($\supset \phi_t(G_0), f_t(G_0)$), leads to the conclusion that $f_t|_{G_0} = \phi_t|_{G_0}$, and consequently $\mathcal{P}|_{G_0} = \phi|_{G_0}$. ∎

(3′.7) THEOREM. *Let $M \subset Z$ be a C^k-submanifold, $k \in \{2, \ldots, \infty, \omega\}$. Fix $z_0 \in \operatorname{dom} \mathcal{P}$ and consider an inverse chart $f : \mathbb{R}^n \twoheadrightarrow M$ for which $f(x_0) = \mathcal{P}(z_0)$. Then the following conditions are equivalent*:

(i) $z_0 \in \Omega$;

(ii) *M is closed near z_0* (see (2.1)) *and the matrix* (3.12) *is nonsingular.*

Moreover, if one of these conditions is satisfied, then $\mathcal{P}$ is of class C^{k-1} in a neighbourhood of z_0 and $\operatorname{im} d_{z_0}\mathcal{P} = T_{\mathcal{P}(z_0)}M$, $\ker d_{z_0}\mathcal{P} = T^{\perp}_{\mathcal{P}(z_0)}M$.

P r o o f. (i)⇒(ii). According to Proposition (2.6), M is closed near z_0. Suppose, contrary to our claim, that $\det A_f(z_0) = 0$. This enables us to choose $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$ and

$$\forall i: \quad \sum_j \left(\frac{\partial f}{\partial x_i}(x_0) \,\middle|\, \frac{\partial f}{\partial x_j}(x_0) \right) \xi_j = \sum_j \left(\mathcal{P}(z_0) - z_0 \,\middle|\, \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right) \xi_j .$$

By the Taylor formula,

$$\frac{f(x_0 + t\xi) - f(x_0) - d_{x_0} f(t\xi)}{t^2} - \frac{1}{2t^2}\, \underline{d}^2_{x_0} f(t\xi) \underset{t \to 0}{\longrightarrow} 0,$$

where $\underline{d}^2_{x_0} f(t\xi) := d^2_{x_0} f(t\xi, t\xi)$. Computations using Remark (3.1) lead to the conclusion that

$$\frac{(z_0 - \mathcal{P}(z_0) \mid f(x_0 + t\xi) - f(x_0))}{t^2} - \frac{1}{2}|d_{x_0} f(\xi)|^2 \underset{t \to 0}{\longrightarrow} 0.$$

Corollary (3.9) gives $\varepsilon > 0$ such that $\mathcal{P}(\widetilde{z}_0) = \mathcal{P}(z_0)$ for $\widetilde{z}_0 := z_0 + \varepsilon(z_0 - \mathcal{P}(z_0))$. So for all $x \in \operatorname{dom} f$ we obtain

$$\begin{aligned} |\widetilde{z}_0 - \mathcal{P}(z_0)|^2 &\le |\widetilde{z}_0 - f(x)|^2 \\ &= |\widetilde{z}_0 - \mathcal{P}(z_0)|^2 + 2(1+\varepsilon)(z_0 - \mathcal{P}(z_0) \mid f(x_0) - f(x)) \\ &\quad + |f(x_0) - f(x)|^2, \end{aligned}$$

which for $x = x_0 + t\xi$ implies

$$2(1+\varepsilon)\frac{(z_0 - \mathcal{P}(z_0) \mid f(x_0+t\xi) - f(x_0))}{t^2} \le \left|\frac{f(x_0+t\xi)-f(x_0)}{t}\right|^2$$

and further, as t tends to zero,

$$\varepsilon|\partial^\xi f(x_0)| \le 0$$

($\partial^\xi f(x_0)$ is the Gateaux derivative of f). This means that $d_{x_0}f$ is not a monomorphism, which is impossible.

(ii)⇒(i). Consider the C^k-mapping $B : Z \times \operatorname{dom} f \to \mathbb{R}^n$,

$$B(z,x) := \left(\left(f(x) - z \;\middle|\; \frac{\partial f}{\partial x_1}(x)\right), \ldots, \left(f(x) - z \;\middle|\; \frac{\partial f}{\partial x_n}(x)\right)\right).$$

Notice that for (z_0, x_0) ($\in \operatorname{dom} B$) the differential $d_{x_0}B(z_0,\cdot)$ is an automorphism of $\mathbb{R}^n$ (for $A_f(z_0)$ is its matrix in the canonical basis of $\mathbb{R}^n$). Since $B(z_0,x_0) = 0$, there exists a C^{k-1}-map $\psi : Z \nrightarrow \mathbb{R}^n$ such that $\psi(z_0) = x_0$ and $\psi \subset \{(z,x) : B(z,x) = 0\}$. Its differential $d_{z_0}\psi$ is an epimorphism. Therefore, the mapping $\phi := f \circ \psi$, of class C^{k-1}, satisfies $\operatorname{im} d_{z_0}\phi = T_{z_0}M$. Obviously, for all $i \in \{1,\ldots,n\}$, $\phi(z) - z \perp \frac{\partial f}{\partial x_i}(\phi(z))$ (see (3.1)), so ϕ satisfies the assumptions of Corollary (3.10) (see (3.6)). Thus $z_0 \in \Omega$ (and $\phi = \mathcal{P}$ in a neighbourhood of z_0), which is our claim.

Now assume that either (i) or (ii) is satisfied. Repeating the construction from the proof of (ii)⇒(i) we conclude that $\mathcal{P}$ ($= \phi$) is of class C^{k-1} in a neighbourhood of z_0. We also know that $\operatorname{im} d_{z_0}\mathcal{P} = T_{z_0}M$. In order to find $\ker d_{z_0}\mathcal{P}$ we choose $\mathcal{O} \in \operatorname{top} T^\perp_{\mathcal{P}(z_0)}$ such that $0 \in \mathcal{O}$ and $\mathcal{P} \equiv \mathcal{P}(z_0)$ on $z_0 + \mathcal{O}$ (see (3.9)). For $u \in T^\perp_{\mathcal{P}(z_0)}$ such that $tu \in \mathcal{O}$, we have

$$0 = \frac{\mathcal{P}(z_0+tu) - \mathcal{P}(z_0)}{t} \underset{t\to 0}{\longrightarrow} \partial^u\mathcal{P}(z_0)\,.$$

Thus $T^\perp_{\mathcal{P}(z_0)} \subset \ker d_{z_0}\mathcal{P}$. Also $\dim T^\perp_{\mathcal{P}(z_0)} = \dim\ker d_{z_0}\mathcal{P}$, so these spaces are indeed equal. ∎

Proof of Theorem (3.11). This follows directly from Theorem (3′.7). ∎

Proof of Theorem (3.13). (a) Fix $(a,a') \in Z \times M$ such that $|a-a'| = \varrho(a,M)$ and an inverse chart $f : \mathbb{R}^n \nrightarrow M$ of class C^2 for which $a' \in \operatorname{im} f$. For any $z \in\,]a,a']$ consider the matrix (3.12) and the polynomial

$$w : [0,1[\; \ni t \mapsto \det A_f(a' + t(a-a'))\,.$$

Clearly, $w(0) \ne 0$; thus $\#\{w = 0\} < \infty$. There is $\delta > 0$ such that $w(t) \ne 0$ for all $t \in\,]1-\delta,1[$. Fix $t \in\,]1-\delta,1[$. Since M is closed near $z_t := a' + t(a-a')$ (see (2.7)), $z_t \in \Omega$ (see (3.11)). We conclude from Theorem (1.5) that $[z_t,a'[\; \subset \Omega$ for all $t \in\,]1-\delta,1[$. Consequently, $]a,a'] \subset \Omega$.

(b) Let $a \in \operatorname{dom}\mathcal{P}$ and $a' \in M$.

(i)⇒(ii). This results from (a) and Remark (3.1).

(ii)⇒(i). Assume that $]a, a'] \subset \Omega$ and $a - a' \perp T_{a'}M$. The set

$$I := \{t \in [0, 1[\, : \mathcal{P}(a' + t(a - a')) = a'\}$$

is non-empty ($0 \in I$) and closed in $[0, 1[$. It is also open in $[0, 1[$, because for fixed $t \in I$ and $x := a' + t(a' - a)$ ($\in \Omega$) one can choose $\delta > 0$ such that $\mathcal{P}(a' + (t+\delta)(a - a')) = \mathcal{P}(\delta(a - a') + x) = a'$ (see (3.9)), which means that $[0, \min\{1, t + \delta\}[\, \subset I$. Hence $I = [0, 1[$, i.e. $\mathcal{P}(z) = a'$ for any $z \in\,]a, a']$. From this we deduce that $\varrho(a, M) = |a - a'|$ and $\mathcal{P}(a) = a'$. ■

Proof of Corollary (3.14). Fix $z \in \overline{\operatorname{dom}\mathcal{P}}$ and a sequence $(z_\nu) \in (\operatorname{dom}\mathcal{P})^{\mathbb{N}}$ convergent to z. By Theorem (3.13), $]z_\nu, \mathcal{P}(z_\nu)] \subset \Omega$ for any $\nu \in \mathbb{N}$. Define

$$x_\nu := z_\nu + \frac{1}{\nu}(\mathcal{P}(z_\nu) - z_\nu) \in \Omega\,, \qquad \nu = 1, 2, \ldots$$

This sequence is convergent to z, since $(|z_\nu - \mathcal{P}(z_\nu)|)_{\nu=1}^{\infty}$ is bounded. Hence $\overline{\operatorname{dom}\mathcal{P}} \subset \overline{\Omega}$ ($\subset \overline{\operatorname{dom}\mathcal{P}}$). ■

4′. Proofs

Proof of Theorem (4.1). We only need to show (i) (see (3′.7)). It follows from the equalities $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$ and $d_z\mathcal{P} \circ d_z\mathcal{P} = d_z\mathcal{P}$ ($\forall z \in M$). ■

(4′.1) Lemma. *Let $M \subset Z$ be a C^2-submanifold. Fix $z \in \Omega$ and an inverse chart $f : \mathbb{R}^n \twoheadrightarrow M$ of class C^2 such that $f(x) = \mathcal{P}(z)$. Let $\widetilde{A}_f(z)$ stand for the endomorphism of $\mathbb{R}^n$ given by the matrix* (3.12) *in the canonical basis. Then*

$$\forall \zeta \in Z: \quad d_z\mathcal{P}(\zeta) = (d_x f \circ \widetilde{A}_f(z)^{-1})\left(\left(\zeta \,\middle|\, \frac{\partial f}{\partial x_1}(x)\right), \ldots, \left(\zeta \,\middle|\, \frac{\partial f}{\partial x_n}(x)\right)\right).$$

Proof. This follows from the proof of Theorem (3′.7) ((ii)⇒(i)). ■

The following two definitions are useful in formulating and proving the next theorems:

Let X, Y be finite-dimensional real linear spaces. Fix $z \in X$, a subspace H in X and a mapping $g : X \twoheadrightarrow Y$ of a neighbourhood of z.

We say that *g is differentiable at z with respect to H* iff $g \circ \tau_z \circ \iota_H$ is differentiable at zero (where $\iota_H : H \ni h \mapsto h \in X$, while τ_z denotes translation by z). We write

(4′.2) $$^{H}d_z g := d_0(g \circ \tau_z \circ \iota_H)\,.$$

Next, we say that a sequence of linear operators $\alpha_\nu : X \twoheadrightarrow Y$ with non-zero domains ($\nu = 1, 2, \ldots$) *is convergent* to a non-zero linear operator

$\alpha : X \nrightarrow Y$ iff $\alpha_\nu \to \alpha$ as $\nu \to \infty$ in $X \times Y$ with respect to the Hausdorff metric.

We will use the following properties of this kind of convergence:

(4′.3) *Let* $l \in L(X,Y)$ *and* $(l_\nu) \in L(X,Y)^{\mathbb{N}}$. *Then* $l_\nu \to l$ *in the Banach space* $L(X,Y)$ *iff* $l_\nu \to l$ *in the Hausdorff metric in* $X \times Y$.

(4′.4) *The map*

$$\left\{(A,B) : \begin{array}{l} A, B \text{ are non-zero linear subspaces} \\ \text{of } X \text{ with } A \cap B = 0 \end{array}\right\} \ni (A,B) \mapsto A + B \in \{E \neq 0 : E \text{ is a linear subspace of } X\}$$

is continuous.

(4′.5) *Let* $L : Y \to W$ *be a linear operator with* W *a finite-dimensional space, and let* $\alpha : X \nrightarrow Y$ *be a partial linear operator. Also, let* $\alpha_\nu : X \nrightarrow Y$ *and* $L_\nu : Y \to W$ *be linear operators* $(\forall \nu \in \mathbb{N})$. *If* $\alpha_\nu \to \alpha$ *and* $L_\nu \to L$ *as* $\nu \to \infty$, *then* $L_\nu \circ \alpha_\nu \to L \circ \alpha$.

(4′.6) *For a linear subspace* A *of* X *let* $\iota_A : A \to X$ *denote the canonical inclusion. Then the map*

$$\{\text{non-zero linear subspaces of } X\} \ni B \mapsto \iota_B \in \{\text{non-zero linear subspaces of } X^2\}$$

is continuous.

The proofs of the above facts are based on the following criterion:

(4′.7) *A function* $f : X \to \{\alpha : Y \nrightarrow W \mid \{(0,0)\} \neq \alpha \text{ is linear}\}$ *is continuous at* $x_0 \in X$ *iff*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X : \{|x - x_0| \le \delta, v_x \in S \cap \operatorname{dom} f(x), v \in S \cap \operatorname{dom} f(x_0), |v - v_x| \le \delta\} \Rightarrow |f(x)(v_x) - f(x_0)(v)| \le \varepsilon,$$

where S *stands for the unit sphere in* Y. (Clearly, the last condition is independent of the choice of norms in X, Y, W.)

The criterion (4′.7) is a consequence of the following facts:

(4′.8) *Let* $C \subset X$ *be a cone in* X, *i.e. by definition*, $]0, \infty[\cdot C \subset C$. *If* C *is open, then the class*

$$\widetilde{C} := \{E \in \mathcal{G}_p(X) : E \setminus \{0\} \subset C\}$$

is open in the Grassmann manifold $\mathcal{G}_p(X)$ *of* p*-dimensional subspaces of* X.

(4′.9) *Let a family* $\{C_\nu\}_{\nu=1}^\infty$ *of closed cones be a base of cone neighbourhoods of a subspace* $E \in \mathcal{G}_p(X)$, *i.e. by definition*:

- $\forall \nu : C_{\nu+1} \subset C_\nu$;

- $\forall\nu : E\setminus\{0\}\subset \operatorname{int} C_\nu$;
- $\bigcap_{\nu=1}^{\infty} C_\nu = E$.

Then $\{\widetilde{\operatorname{int} C_\nu}\}_{\nu=1}^{\infty}$ *is a neighbourhood base of* E *in the topological space* $\mathcal{G}_p(X)$.

Let $\alpha_0 := f(x_0) : Y \nrightarrow W$ be a partial linear operator from the criterion (4′.7), while $Q : Y \to \operatorname{dom}\alpha_0$ the linear orthogonal projection in the sense of a fixed inner product in Y. Then the family

$$\begin{aligned} C_\nu := \{(y,w)\in Y\times W :&\\ &||Q(y)|y-|y|Q(y)| + ||Q(y)|w-\alpha_0(|y|Q(y))| \le \tfrac{1}{\nu}|y|\,|Q(y)|\}\\ &\cap\{(y,w) : |w-\alpha_0(Q(y))| \le \tfrac12|y| \le |Q(y)|\}\end{aligned}$$

($\nu = 1,2,\ldots$) of closed cones is a base of cone neighbourhoods of the subspace $\alpha_0\subset Y\times W$. The proof of (4′.7) rests on this fact.

(4′.10) LEMMA. *Let* $M\subset Z$ *be a* C^2*-submanifold. Let* $z\in\Omega$ *and let* $f:\mathbb{R}^n \nrightarrow M$ *be an inverse chart for which* $f(x)=\mathcal{P}(z)$. *Fix* $\zeta\in Z$ *and put* $\psi := f^{-1}\circ\mathcal{P}_\Omega$. *Then* $\partial^\zeta\psi$ *is differentiable at* z *with respect to* $T^\perp_{\mathcal{P}(z)}$ *and for* $v\in T^\perp_{\mathcal{P}(z)}$,

$$^{T^\perp_{\mathcal{P}(z)}}d_z(\partial^\zeta\psi)(v) = \partial^v(\partial^\zeta\psi)(z) = \sum_{i,j}\left(v \,\middle|\, \frac{\partial^2 f}{\partial x_i\partial x_j}(x)\right)\partial^\zeta\psi_j(z)\cdot\widetilde{A}_f(z)^{-1}(e_i)\,,$$

where ψ_j *denotes the* j*-th coordinate function of* ψ, $\{e_1,\ldots,e_n\}$ *is the canonical basis in* $\mathbb{R}^n$ *and* $\widetilde{A}_f(z)$ *is the endomorphism of* $\mathbb{R}^n$ *with matrix* $A_f(z)$ (*see* (3.12)), *i.e.* $(\widetilde{A}_f(z))(\xi) = \sum_{i,j} A_f(z)_{ij}\cdot\xi_j e_i$ *for* $\xi\in\mathbb{R}^n$.

Proof. By (3.6) we can apply Corollary (3.9) to find $\mathcal{O}\in\operatorname{top} T^\perp_{\mathcal{P}(z)}$ such that $0\in\mathcal{O}$ and $\mathcal{P}\equiv\mathcal{P}(z)$ on $z+\mathcal{O}$. In view of Lemma (4′.1) for every $v\in\mathcal{O}$ we have $z+v\in\operatorname{dom}\psi$ and

$$(4'.11)\quad \partial^\zeta\psi(z+v) = \widetilde{A}_f(z+v)^{-1}\left(\left(\zeta \,\middle|\, \frac{\partial f}{\partial x_1}(x)\right),\ldots,\left(\zeta \,\middle|\, \frac{\partial f}{\partial x_n}(x)\right)\right).$$

Also

$$\widetilde{A}_f(z+v) = \widetilde{A}_f(z) - \sum_{i,j}\left(v \,\middle|\, \frac{\partial^2 f}{\partial x_i\partial x_j}(x)\right)e_j^*\cdot e_i\,,$$

where $\{e_1^*,\ldots,e_n^*\}$ is the dual basis to $\{e_1,\ldots,e_n\}$. The mapping $\mu : \operatorname{Aut}\mathbb{R}^n \ni E\mapsto E^{-1}\in\operatorname{Aut}\mathbb{R}^n$ is analytic (as the solution of the implicit equation $R(E,\mu(E)) = 0$, where $R : (\operatorname{Aut}\mathbb{R}^n)^2\ni(E,F)\mapsto E\circ F-\mathrm{id}_{\mathbb{R}^n}$). For $L\in\operatorname{End}\mathbb{R}^n$ we have $d_E\mu(L) = -E^{-1}\circ L\circ E^{-1}$. Also each $\xi\in\mathbb{R}^n$ defines the analytic mapping $\xi^{**} : \operatorname{End}\mathbb{R}^n\in L\mapsto L(\xi)\in\mathbb{R}^n$. In this notation, for

$$\xi := \left(\left(\zeta \,\middle|\, \frac{\partial f}{\partial x_1}(x)\right),\ldots,\left(\zeta \,\middle|\, \frac{\partial f}{\partial x_n}(x)\right)\right)$$

the relation (4′.11) takes the form

$$\begin{aligned}\partial^\zeta\psi(z+v) &= (\xi^{**}\circ\mu)(\widetilde{A}_f(z+v))\\ &= (\xi^{**}\circ\mu)\Big(\widetilde{A}_f(z) - \sum_{i,j}\Big(v\;\Big|\;\frac{\partial^2 f}{\partial x_i\partial x_j}(x)\Big)e_j^*\cdot e_i\Big).\end{aligned}$$

So $\partial^\zeta\psi$ is differentiable at z with respect to $T^\perp_{\mathcal{P}(z)}$.

In order to find the explicit form of $\partial^v(\partial^\zeta\psi)(z)$ for $v\in T^\perp_{\mathcal{P}(z)}$, put

$$\gamma(t) := \widetilde{A}_f(z) - t\sum_{i,j}\Big(v\;\Big|\;\frac{\partial^2 f}{\partial x_i\partial x_j}(x)\Big)e_j^*\cdot e_i \quad \text{for } t\in\mathbb{R}.$$

Then

$$\begin{aligned}\partial^v(\partial^\zeta\psi)(z) &= \frac{d}{dt}\partial^\zeta\psi(z+tv)\Big|_{t=0} = (d_{\gamma(0)}(\xi^{**}\circ\mu))(\gamma'(0))\\ &= -\,\xi^{**}(\gamma(0)^{-1}\circ\gamma'(0)\circ\gamma(0)^{-1})\\ &= \widetilde{A}_f(z)^{-1}\Big(\sum_{i,j}\Big(v\;\Big|\;\frac{\partial^2 f}{\partial x_i\partial x_j}(x)\Big)\partial^\zeta\psi_j(z)\cdot e_i\Big),\end{aligned}$$

which is the desired conclusion. ∎

(4′.12) LEMMA (the curvilinear version of the theorem on the existence of the Fréchet differential). *Let $M\subset Z$ be a C^2-submanifold and $a\in M$. Let Y denote a finite-dimensional linear space, and $g: Z \multimap Y$ a mapping of a neighbourhood of a such that $g_M := g|_M$ is differentiable at a. Assume that in a neighbourhood of a the differentials*

$$L_z := {}^{T^\perp_{\mathcal{P}(z)}}d_z g$$

exist, and

$$L_z \xrightarrow[z\to a]{} L_a.$$

Then g is differentiable at a.

Proof. We will reduce this problem to the classical theorem. Consider a Euclidean space X of dimension $\dim Z$, its subspace H of dimension $\dim M$ and an inverse chart $f: H \multimap M$ of class C^2 for which $f(0)=a$. There exists a C^2-diffeomorphism $\Phi: X\multimap Z$ such that $\Phi(0)=a$, $f|_{H\cap\operatorname{dom}\Phi}\subset\Phi$ and for all $x\in\operatorname{dom}\Phi$, the map $H^\perp\ni v\mapsto\Phi(x+v)-\Phi(x)$ is contained in the linear isometry of $H^\perp$ and $T^\perp_{f(x')}$ (where x' stands for the orthogonal projection of x onto H). In view of Theorem (3.8) we can assume that $\operatorname{im}\Phi\subset\operatorname{dom}\mathcal{P}$ and $\mathcal{P}(z)=w$ whenever $z\in\operatorname{im}\Phi$, $w\in M\cap\operatorname{im}\Phi$ and $z-w\perp T_wM$. We will show that $g\circ\Phi$ satisfies the assumptions of the classical theorem on the existence of the Fréchet differential, which will complete the proof.

Obviously, the differential ${}^{H}d_0(g\circ\Phi)$ exists. Write $\mathcal{O} := (\operatorname{dom}\Phi)\cap \Pi^{-1}(\operatorname{dom} f\cap\operatorname{dom}\Phi)$ $(\in \operatorname{top} X)$, where $\Pi : X\to H$ is the linear orthogonal projection. Also, fix $x\in\mathcal{O}$ and put $x' := \Pi(x)$. Then $\Phi(x)-f(x') = \Phi(x'+(x-x'))-\Phi(x')\in T^{\perp}_{f(x')}$, therefore $\mathcal{P}(\Phi(x)) = f(x')$ and $L_{\Phi(x)} = d_0(g\circ\tau_{\Phi(x)}\circ\iota_{T^{\perp}_{f(x')}})$. Consider $u\in H^{\perp}$ such that $x+u\in\operatorname{dom}\Phi$. Then

$$(g\circ\Phi)(u+x) = (g\circ\tau_{\Phi(x)}\circ\iota_{T^{\perp}_{f(x')}})(\Phi(x+u)-\Phi(x))\,.$$

The differential of $H^{\perp}\ni u\mapsto\Phi(x+u)-\Phi(x)$ at zero is an isometry of $H^{\perp}$ and $T^{\perp}_{f(x')}$, contains the function itself and is the differential of Φ at x with respect to $H^{\perp}$. Thus in a neighbourhood of zero we have

$$(g\circ\Phi)\circ\tau_x\circ\iota_{H^{\perp}} = (g\circ\tau_{\Phi(x)}\circ\iota_{T^{\perp}_{f(x')}})\circ d_x\Phi\circ\iota_{H^{\perp}}\,,$$

from which it follows that ${}^{H^{\perp}}d_x(g\circ\Phi)$ exists for all $x\in\mathcal{O}$ and is equal to $\iota_{\Phi(x)}\circ d_x\Phi\circ\iota_{H^{\perp}}$.

Knowing that ${}^{H}d_0(g\circ\Phi)$ exists we are reduced to proving that

$${}^{H^{\perp}}d_x(g\circ\Phi)\underset{x\to 0}{\longrightarrow}{}^{H^{\perp}}d_0(g\circ\Phi)$$

in the Banach space $L(H^{\perp},Y)$ or, which is equivalent, in the Hausdorff metric (see (4′.3)). We will use the criterion (4′.7). Fix $\varepsilon>0$. Since $L_z\to L_a$ as $z\to a$, there is $\delta>0$ such that

$$\forall |z-a|\le\delta,\ v_z\in S\cap T^{\perp}_{f(z')},\ v\in S\cap T^{\perp}_a,\ |v-v_z|\le\delta:\quad |L_z(v_z)-L_a(a)|\le\varepsilon.$$

Since $0\in\mathcal{O}$ and Φ and Φ' (the derivative of Φ) are continuous, there is $\vartheta>0$ such that if $|x|<\vartheta$, then $x\in\mathcal{O}$, $|\Phi(x)-a|\le\delta$ and $|d_x\Phi-d_0\Phi|\le\delta$. Now fix $x\in X$ such that $|x|\le\delta$ and define $v_{\Phi(x)} := {}^{H^{\perp}}d_x\Phi(u)$, $v := {}^{H^{\perp}}d_0\Phi(u)$. The functions $d_x\Phi\circ\iota_{H^{\perp}}$ and $d_0\Phi\circ\iota_{H^{\perp}}$ are isometries, therefore $|v_{\Phi(x)}| = |v| = 1$. Hence $|{}^{H^{\perp}}d_x(g\circ\Phi)(u) - {}^{H^{\perp}}d_0(g\circ\Phi)(u)|\le\varepsilon$. ∎

(4′.13) Theorem (global version of (4′.12)). *Consider a C^2-submanifold $M\subset Z$ and a mapping $g : Z \to Y$ of a subset $\operatorname{dom} g\in\operatorname{top} Z$ of the domain of $\mathcal{P}$ with values in a finite-dimensional linear space Y. Suppose that $g_M := g|_M$ is of class C^1 and ${}^{T^{\perp}_{\mathcal{P}(z)}}d_z g$ exists for any $z\in\operatorname{dom} g$. Moreover, assume that the function*

$$\operatorname{dom} g\ni z\mapsto {}^{T^{\perp}_{\mathcal{P}(z)}}d_z g$$

is continuous in the Hausdorff metric at any point of $M\cap\operatorname{dom} g$. Then

(i) *g is differentiable at a for all $a\in M\cap\operatorname{dom} g$;*
(ii) *the function $M\cap\operatorname{dom} g\ni a\mapsto d_a g$ is continuous.*

Proof. By Lemma (4′.12) it remains to prove (ii). The map $G := g_M\circ\mathcal{P}_\Omega$ $(\mathcal{P}_\Omega := \mathcal{P}|_\Omega)$ is of class C^1 (see (4.1)) and $M\cap\operatorname{dom} G = M\cap\operatorname{dom} g$.

For $z \in M \cap \operatorname{dom} g$ we denote by $\alpha_z : T_z \hookrightarrow Z$ the inclusion. Theorem (4.1) states that $d_z\mathcal{P}$ is the orthogonal projection onto T_z, so

$$\forall z \in M \cap \operatorname{dom} g : \quad d_zG \circ \alpha_z = d_zg \circ \alpha_z \ (= {}^{T_z}d_zg)\,.$$

The map $M \ni z \mapsto T_zM$ is of class C^1. On the other hand, $\mathcal{G}_{\dim M}(Z) \ni U \mapsto \iota_U \in L(U, Z)$ is continuous (see (4′.6)). This yields the continuity of

$$M \cap \operatorname{dom} G \ni z \mapsto d_zG \circ \alpha_z = {}^{T_z}d_zg \in \mathcal{G}_{\dim Z}(Z \times Y)$$

(see (4′.5)). From this we conclude that

$$d_zg = {}^{T_z}d_zg \oplus {}^{T_z^\perp}d_zg \underset{M\ni z\mapsto a}{\longrightarrow} {}^{T_a}d_ag \oplus {}^{T_a^\perp}d_ag = d_ag$$

(see (4′.4)). ∎

P r o o f o f T h e o r e m (4.2). It is sufficient to show that the assertion holds locally. Let $f : \mathbb{R}^n \nrightarrow M$ be an inverse chart of class C^2 and, according to the notation of Lemma (4′.10), set $\psi := f^{-1} \circ \mathcal{P}$. This is a C^1-mapping. The proof will be completed when we prove that for any $\zeta \in Z$, $\partial^\zeta\psi$ is differentiable and the function $\operatorname{im} f \ni x \mapsto d_x(\partial^\zeta\psi)$ is continuous.

Fix $\zeta \in Z$. Intending to make use of Theorem (4′.13) we have to show that $(\partial^\zeta\psi)_M$ is of class C^1 and that for $z \in \operatorname{dom}\psi$, the differentials ${}^{T^\perp_{\mathcal{P}(z)}}d_z(\partial^\zeta\psi)$ exist and converge to ${}^{T^\perp_{z_0}}d_{z_0}(\partial^\zeta\psi)$ as $z \to z_0$ $(\in M)$. In the notation of (4′.11),

$$\partial^\zeta\psi(z) = \Big(\sum_{i,j}\Big(\frac{\partial f}{\partial x_i}(x)\,\Big|\,\frac{\partial f}{\partial x_j}(x)\Big)e_j^*\cdot e_i\Big)\Big(\Big(\zeta\,\Big|\,\frac{\partial f}{\partial x_1}(x)\Big),\ldots,\Big(\zeta\,\Big|\,\frac{\partial f}{\partial x_n}(x)\Big)\Big)$$

for $z = f(x)$ $(x \in \operatorname{dom} f)$. The right-hand side is a C^1-function of x, thus $(\partial^\zeta\psi)_M$ is of class C^1. Now we only have to show that

$${}^{T^\perp_{\mathcal{P}(z)}}d_z(\partial^\zeta\psi) \underset{z\to z_0}{\longrightarrow} {}^{T^\perp_{z_0}}d_{z_0}(\partial^\zeta\psi)\,,$$

where $z_0 \in \operatorname{im} f$. This follows from the criterion (4′.7) and from $T_{\mathcal{P}(z)} \to T_{z_0}$ and $T^\perp_{\mathcal{P}(z)} \to T^\perp_{z_0}$ as $z \to z_0$ (see (3.5)). ∎

D i s c u s s i o n o f E x a m p l e (4.3). Consider the C^2-embedding $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) := \begin{cases} (t, \frac{1}{3}t^3)\,, & t \geq 0,\\ (t, 0)\,, & t < 0,\end{cases}$$

and the submanifold $M := f(\mathbb{R})$. Suppose, contrary to our claim, that

(4′.14) $\quad\mathcal{P}$ is twice differentiable in a neighbourhood of $(0, 0)$.

Then there is $r > 0$ such that $\mathcal{P}$ is twice differentiable in $U := {]-r, r[}^2$ and

$$\forall z \in U\ \forall w \in U \cap M : \quad z - w \perp T_wM \ \Rightarrow\ \mathcal{P}(z) = w$$

(see (3.8)). Thus, for $z = (z_1, z_2) \in U \cap \{(x, y) : x \le 0\}$ we have

$$z_1 = (f^{-1} \circ \mathcal{P})(z)\,. \tag{4'.15}$$

Fix $\lambda \in\,]0, r[$ and define the C^2-curve

$$g_\lambda : \mathbb{R} \ni t \mapsto f(t) + (-\lambda t^2, \lambda)\,.$$

There is $\delta > 0$ such that for all $t \in [0, \delta]$, $t^4 - 2\lambda t + 1 > 0$ and $g_\lambda(t), f(t) \in U$ and $\mathcal{P}(g_\lambda(t)) = f(t)$. In view of (4′.14) the map

$$\kappa(t) := \partial^{e_1}(f^{-1} \circ \mathcal{P}_\Omega)(g_\lambda(t))$$

is differentiable in a neighbourhood of zero. We have $\det A_f(g_\lambda(t)) = t^4 - 2\lambda t + 1 \ne 0$ for $t \in [0, \delta]$ (see (3.12)), so by (4′.11), $\kappa(t) = 1/(t^4 - 2\lambda t + 1)$ for any $t \in [0, \delta]$. Therefore $\kappa'(0^+) = 2\lambda \ne 0$. On the other hand, (4′.15) yields $\partial^{e_1}(f^{-1} \circ \mathcal{P}_\Omega) \equiv 1$ in $U \cap \{(x, y) : x \le 0\}$, thus $\kappa'(0^-) = 0$, a contradiction. ■

Proof of Proposition (4.4). The mapping

$$\varrho : \Omega \setminus M \ni x \mapsto |x - \mathcal{P}(x)| \in \mathbb{R}$$

is of class C^1. For all $b \in Z \setminus \{0\}$ one obtains

$$\begin{aligned}\partial^b \varrho^2(x) &= 2(x - \mathcal{P}(x) \mid b - \partial^b \mathcal{P}(x)) \\ &= 2(x - \mathcal{P}(x) \mid b) - 2(x - \mathcal{P}(x) \mid d_x\mathcal{P}(b)) \\ &= 2(x - \mathcal{P}(x) \mid b)\,,\end{aligned}$$

since $x - \mathcal{P}(x) \in T^{\perp}_{\mathcal{P}(x)}$ and $d_x\mathcal{P}(b) \in T_{\mathcal{P}(x)}$. On the other hand, $\partial^b(\lambda^2) = 2\varrho \cdot \partial^b \varrho$, so $2\varrho(x) \cdot \partial^b \varrho(x) = 2(x - \mathcal{P}(x) \mid b)$, and finally,

$$\partial^b \varrho(x) = \left(\frac{x - \mathcal{P}(x)}{|x - \mathcal{P}(x)|} \,\middle|\, b \right). \ ■$$

5′. Proofs

(5′.1) REMARK. *For every submanifold $M \subset Z$ the following conditions are equivalent*:

(i) *M is non-empty, convex and closed*;
(ii) *M is an affine subspace of Z.*

Proof. (ii)⇒(i) is obvious.

(i)⇒(ii). One can assume that $0 \in M$. Let $H := \operatorname{Lin} M$, $n := \dim H$ ($\ge \dim M$), $\{e_1, \ldots, e_n\} \subset M$ be a basis of H and $\{e_1^*, \ldots, e_n^*\}$ the dual basis. The set

$$\{h \in H : (e_1^* + \ldots + e_n^*)(h) < 1\} \cap \bigcap_{i=1}^{n} \{h \in H : e_i^*(h) > 0\} \subset \operatorname{conv}\{0, e_1, \ldots, e_n\}$$

is non-empty, open in H and contained in M. Therefore $\dim M = n$, and consequently $M \in \operatorname{top} H$. Finally, $H = M$. ■

Proof of Theorem (5.3). This follows (even for M being a C^1-submanifold) immediately from (5.2) and (5′.1). We will present a more elementary proof without the use of Theorem (5.2). The set M is closed, because if $(x_n) \in M^{\mathbb{N}}$ converges to $z \in Z$, then $0 = |x_n - \mathcal{P}(x_n)| \to |z - \mathcal{P}(z)|$ as $n \to \infty$. Hence $z = \mathcal{P}(z) \in M$. In view of Remark (5′.1) it suffices to prove the convexity of M. Fix $a \in Z$ and set

$$J := \{t \in \mathbb{R}_+ : \mathcal{P}(\mathcal{P}(a) + t(a - \mathcal{P}(a))) = \mathcal{P}(a)\},$$

where $\mathbb{R}_+ := [0, \infty[$. Notice that $J \in (\operatorname{cotop} \mathbb{R}_+) \setminus \{\emptyset\}$. Simultaneously $J \in \operatorname{top} \mathbb{R}_+$. In order to prove this, fix $t_0 \in J$ and put $b := \mathcal{P}(a) + t_0(a - \mathcal{P}(a))$. By Corollary (3.9) there exists $\mathcal{O} \in \operatorname{top} T^{\perp}_{\mathcal{P}(a)}$ such that $0 \in \mathcal{O}$ and $\mathcal{P} \equiv \mathcal{P}(a)$ on $b + \mathcal{O}$. Moreover, there is $\delta > 0$ for which $\delta(a - \mathcal{P}(a)) \in \mathcal{O}$. Thus $\mathcal{P}(a) = \mathcal{P}(b + \delta(a - \mathcal{P}(a)))$, which means that $[0, t_0 + \delta] \subset J$. Consequently, $J = \mathbb{R}_+$.

Now we show that M is contained in the half-space $\Pi(a) := \{z \in Z : (z - \mathcal{P}(a) \mid a - \mathcal{P}(a)) \le 0\}$. Fix $x \in M$ and $t \in J$ $(= \mathbb{R}_+)$. We have

$$|\mathcal{P}(a) + t(a - \mathcal{P}(a)) - x| \ge |\mathcal{P}(a) + t(a - \mathcal{P}(a)) - \mathcal{P}(a)|,$$

which is equivalent to the inequality

$$(x - \mathcal{P}(a) \mid a - \mathcal{P}(a)) \le \frac{1}{2t} |\mathcal{P}(a) - x|^2$$

for all $t > 0$, so indeed $x \in \Pi(a)$ $(\forall a \in Z)$. Therefore $M \subset \bigcap_{a \in Z} \Pi(a)$. The inverse inclusion also holds, since for $z \in \bigcap_{a \in Z} \Pi(a)$ we have, in particular, $z \in \Pi(z)$. This implies $z = \mathcal{P}(z) \in M$. As a consequence, M $(= \bigcap_{a \in Z} \Pi(a))$ is convex. ■

6′. Proofs

(6′.1) Remark. *Consider a concave function* $f : \mathbb{R}^k \to \mathbb{R}$ *(i.e. such that the set* $\{(x, y) \in \mathbb{R}^k \times \mathbb{R} : f(x) \ge y\}$ *is convex) and the orthogonal projection* $\mathcal{P} : \mathbb{R}^k \times \mathbb{R} \nrightarrow f$. *Then* $\{(x, y) \in \mathbb{R}^k \times \mathbb{R} : f(x) \le y\} \subset \operatorname{dom} \mathcal{P}$.

(6′.2) Remark. *Consider a Euclidean space* Z, *a set* $M \subset Z$, *the orthogonal projection* $\mathcal{P} : Z \nrightarrow M$ *and a subset* $U \subset Z$. *Assume that* $I : Z \to Z$ *is an isometry such that* $I(M) = M$. *Then* $I(U \setminus \operatorname{dom} \mathcal{P}) = I(U) \setminus \operatorname{dom} \mathcal{P}$.

Let us introduce the symbols and notions constantly used throughout this section. We view $Z = \mathbb{R}^2$ as the complex plane $\mathbb{C}$. For all $a \in \mathbb{R}$ for which the following definition makes sense, we denote by $L(a)$ the affine line $(a + f(a)i) + T^{\perp}_{a+f(a)i} M$, where $f : \mathbb{R} \nrightarrow \mathbb{R}$ is the function under consideration and M $(= f)$ is its graph.

D i s c u s s i o n o f E x a m p l e (6.1). For $a \in \mathbb{R} \setminus \{0\}$ the normal line $L(a) = \{x + iy : y - a^2 = \frac{-1}{2a}(x - a)\}$ intersects $\mathbb{R}i$ at $(a^2 + 1/2)i$. Therefore $[0, i/2] \subset \operatorname{dom} \mathcal{P}$ and $\mathcal{P}|_{[0,i/2]} \equiv 0$. Also, for $z \in \,]1/2, \infty[\, \cdot i$ there exists a unique $a > 0$ such that $z \in L(a)$. Moreover, $\{b \in \mathbb{R} : z \in L(b)\} = \{-a, 0, a\}$. But since $|a + a^2 i - z| = |-a + a^2 i - z| < |z|$, we have $z \notin \operatorname{dom} \mathcal{P}$. Hence

$$\mathbb{R}_+ i \cap \operatorname{dom} \mathcal{P} = [0, i/2]\,.$$

By Remark (6′.1), $\{x + iy : y \le x^2\} \subset \operatorname{dom} \mathcal{P}$. We are left with considering points $z_0 = x_0 + iy_0$ for which $x_0 \ne 0$ and $y_0 > x_0^2$. For such z_0 there exists $a + a^2 i \ne 0$ realizing the distance of z_0 from f. Thus $z_0 \in \,]a + a^2 i, (a^2 + 1)i[\, \subset \operatorname{dom} \mathcal{P}$ and $\mathcal{P}(z_0) = a + a^2 i$ (see (1.5)). Finally, $\operatorname{dom} \mathcal{P} = \mathbb{C} \setminus (]1/2, \infty[\, \cdot i)$. ∎

D i s c u s s i o n o f E x a m p l e (6.2). The results obtained in the discussion of Example (6.1) imply

$$\{z \in \mathbb{C} : \operatorname{Re} z \ge 0\} \subset \operatorname{dom} \mathcal{P}\,.$$

Next, for $z = x + iy$ such that $x < 0$ define

$$h = h_z : [0, \infty[\, \ni a \mapsto |x + iy - (a + a^2 i)|^2 = a^4 + (1 - 2y)a^2 - 2ax + x^2 + y^2\,,$$

which is either increasing or reaches one minimum. Therefore

$$x + iy \notin \operatorname{dom} \mathcal{P} \;\Leftrightarrow\; \exists a_0 > 0 : \; h'(a_0) = 0 \text{ and } h(0) = h(a_0)\,.$$

The system of equations on the right-hand side of the above equivalence has a solution $a_0 > 0$ iff $y = \frac{1}{2}(3(-x)^{2/3} + 1)$. Then $a_0 = (-x)^{1/3}$. So the graph of the function

$$G : \,]-\infty, 0[\, \ni x \mapsto \frac{1}{2}(3(-x)^{2/3} + 1) \in \mathbb{R} \tag{6′.3}$$

coincides with $\mathbb{R}^2 \setminus \operatorname{dom} \mathcal{P}$. ∎

D i s c u s s i o n o f E x a m p l e (6.3). When we delete zero from the M of Example (6.2), all the points that were previously projected onto zero disappear from $\operatorname{dom} \mathcal{P}$. These are exactly the points $z \in \{w : \operatorname{Re} w \le 0\}$ for which, in the notation of Discussion of (6.2), either $z \in \,]-\infty, 1/2[\, \cdot i$, or $h'(a_0) = 0$ and $h(0) = h(a_0)$ for some $a_0 > 0$. Finally,

$$\operatorname{dom} \mathcal{P} = \mathbb{C} \setminus (\{i/2\} \cup \{x + iy : x \le 0, y < G(x)\}) \quad \text{(see (6′.3))}\,. \;∎$$

D i s c u s s i o n o f E x a m p l e (6.4). First consider points of $\mathbb{R}i$. If $a + i(\cos a)$ realizes such a point's distance from $f = \cos$, then $|a| < \pi$. The normal at $a + i(\cos a)$,

$$L(a) = \left\{x + iy : y - \cos a = \frac{x - a}{\sin a}\right\} \quad (\text{for } 0 < |a| < \pi)\,,$$

intersects $\mathbb{R}i$ at $y(a) := (\cos a - a/\sin a)i$. Since the function

$$y : \,]0, \pi[\, \ni a \mapsto y(a) \in \,]-\infty, 0[\, \cdot i$$

is continuous and bijective, for any $z \in [0,\infty[\cdot i$ the point $a = 0$ is the unique point a in $]-\pi,\pi[$ for which z lies on $L(a)$. Therefore $[0,\infty[\cdot i \subset \operatorname{dom}\mathcal{P}$.

Next, for fixed $y_0 < 0$ there exists a unique $a \in]0,\pi[$ such that $y_0 i \in L(a)$. Simultaneously, $\{b \in]-\pi,\pi[: y_0 i \in L(b)\} = \{-a, 0, a\}$. If we had $|y_0 i - (a + i\cos a)| > |y_0 i - i|$, then $y_0 i \in \operatorname{dom}\mathcal{P}$ and $\mathcal{P}(y_0 i) = i$, hence $0 \in]y_0 i, i] \subset \Omega$ (see (3.13)) and, by Theorem (3.11), $0 \neq \det A_F(0)$, where $F : \mathbb{R} \ni a \mapsto (a, \cos a) \in \mathbb{R}^2$. But the matrix $A_F(0)$ is singular. Therefore, the distance of $y_0 i$ from f is realized by two points: $a + i\cos a$, $-a + i\cos a$. This means that $\mathbb{R}i \cap \operatorname{dom}\mathcal{P} = [0,\infty[\cdot i$.

Now, consider $z_0 = x_0 + iy_0$ such that $x_0 \in]0,\pi[$ and $y_0 < \cos x_0$. As mentioned at the beginning, there exists $a \in]0,\pi[$ such that $a + i\cos a$ realizes the distance of z_0 from f. But $z_0 \in]y(a), a + i\cos a[\subset \operatorname{dom}\mathcal{P}$. Therefore, for

$$U := \mathbb{R}i \cup \{x + iy : x \in]0,\pi[,\ y \leq \cos x\}$$

we have $U \setminus \operatorname{dom}\mathcal{P} =]-\infty,0[\cdot i$. Put $M := f$. Consider the family $\mathcal{J} := \{\tau_{2k\pi} \circ I\}_{k\in\mathbb{Z}}$ of isometries of the plane $\mathbb{C}$, where $\tau_{2k\pi}$ is translation by $2k\pi$, while I is either $\mathrm{id}_{\mathbb{C}}$, $S_{\pi+\mathbb{R}i}$, $R_{\pi/2}$, or $S_{\pi+\mathbb{R}i} \circ R_{\pi/2}$ (where S_b is the symmetry about a straight line b and R_B the symmetry about a point B). Then

$$\bigcup_{J\in\mathcal{J}} J(U) = \mathbb{C} \quad \text{and} \quad \forall J \in \mathcal{J} :\ J(M) = M\,.$$

Applying Remark (6′.2) to our U we obtain

$$\mathbb{C} \setminus \operatorname{dom}\mathcal{P} = \Big(\bigcup_{J\in\mathcal{J}} J(U)\Big) \setminus \operatorname{dom}\mathcal{P} = \bigcup_{k\in\mathbb{Z}} (k\pi +]0,\infty[\cdot(-1)^{k+1} i)\,. \ \blacksquare$$

Discussion of Example (6.5). Remark (6′.1) yields $\{x + iy : y \leq e^x\} \subset \operatorname{dom}\mathcal{P}$. Fix $z = x + iy \in \mathbb{C}$ and consider the function

$$h = h_z : \mathbb{R} \ni a \mapsto |z - (a + ie^a)|^2 = (x-a)^2 + (y - e^a)^2\,.$$

For $|y| \leq 2\sqrt{2}$, h reaches one minimum, so $\{w \in \mathbb{C} : \operatorname{Im} w \leq 2\sqrt{2}\} \subset \operatorname{dom}\mathcal{P}$. If $y = \operatorname{Im} z > 2\sqrt{2}$, then $h''(a) = 0$ has two solutions $a_1 < a_2$. A sufficient condition for $z \notin \operatorname{dom}\mathcal{P}$ is:

$$h'(a_1) > 0 \quad \text{and} \quad h'(a_2) < 0\,. \tag{6′.4}$$

This is equivalent to

$$-\ln\sqrt{2e} - u_2(t) < x < -\ln\sqrt{2e} - u_1(t)\,,$$

where $t := y/2\sqrt{2}$ (> 1) and

$$u_1(t) := t(t - \sqrt{t^2-1}) + \ln(t + \sqrt{t^2-1})\,,$$
$$u_2(t) := t(t + \sqrt{t^2-1}) - \ln(t + \sqrt{t^2-1})\,.$$

Moreover, $\lim_{t\to\infty} u_1(t) = \lim_{t\to\infty} u_2(t) = \infty$ and the graphs of u_1 and u_2

are tangent at the point corresponding to $t = 1$, i.e. at $-1 - \ln\sqrt{2e} + 2\sqrt{2}i$. The condition $(6'.4)$ is satisfied by points of the set

$$\{x + iy : y > 2\sqrt{2} \text{ and } -u_2(y/2\sqrt{2}) < x < -u_1(y/2\sqrt{2})\}$$

which contains $\mathbb{C} \setminus \operatorname{dom} \mathcal{P}$. ∎

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