Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions

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Abstract. The purpose of this paper is to study the periodic boundary value problem
\[-u''(t) = f(t, u(t), u'(t)), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\]
when \(f\) satisfies the Carathéodory conditions. We show that a generalized upper and lower solution method is still valid, and
develop a monotone iterative technique for finding minimal and maximal solutions.

1. Introduction. In this paper we consider the following periodic boundary value problem (PBVP for short) of second order:

\[
\begin{align*}
(P) \quad \begin{cases}
-u''(t) = f(t, u(t), u'(t)), \\
u(0) = u(2\pi), \quad u'(0) = u'(2\pi).
\end{cases}
\end{align*}
\]

As is well known, the method of upper and lower solutions has been successfully applied to study this PBVP when \(f\) is a continuous function (see [2–6, 12] and the monograph [9] and the references therein).

Here, we generalize the method of upper and lower solutions to the case when \(f\) is a Carathéodory function. We point out that for \(f\) continuous the classical arguments of [2–6, 9, 12] are no longer valid since the solutions are in the Sobolev space \(W^{2,1}(I)\), \(I = [0, 2\pi]\). Thus, if \(u\) is a solution, \(u''\) is not necessarily continuous on \(I\) but only \(u'' \in L^1(I)\).

Our ideas are in the spirit of [7, 10] where \(f(t, u(t), u'(t)) = f(t, u(t))\). There, when \(u\) is bounded we deduce that \(u''\) is bounded, and so is \(u'\). In our situation, we have to find a bound for the derivative of a solution since the

1991 Mathematics Subject Classification: Primary 34B15, 34C25.

Key words and phrases: upper and lower solutions, monotone iterative technique, Carathéodory function.

Research of the second and third authors partially supported by DGCYT, projects PS88-0054 and PB91-0793, and by Xunta de Galicia, project XUGA 20701A90.
derivative of the modified problem relative to (P) may be unbounded. To this purpose we prove a new result (Theorem 1). Thus, we improve the results of [8] where we require \( f \) to be locally Lipschitzian or locally equicontinuous in some variables. The proof of some known results are included for the convenience of the reader: For instance, Lemma 4 is taken from [10]. Also we note that part (c) of Lemma 1 is proved in [4] and Theorem 2 is related to the results of Adje in [1] but our proof is simpler using a convenient modified problem.

When \( v \) and \( w \) are (generalized) lower and upper solutions relative to (P) and \( v \leq w \), we denote by \( S[v, w] \) the set of solutions of (P) in the sector \([v, w] = \{ u \in W^{2,1}(I) : v(t) \leq u(t) \leq w(t) \text{ for } t \in I \} \) (see [7, 10]). We generalize the monotone method [9] to obtain minimal and maximal solutions as limits of monotone iterates.

2. The method of upper and lower solutions. We shall suppose that \( f : I \times \mathbb{R}^2 \to \mathbb{R}, I = [0, 2\pi] \), is a Carathéodory function, that is:

(i) for a.e. \( t \in I \), the function \((u, s) \in \mathbb{R}^2 \to f(t, u, s) \in \mathbb{R}\) is continuous,

(ii) for every \((u, s) \in \mathbb{R}^2\), the function \( t \in I \to f(t, u, s) \) is measurable,

(iii) for every \( R > 0 \), there exists a real-valued function \( h(t) = h_R(t) \in L^1(I) \) such that

\[
|f(t, u, s)| \leq h(t)
\]

for a.e. \( t \in I \) and every \((u, s) \in \mathbb{R}^2\) satisfying \(|u| \leq R, |s| \leq R\).

A function \( u \in W^{2,1}(I) \) is a solution of (P) if (1.1) holds for a.e. \( t \in I \), and \( u \) satisfies (1.2). When \( f \) is continuous, any solution of (P) is a classical solution, that is, a \( C^2 \)-solution. If, in addition, \( f \) is \( 2\pi \)-periodic in \( t \), then any solution can be extended by periodicity to \( \mathbb{R} \), and then it is a periodic solution of (1.1).

Let us say that a function \( v : I \to \mathbb{R} \) is a lower solution of (P) if \( v \in W^{2,1}(I) \),

\[
-v''(t) \leq f(t, v(t), v'(t)) \quad \text{for a.e. } t \in I
\]

and

\[
v(0) = v(2\pi), \quad v'(0) \geq v'(2\pi).
\]

Similarly, \( w : I \to \mathbb{R} \) is an upper solution of (P) if \( w \in W^{2,1}(I) \),

\[
-w''(t) \geq f(t, w(t), w'(t)) \quad \text{for a.e. } t \in I
\]

and

\[
w(0) = w(2\pi), \quad w'(0) \leq w'(2\pi).
\]

Throughout we shall suppose that \( v \leq w \) on \( I \). We shall consider the following condition:
There exists $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous such that
\[ |f(t, u, s)| \leq g(|s|) \]
for a.e. $t \in I$ with $v(t) \leq u \leq w(t)$, and $s \in \mathbb{R}$, satisfying
\[ (2.6) \quad \int_{-\infty}^\infty \frac{s}{\lambda} \frac{g(s)}{g(s) + C} \, ds = \infty \quad \forall \lambda > 0 \text{ and } \forall C > 0. \]

Note that the usual Nagumo condition $\int_{-\infty}^{\infty} (s/g(s)) \, ds = \infty$ implies (2.6) when either $\limsup_{s \to \infty} g(s) < \infty$ or $\liminf_{s \to \infty} g(s) > 0$.

Now, we give a priori estimates for the derivative of solutions of (P).

**Lemma 1.** Let $0 \leq t_1 < t_2 \leq 2\pi$, $u \in W^{2,1}([t_1, t_2])$ and assume that $v \leq u \leq w$ on $[t_1, t_2]$ and (1.1) is satisfied for a.e. $t \in [t_1, t_2]$. If (H1) holds, then there exists a positive constant $N$ which depends only on $v$, $w$, $g$ and a constant $C$, such that:

(a) $u'(t_1) \leq C$ or $u'(t_2) \leq C$ implies $u'(t) \leq N$ on $[t_1, t_2]$.
(b) $u'(t_1) \geq C$ or $u'(t_2) \geq C$ implies $u'(t) \geq -N$ on $[t_1, t_2]$.
(c) $u(t_1) - u(t_2) = u'(t_1) - u'(t_2) = 0$ implies $|u'(t)| \leq N$ on $[t_1, t_2]$.

**Proof.** (a) Suppose that $u'(t_1) \leq C$ and that
\[ (2.7) \quad \forall n \in \mathbb{N}, \exists T_n \in [t_1, t_2] \text{ such that } u'(T_n) = n. \]

Let $n_0 \in \mathbb{N}$ be such that
\[ \int_{|C|}^{n_0} \frac{s}{g(s)} \, ds > \max_{t \in I} w(t) - \min_{t \in I} v(t). \]

By (2.7) there exists $T \in [t_1, T_{n_0}]$ such that $u'(T) = |C|$ and $0 \leq |C| \leq u'(t) \leq n_0$ for all $t \in [T, T_{n_0}]$. In this interval we obtain
\[ |u''(t)| = |f(t, u(t), u'(t))| \leq g(|u'(t)|) \]
and
\[ \frac{u''u'}{g(u')} \leq \frac{|u''|u'}{g(|u'|)} \leq u'. \]

Thus,
\[ \int_{t}^{T_{n_0}} \frac{u'(t)u''(t)}{g(u'(t))} \, dt \leq \int_{t}^{T_{n_0}} \frac{u'(t)}{g(u'(t))} \, dt = u(T_{n_0}) - u(T) \]
\[ \leq w(T_{n_0}) - v(T) \leq \max_{t \in I} w(t) - \min_{t \in I} v(t). \]

On the other hand,
\[ \int_{t}^{T_{n_0}} \frac{u'(t)u''(t)}{g(u'(t))} \, dt = \int_{|C|}^{n_0} \frac{s}{g(s)} \, ds > \max_{t \in I} w(t) - \min_{t \in I} v(t). \]
As a consequence, there exists \( N > 0 \) such that \( u'(t) \leq N \) on \([t_1, t_2] \).

If \( u'(t_2) \leq C \) and the assertion of (a) is not satisfied, then we deduce that property (2.7) holds.

Let \( n_1 \in \mathbb{N} \) be such that

\[
\int_{|C|}^{n_1} \frac{s}{g(s)} \, ds > \max_{t \in I} w(t) - \min_{t \in I} v(t).
\]

By (2.7) there exists \( T \in [T_{n_1}, t_2] \) such that \( u'(T) = |C| \) and \( 0 \leq |C| \leq u'(t) \leq n_1 \) for all \( t \in [T_{n_1}, T] \). Thus,

\[
-\frac{u''u'}{g(u')} \leq \frac{|-u''|u'}{g(|u'|)} \leq u'
\]
on \([T_{n_1}, T]\)

and

\[
-\int_{T_{n_1}}^{T} \frac{u'(t)u''(t)}{g(u'(t))} \, dt \leq \int_{T_{n_1}}^{T} u'(t) \, dt \leq \max_{t \in I} w(t) - \min_{t \in I} v(t).
\]

On the other hand,

\[
-\int_{T_{n_1}}^{T} \frac{u'(t)u''(t)}{g(u'(t))} \, dt = -\int_{n_1}^{|C|} \frac{s}{g(s)} \, ds = \int_{n_1}^{n_1} \frac{s}{g(s)} \, ds > \max_{t \in I} w(t) - \min_{t \in I} v(t).
\]

Therefore there exists \( N > 0 \) such that \( u'(t) \leq N \) on \([t_1, t_2]\).

Analogously we prove (b). The proof of (c) is given in Lemma 3.2 of [4].

For any \( u \in X = C^1(I) \), we define

\[
p(t, u) = \begin{cases} 
v(t), & u < v(t), \\
u, & v(t) \leq u \leq w(t), \\
w(t), & u > w(t). \end{cases}
\]

We obtain the following series of results:

**Lemma 2.** For \( u \in X \), the following two properties hold:

(a) \( \frac{d}{dt} p(t, u(t)) \) exists for a.e. \( t \in I \).

(b) If \( u, u_m \in X \) and \( u_m \overset{X}{\longrightarrow} u \), then

\[
\left\{ \frac{d}{dt} p(t, u_m(t)) \right\} \overset{\text{a.e.} \ t \in I}{\longrightarrow} \frac{d}{dt} p(t, u(t))
\]

**Proof.** Note that if \( u \in X \) then \( u^+ = \max \{u, 0\} \) and \( u^- = \max \{-u, 0\} \)

are absolutely continuous. We rewrite \( p(t, u) = [u - v(t)]^+ - [u - w(t)]^+ + u \).

Because \( u, v, w \in X \), it is enough to prove that if \( u, u_m \in X \) and \( u_m \overset{X}{\longrightarrow} u \),
then
\[ \left\{ \frac{d}{dt} p(t, u_m^+(t)) \right\} \to \frac{d}{dt} p(t, u^+(t)) \quad \text{for a.e. } t \in I. \]

Since \( \frac{d}{dt} u_m^+(t) \) and \( \frac{d}{dt} u^+(t) \) exist for a.e. \( t \in I \), suppose that \( t_0 \in I \) is such that \( \frac{d}{dt} u_m^+(t_0) \) and \( \frac{d}{dt} u^+(t_0) \) exist for all \( m = 1, 2, \ldots \).

If \( u(t_0) > 0 \), then \( u(t_0) = u^+(t_0) > 0 \). Therefore \( \frac{d}{dt} u^+(t_0) = \frac{d}{dt} u(t_0) \) and there exists \( M \in \mathbb{N} \) such that \( u_m(t_0) = u_m^+(t_0) > 0 \) for all \( m \geq M \). Thus
\[ \frac{d}{dt} u_m^+(t_0) = \frac{d}{dt} u(t_0) \Rightarrow d \frac{d}{dt} u_m(t_0) = \frac{d}{dt} u(t_0). \]

If \( u(t_0) < 0 \), then there exists \( M > 0 \) such that \( u_m(t_0) < 0 \) for all \( m \geq M \). Therefore \( u_m(t) < 0 \) on \( (t_0 - \delta_m, t_0 + \delta_m) \) for some \( \delta_m > 0 \) and then \( u_m^+(t) = 0 \) on \( (t_0 - \delta_m, t_0 + \delta_m) \). Hence \( \frac{d}{dt} u_m^+(t_0) = \frac{d}{dt} u^+(t_0) = 0 \) and then obviously
\[ \frac{d}{dt} u_m^+(t_0) \to \frac{d}{dt} u^+(t_0) \quad \text{as } m \to \infty. \]

If \( u(t_0) = 0 \), then \( u^+(t_0) = 0 \). Since \( \frac{d}{dt} u^+(t_0) \) exists, we have \( \frac{d}{dt} u(t_0) \equiv 0 \).

Because \( \frac{d}{dt} u_m^+(t_0) \) exists, we find that
\[ \frac{d}{dt} u_m^+(t_0) = \begin{cases} u_m'(t_0), & u_m(t_0) > 0, \\ 0, & u_m(t_0) \leq 0. \end{cases} \]

Therefore
\[ \left| \frac{d}{dt} u_m^+(t_0) \right| \leq \left| \frac{d}{dt} u_m(t_0) \right| \to \left| \frac{d}{dt} u(t_0) \right| = 0 = \frac{d}{dt} u^+(t_0). \]

Similarly, we can prove the conclusions about \( u^-(t) \), and thus the proof of Lemma 2 is complete.

Now, consider the following modified problem:
\[ \begin{cases} -u'' + u = f^*(t, u, u') + p(t, u), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{2.8} \]

where \( f^*(t, u(t), \dot{u}(t)) = f(t, p(t, u(t)), \frac{d}{dt} p(t, u(t)) \).

Since \( u \in X, \frac{d}{dt} p(t, u(t)) \) exists for a.e. \( t \in I \). If \( t_0 \in I \) is such that \( \frac{d}{dt} p(t_0, u(t_0)) \) does not exist, then it is easy to prove that the left and right derivatives of \( p(t, u(t)) \) at \( t_0 \) must exist and both values depend only on the X-norms of \( u, v \) and \( w \). Therefore we can complement the values of \( \frac{d}{dt} p(t, u(t)) \) in such a way that it is bounded and the bound depends only on the X-norm of \( u, v \) and \( w \). For any \( z \in X \), the linear problem
\[ \begin{cases} -u'' + u = f^*(t, z(t), \dot{z}(t)) + p(t, z(t)) \equiv \sigma(t), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{2.9} \]
has a unique solution \( u \) given by the formula

\[
(2.10) \quad u(t) = C_1 e^t + C_2 e^{-t} - \frac{e^t}{2} \int_0^t \sigma(s)e^{-s} \, ds + \frac{e^{-t}}{2} \int_0^t \sigma(s)e^s \, ds
\]

where

\[
C_1 = \frac{1}{2(e^{2\pi} - 1)} \int_0^{2\pi} \sigma(s)e^{2\pi - s} \, ds,
\]

\[
C_2 = \frac{1}{2(e^{2\pi} - 1)} \int_0^{2\pi} \sigma(s)e^s \, ds.
\]

Note that \( \sigma(t) = f^*(t, z(t), z'(t)) + p(t, z(t)) \) is measurable, \( |p(t, z(t))| \leq R \) and \( \frac{d}{dt}p(t, z(t)) \leq R \), which implies that \( |f^*(t, z(t), z'(t))| \leq h(t) \in L^1(I) \) and \( \sigma \in L^1(I) \).

From (2.10) and the formula

\[
(2.11) \quad u'(t) = C_1 e^t - C_2 e^{-t} - \frac{e^t}{2} \int_0^t \sigma(s)e^{-s} \, ds + \frac{e^{-t}}{2} \int_0^t \sigma(s)e^s \, ds
\]

it is clear that \( u \in X \).

Define the operator \( T : X \to X \), where \( T(z) = u \), with \( u \) defined by (2.10). For this operator we obtain the following result

**Lemma 3.** \( T : X \to X \) is compact.

**Proof.** Let \( z_m \in X, m \in \mathbb{N}, z_m \xrightarrow{X} z \), \( T(z_m) = u_m, T(z) = u \). We have \( \|z_m\|_X \leq M \) for some \( M > 0 \). Then \( p(t, z_m(t)) \to p(t, z(t)) \) for a.e. \( t \in I \) and \( |p(t, z_m(t))| \leq M, \left| \frac{d}{dt}p(t, z_m(t)) \right| \leq N \) for a.e. \( t \in I \) and for some \( N \) depending only on \( M, v \) and \( w \).

Now, \( |f^*(t, z(t), z'(t))| \leq h(t) \in L^1(I) \) follows from (2.1). By the hypothesis on \( f \) and Lemma 2 we know that \( f^*(t, z_m(t), z'_m(t)) \) converges to \( f^*(t, z(t), z'(t)) \) in measure. Hence, by the Lebesgue dominated convergence theorem,

\[
\lim_{m \to \infty} \int_0^t \sigma_m(s)e^{\pm s} \, ds = \int_0^t \sigma(s)e^{\pm s} \, ds
\]

where \( \sigma_m(t) = f^*(t, z_m(t), z'_m(t)) + p(t, z_m(t)) \).

By (2.10) and (2.11) we have \( \lim_{m \to \infty}(u_m(t), u'_m(t)) = (u(t), u'(t)) \) for every \( t \in I \). Because \( |\sigma_m(s)| \leq h(s) + |v(s)| + |w(s)| \in L^1(I) \), the sequence \( \{\sigma_m(t)\} = \{\int_0^t \sigma_m(s)e^{\pm s} \, ds\} \) is equicontinuous, and so are \( \{u_m(t)\} \) and \( \{u'_m(t)\} \).

It is obvious that \( \{u_m(t), u'_m(t)\} \) is uniformly bounded. Therefore, by the Ascoli theorem \( u_m \xrightarrow{X} u \). Hence \( T \) is continuous.
Similarly, for any bounded set \( B \subset X \), let \( B_1 = \{ u : u = T(z) \text{ for some } z \in B \} \) and \( B_2 = \{ u' : u \in B_1 \} \). Then \( B_1 \) and \( B_2 \) are equicontinuous and uniformly bounded. Thus, there exist subsequences \( \{ u_m \} \subset B_1 \) and \( \{ u_m' \} \subset B_2 \) such that \( u_m \rightharpoonup u \) and \( u_m' \rightharpoonup v \) uniformly on \( I \). Using (2.10) and (2.11) it is easy to prove that \( \exists (t) = u'(t) \). In consequence, \( u_m \rightharpoonup u \), and this shows that \( T \) is compact. 

**Lemma 4.** Let \( y \in W^{2,1}(I) \) and suppose that there exists \( M \in L^1(I) \) such that \( M(t) > 0 \) for a.e. \( t \in I \) and \( y''(t) \geq M(t)y(t) \) for a.e. \( t \in I \), \( y(0) = y(2\pi), y'(0) \geq y'(2\pi) \). Then \( y(t) \leq 0 \) for every \( t \in I \).

**Proof.** The proof can be found in [10, Lemma 3.1] and we present it for the sake of completeness. If \( X \subset I \) is such that \( y(t) > 0 \) for a.e. \( t \in X \), then \( y''(t) > 0 \) for a.e. \( t \in X \). In consequence, there exists at least one \( \tau \in I \) with \( y(\tau) \leq 0 \). If \( y(0) > 0 \), then there exist \( 0 \leq s_1 \leq s_2 \leq 2\pi \) with \( y(s_1) = y(s_2) = 0 \) and \( y(s) > 0 \) for \( s \in J = [0, s_1] \cup (s_2, 2\pi] \subset X \). Thus, \( y' \) is nondecreasing on \( J \) and we get a contradiction since \( y'(0) \geq y'(2\pi) \). Hence, \( y(0) \leq 0 \).

Now, if \( \max \{ y(s) : s \in I \} = y(t_0) > 0 \), then there exist \( t_1, t_2 \in (0, 2\pi) \) such that \( t_1 < t_0 < t_2 \), \( y(t_1) = y(t_2) = 0 \), and \( y(s) > 0 \) for \( s \in (t_1, t_2) \). In consequence, \( y' \) is nondecreasing on \( (t_1, t_2) \), and this is not possible since \( y(t_1) = y(t_2) = 0 \) and \( y(t_0) > 0 \).

**Lemma 5.** Let \( u \in W^{2,1}([t_1, t_2]) \), \( h \in L^1([t_1, t_2]) \) and \( c \) be constant, \( -u''(t) = f(t) \), with \( |f(t)| \leq h(t) \) for a.e. \( t \in [t_1, t_2] \). Then there exists a constant \( N > 0 \) depending only on \( c \) and \( h \) such that:

(a) \( u'(t_1) \leq c \) or \( u'(t_2) \leq c \) implies \( u'(t) \leq N \) on \([t_1, t_2] \).

(b) \( u'(t_1) \geq c \) or \( u'(t_2) \geq c \) implies \( u'(t) \geq -N \) on \([t_1, t_2] \).

**Proof.** (a) If \( u'(t_1) \leq c \), taking into account that \( u''(t) \leq |u''(t)| = |f(t)| \leq h(t) \), we obtain

\[
u'(t) \leq u'(t_1) + \int_{t_1}^{t} h(s) \, ds \leq c + \|h\|_1 \quad \text{on } [t_1, t_2].
\]

If \( u'(t_2) \leq c \), then from \( -u''(t) \leq -u''(t) \leq |f(t)| \leq h(t) \) we get

\[
u'(t) \leq u'(t_2) + \int_{t_2}^{t} h(s) \, ds \leq c + \|h\|_1 \quad \text{on } [t_1, t_2].
\]

(b) If \( u'(t_1) \geq c \), then \( -u''(t) \leq -u''(t_1) + \|h\|_1 \leq -c + \|h\|_1 \) on \([t_1, t_2] \), that is, \( u'(t) \geq -N \) on \([t_1, t_2] \).

If \( u'(t_2) \geq c \), then \( -u'(t) \leq -u'(t_2) + \|h\|_1 \leq -c + \|h\|_1 \), i.e. \( u'(t) \geq -N \) on \([t_1, t_2] \).
Using the previous lemmas we obtain the following a priori estimate for the solutions of problem (2.8).

**Theorem 1.** There exists a constant $M > 0$ such that if $\lambda \in [0, 1]$, $u \in X$ and $u = \lambda Tu$, then $\|u\|_X \leq M$.

**Proof.** The equation $u = \lambda Tu$ is equivalent to

$$
\begin{align*}
\left\{ \begin{array}{ll}
- u'' + u = \lambda f^*(t, u(t), u'(t)) + \lambda p(t, u(t)), \\
u(0) = u(2\pi), \quad u'(0) = u'(2\pi).
\end{array} \right.
\end{align*}
$$

(2.12)

We divide the proof into two parts:

**Step 1:** Estimate for $u(t)$. Let $I^0 = [0, 2\pi]$, $A_1 = \{ t \in I^0 : u(t) > w(t) \}$. We distinguish two cases:

1. $I^0 = A_1$. Then, for a.e. $t \in I^0$ we have

$$
- u''(t) + u(t) = \lambda f(t, u(t), u'(t)) + \lambda w(t) \leq -\lambda w''(t) + \lambda w(t).
$$

Hence, $y(t) = u(t) - \lambda w(t)$ satisfies

$$
\begin{align*}
\left\{ \begin{array}{ll}
y''(t) &\geq y(t) \quad \text{for a.e. } t \in I^0, \\
y(0) = y(2\pi), \quad y'(0) > y'(2\pi).
\end{array} \right.
\end{align*}
$$

From Lemma 4 we conclude that $y \leq 0$, that is, $u \leq \lambda w \leq C$ on $I$.

2. $I^0 \neq A_1$. Thus, there exists $s_1 \in I^0$ such that $u(s_1) \leq w(s_1)$. We first prove that there exists a positive constant $C$ depending only on $w$ such that $u(0) \leq C$. Obviously this is true if $w(0) \leq w(0)$.

In case $u(0) > w(0)$, let $y(t) = u(t) - \lambda w(t)$. We suppose that $y(0) > 0$ since $y(0) \leq 0$ implies that $u(0) \leq \lambda w(0)$.

For $y(0) > 0$, let $t_0 = \sup\{ t \in I : y(s) > 0 \text{ for } s \in [0, t] \}$ and

$$
t^* = \sup\{ t \in [0, s_1] : u(s) > w(s) \text{ for } s \in [0, t] \}.
$$

Then $t^* \leq s_1 < 2\pi$, $u(t^*) = w(t^*)$ and $u > w$ on $[0, t^*)$.

We shall prove that $t_0 > t^*$. If not, $y''(t) \geq y(t) > 0$ for a.e. $t \in [0, t_0]$ and $y'(t) > y'(0) \geq 0$. Hence, $y'(t_0) > y'(0) > 0$. By the definition of $t_0$ we see that $t_0 = 2\pi$, and $y'(2\pi) > y'(0)$. This implies that $w'(2\pi) < w'(0)$, a contradiction with (2.5). This shows that $t_0 > t^*$.

Therefore $y''(t) \geq y(t) > 0$ for a.e. $t \in [0, t^*)$ and thus $y'(t) > y'(0) \geq 0$. This implies that $y(0) \leq y(t^*) = u(t^*) - \lambda w(t^*) = (1 - \lambda)w(t^*)$ and that $u(0) \leq \lambda w(0) + (1 - \lambda)w(t^*) \leq C$.

For $y(0) < 0$, we have $y'(2\pi) \leq y'(0) < 0$, $y(2\pi) = y(0) > 0$, $u(2\pi) - w(2\pi) = u(0) - w(0) > 0$. Choosing $t_1 = \inf\{ t \in I : y(s) > 0 \text{ for } s \in (t, 2\pi) \}$ and $t = \inf\{ t \in (s, 2\pi) : u(s) > w(s) \text{ for } s \in (t, 2\pi) \}$ and reasoning as in the previous case we again obtain $u(0) \leq C$.

We decompose $A_1 = \bigcup (a_i, b_i)$ so that $u(t) > w(t)$ for $t \in (a_i, b_i)$ and

$$
- y''(t) + y(t) \leq 0 \quad \text{for a.e. } t \in (a_i, b_i).
$$

(2.13)
By the definition of \( a_i \) and \( b_i \) we obtain \( y(a_i) = (1 - \lambda)w(a_i) \) and \( y(b_i) = (1 - \lambda)w(b_i) \). In consequence, there exists \( C \in \mathbb{R} \) such that
\[
(2.14) \quad y(a_i) \leq C \quad \text{and} \quad y(b_i) \leq C.
\]

Now, (2.13) and (2.14) imply that \( y(t) \leq C + 1 \) for \( t \in (a_i, b_i) \). Therefore, \( u(t) \leq C + 1 + \lambda w(t) \leq M \) on \( \mathring{A}_1 \). Obviously, \( u \leq M \) on \( I \setminus \mathring{A}_1 \) and thus \( u \leq M \) on \( I \).

Similarly, we can prove that \( u \geq -M \) on \( I \). Hence \( |u(t)| \leq M \) for any \( t \in I \).

**Step 2: Estimate for \( u'(t) \).** Let \( B = \{ t \in I : v(t) < u(t) < w(t) \} \). Suppose that \( B \neq \emptyset \). Then \( p(t, u(t)) = u(t) \) for \( t \in B \) and \( u(t) \leq v(t) \) or \( u(t) \geq w(t) \) for \( t \in I \setminus B \). We write \( B = \bigcup (a_i, b_i) \) since \( B \) is an open set. For \( (a_i, b_i) \), only one of the following situations hold:

- **(2.i) \( 0 < a_i < b_i < 2\pi \), \( [u(a_i) - v(a_i)] - [w(a_i) - u(a_i)] = 0 \), \( [w(b_i) - u(b_i)] = 0 \) and \( v(t) < u(t) < w(t) \) for \( t \in (a_i, b_i) \).**

- **(2.ii) \( a_i = 0 \) or \( b_i = 2\pi \).**

In the first situation we have \( p(t, u(t)) = u(t) \) and \( \frac{d}{dt} p(t, u(t)) = u'(t) \).

Now, consider the following four cases:

- **(2.i.I) \( u(a_i) = v(a_i) \) and \( u(b_i) = v(b_i) \).** Then \( u'(a_i) \geq v'(a_i) \) and \( u'(b_i) \leq v'(b_i) \). Thus,
  \[
  -u'' = \lambda f(t, u, u') + (\lambda - 1)u \equiv \tilde{f}(t, u, u'),
  \]
  \[
  |\tilde{f}(t, u, u')| \leq g(|u'|) + C \equiv \tilde{g}(|u'|)
  \]
  and, by the hypothesis (H1),
  \[
  \int_{\lambda}^{\infty} \frac{s}{\tilde{g}(s) + K} \, ds = \infty \quad \forall \lambda > 0 \text{ and } \forall K > 0.
  \]

By Lemma 1 we know that there exists a constant \( N \) depending only on \( g, v \) and \( w \) such that \( |u'| \leq N \) on \( [a_i, b_i] \).

- **(2.i.II) \( u(a_i) = w(a_i) \) and \( u(b_i) = w(b_i) \).** Then \( |u'| \leq N \) on \( [a_i, b_i] \).

- **(2.i.III) \( u(a_i) = v(a_i) \) and \( u(b_i) = w(b_i) \).** Then \( u'(a_i) \geq v'(a_i) \) and \( u'(b_i) \geq w'(b_i) \). By Lemma 1, \( u'(t) \geq -N \) on \( [a_i, b_i] \).

If \( u'(a_i) > v'(a_i) \) or \( u'(b_i) > w'(b_i) \), then by Lemma 1, \( u' \leq N \) on \( [a_i, b_i] \).

Otherwise \( u'(a_i) > v'(a_i) \) and \( u'(b_i) > w'(b_i) \). Let \( a = \inf \{ t : u'(s) > v'(s) \text{ for } s \in (t, a_i) \} \) and \( b = \sup \{ t : u'(s) > w'(s) \text{ for } s \in (b_i, t) \} \). Then \( a < a_i < b_i < b \), \( u'(a) \geq v'(a) \) and \( u'(b) \geq w'(b) \). Moreover, \( u' > v' \) on \( (a_i, a) \) and \( u' > w' \) on \( (b_i, b) \).

Now, \( u(a_i) = v(a_i) \) and \( u(b_i) = w(b_i) \) imply that \( u > w \) on \( (b_i, b) \) and \( u < v \) on \( (a, a_i) \). We conclude that \( (u'(a) - v'(a)) \cdot (u'(b) - w'(b)) = 0 \). Otherwise, \( u'(a) > v'(a) \) and \( u'(b) > w'(b) \). Therefore \( a = 0 \) and \( b = 2\pi \) by
Thus \( u(0) < v(0) \leq w(0) = w(2\pi) < u(2\pi) \), and this is a contradiction.

If \( u'(b) = w'(b) \), then \(-u'' = \lambda f(t, w, w') + \lambda w - u \leq \lambda h(t) + c \) for a.e. \( t \in [b, b] \). By integration,

\[
u'(t) \leq \lambda \int_{t}^{b} h(s) \, ds + 2\pi c + u'(b) = u'(b) + \lambda \int_{t}^{b} h(s) \, ds + 2\pi c \leq C \quad \text{on} \ (b, b).
\]

Hence \( u'(b) \leq C \). Using again Lemma 1 we have \( u'(t) \leq N \) on \([a, b] \). If \( u'(a) = v'(a) \), then similarly we see that \( u'(t) \leq N \) on \([a, b] \). Hence \( |u'| \leq N \) on \([a, b] \).

(2.i.IV) If \( u(b) = v(b) \) and \( u(a) = w(a) \), then analogously to (2.i.III), \( |u'| \leq N \) on \([a, b] \).

To show (2.ii), suppose \( a_i = 0 \); the boundary conditions for \( v, u \) and \( w \) imply that \( b_i = 2\pi \).

Let \( a = \sup \{ t \in I : v(s) < u(s) < w(s) \text{ for } s \in [0, t] \} \). Then \( u(a) = v(a) \) or \( u(a) = w(a) \).

If \( u'(a) = v'(a) \), then it is clear that \( u'(a) \leq v'(a) \). Lemma 1 implies \( u'(a) \leq N \) for a.e. \( t \in [0, a] \). If \( u'(a) = v'(a) \) we obtain \( u'(t) \geq -N \); therefore \( u'(a) < v'(a) \).

Now, let \( t_0 = \sup \{ t \in I : u'(s) < v'(s) \text{ for } s \in (a, t) \} \).

If \( u'(t_0) < v'(t_0) \) we obtain \( t_0 = 2\pi \) and \( u(2\pi) < v(2\pi) \), which is a contradiction. In consequence, \( u'(t_0) = v'(t_0) \) and \( t_0 < 2\pi \). In the interval \((a, t_0)\) we have

\[-u'' = \lambda f(t, v, v') + \lambda v - u \geq -\lambda h + C.
\]

Thus

\[ -\int_{t}^{t_0} u''(s) \, ds \geq K \]

and \( u'(t) \geq K + v'(t_0) = K_1 \) on \((a, t_0)\). By continuity \( u'(a) \geq K_1 \), and Lemma 1 implies \( |u'| \leq N \) on \([0, a] \).

If \( u(a) = w(a) \), the reasoning is analogous.

If \( b_i = 2\pi \), we obtain \( |u'| \leq N \) on \([b, 2\pi] \) for

\[ b = \inf \{ t \in I : v(s) < u(s) < w(s) \text{ for } s \in (t, 2\pi) \}. \]

Thus, we obtain \( |u'(t)| \leq N \) for all \( t \in B \cup D \), with

\[ D = \{ a_i, b_i \in (0, 2\pi) : \text{either } (a_i, b_i) \in B; \text{ or } [0, b_i) \in B; \text{ or } (a_i, 2\pi] \in B \}. \]
If \( B \neq I^0 \), let \( B_1 = \{ t \in I : u(t) < v(t) \} \), \( B_2 = \{ t \in I : u(t) > w(t) \} \). Then \( B_1 \neq I \) and \( B_2 \neq I \).

First we suppose that \( B_1 \neq \emptyset \) and \( B_2 \neq \emptyset \). Decompose \( B_1 = \bigcup (a_i, b_i) \) and \( B_2 = \bigcup (c_i, d_i) \).

For \((a_i, b_i)\), we have one of the following possibilities:

1. \((2.A)\) \(0 < a_i < b_i < 2\pi\).
2. \((2.B)\) \(a_i = 0 \) or \( b_i = 2\pi\).

In the first case \( u'(a_i) \leq v'(a_i) \) and \( u'(b_i) \geq v'(b_i) \). Since \(-u'' = \lambda f(t, u, v') + \lambda v - u\), Lemma 5 implies \(|u'| \leq N\) on \([a_i, b_i]\).

In the second situation, we first consider \( a_i = 0 \). Then \( u(0) < v(0) \) and \( u(2\pi) < v(2\pi) \), that is, \( b_i = 2\pi\). In consequence, there exists \( a \in (0, 2\pi) \) such that \( u(a) = v(a) \) and \( u(t) < v(t) \) on \([0, a]\). Thus, without loss of generality, we can assume \( u'(a) > v'(a) \) (otherwise, Lemma 5 implies \(|u'| \leq N\) on \([0, a]\)).

Now, if \( v(a) < w(a) \), let \( b = \sup\{t \in I : v(s) < w(s) \text{ for } s \in [a, t]\} \). Hence, there exists \( t \leq b \) such that \( u'(t) > v'(t) \) on \([a, t]\). Therefore \( v(t) < u(t) < w(t) \) on \([a, t]\), and consequently \( a \in D \). Thus \(|u'(a)| \leq N \) and Lemma 5 assures that \(|u'| \leq N\) on \([0, a]\).

On the other hand, if \( v(a) = w(a) \) and \( v'(a) < u'(a) < w'(a) \) there exists a subinterval \((a, a + \delta) \subset (0, 2\pi) \) such that \( v < u < w \) on \((a, a + \delta) \); then \( a \in D \) and \(|u'(a)| \leq N\). Again, Lemma 5 implies \(|u'| \leq N\) on \([0, a]\).

Finally, if \( v(a) = w(a) \) and \( u'(a) > w'(a) \) there exists \( t_0 \in (0, 2\pi) \) such that \( u > w \) on \((a, t_0) \) with \( u'(t_0) = w'(t_0) \). Therefore \(-u'' = \lambda f(t, v, v') + \lambda v - u\) and \( u'(t) \leq w'(t) + c \) for all \( t \in (a, t_0) \). The continuity of \( u' \) and Lemma 5 imply \(|u'| \leq N\) on \([0, a]\).

If \( b_i = 2\pi \) the proof is analogous.

For the set \( B_2 \) the reasoning is similar.

Thus, we obtain \(|u'(t)| \leq N\) for all \( t \in E \cup F \equiv S \), where \( E = B \cup B_1 \cup B_2 \) and

\[
F = \{a_i, b_i \in (0, 2\pi) : \text{ either } (a_i, b_i) \in E; \text{ or } [0, b_i) \in E; \text{ or } (a_i, 2\pi) \in E\}.
\]

If \( t \in I \setminus S \), then obviously either \( u(t) = v(t) \) or \( u(t) = w(t) \). Also there exists \( \{x_n\} \subset F, x_n \neq t \) for all \( n \in \mathbb{N} \), such that \( t = \lim_{n \to \infty} x_n \) because if there exists \( \delta > 0 \) such that \( t \in (t - \delta, t + \delta) \cap F = \emptyset \) then \( t \in S \). Since \(|u'(x_n)| \leq N\) for all \( \{x_n\} \subset F \) we obtain \(|u'(t)| = |\lim_{n \to \infty} u'(x_n)| \leq N\) for all \( t \in I \setminus S \).

This completes the proof of Theorem 1. □

**Theorem 2.** Suppose that \( v(t) \leq w(t) \) are lower and upper solutions of \((P)\), respectively. If \((H1)\) holds, then there exists a solution \( u \) of \((P)\) such that \( u \in [v, w] \).
Proof. Let $X = C^1(I)$. By Lemma 2, $\frac{d}{dt}p(t, u(t))$ exists for a.e. $t \in I$. Problem (2.8) is equivalent to the functional equation $u = Tu$, with $T$ defined as in Lemma 3. By Theorem 1 we know that every solution of $u = \lambda Tu$ satisfies $|u|_X \leq M$ for some constant $M > 0$. In consequence, the Sader theorem [11] implies that there exists a solution $u$ of problem (2.8).

Finally, we prove that every solution $u$ of (2.8) is such that $u \in [v, w]$, that is, $u$ is a solution in $[v, w]$ of problem (P). Indeed, suppose that $u > w$ on $[0, 2\pi]$. Then

$$-u'' + u = f(t, w, w') + w \leq -u'' + w.$$ 

Since $(u - w)(0) = (u - w)(2\pi)$ and $(u - w)'(0) \geq (u - w)'(2\pi)$, Lemma 4 implies that $u \leq w$ on $[0, 2\pi]$, which is a contradiction. Consequently, there exists $s \in [0, 2\pi]$ such that $u(s) \leq w(s)$. If there exists $s_1 \in [0, 2\pi]$ with $u(s_1) > w(s_1)$, and there exists $t_1 < t_2 \in (0, 2\pi)$ such that $u > w$ on $(t_1, t_2)$, with $(u - w)(t_1) = (u - w)(t_2) = 0$, then in the interval $(t_1, t_2)$ we have

$$-u'' + u = f(t, w, w') + w \leq -u'' + w.$$ 

This, together with the boundary conditions, implies that $u \leq w$ on $(t_1, t_2)$, which is a contradiction.

Therefore, suppose that there exist $t_1 \leq t_2 \in (0, 2\pi)$ such that $u > w$ on $[0, t_1] \cup (t_2, 2\pi]$, with $(u - w)(t_1) = (u - w)(t_2) = 0$. In both intervals we have $(u - w)'' \geq u - w > 0$.

If $(u - w)'(0) \geq 0$ then $(u - w)'(t) > 0$ for any $t \in [0, t_1)$ and $(u - w)(t_1) > (u - w)(0) > 0$, which is not possible.

On the other hand, if $(u - w)'(0) < 0$, we obtain $(u - w)'(2\pi) < 0$. In consequence, $(u - w)' < 0$ on $(t_2, 2\pi]$ and $(u - w)(t_2) > (u - w)(2\pi) > 0$.

Therefore $u \leq w$ on the interval $I$. Analogously we can prove that $u \geq v$ on $I$. Hence, every solution of (2.8) is a solution of problem (P) in the sector $[v, w]$.

This completes the proof of Theorem 2.

3. Monotone iterative technique. Throughout this section we suppose that $v \leq w$ are lower and upper solutions of (P), respectively. We introduce the following hypotheses:

(H2) There exists $M \in L^1(I)$ such that $M(t) > 0$ for a.e. $t \in I$ and

\begin{equation}
(f(t, \phi, s) - f(t, \varphi, s) \geq -M(t)(\phi - \varphi)
\end{equation}

for a.e. $t \in I$ and every $v(t) \leq \varphi \leq \phi \leq w(t)$, $s \in \mathbb{R}$.

(H3) There exists $N \in L^1(I)$ such that $N(t) \geq 0$ for a.e. $t \in I$ and

\begin{equation}
(f(t, u, s) - f(t, u, y) \geq -N(t)(s - y)
\end{equation}

for a.e. $t \in I$ and every $v(t) \leq u \leq w(t)$, $s \geq y, s, y \in \mathbb{R}$.
Theorem 3. Suppose that (H1)–(H3) hold. Then there exist monotone sequences \( v_n \nearrow x \) and \( w_n \searrow z \) as \( n \to \infty \), uniformly on \( I \), with \( v_0 = v \) and \( w_0 = w \). Here, \( x \) and \( z \) are the minimal and maximal solutions of (P) respectively on \([v,w]\), that is, if \( u \in [v,w] \) is a solution of (P), then \( u \in [x,z] \). Moreover, the sequences \( \{v_n\} \) and \( \{w_n\} \) satisfy \( v = v_0 \leq \ldots \leq v_n \leq \ldots \) \( \ldots \leq w_n \leq \ldots \leq w_0 = w \).

Proof. For any \( q \in [v,w] \cap X \), consider the following quasilinear periodic boundary value problem:

\[
\begin{cases}
-u''(t) = f(t, q(t), \frac{d}{dt}p(t, u(t))) + M(t)q(t) - u(t), \\
u(0) = u(2\pi), \quad u'(0) = u'(2\pi).
\end{cases}
\]

Using (3.1), we deduce that if \( u \) is a solution of (3.3), then

\[
f\left(t, v(t), \frac{d}{dt}p(t, u(t))\right) + Mv(t) \leq -u''(t) + Mu(t)
\]

\[
\leq f\left(t, w(t), \frac{d}{dt}p(t, u(t))\right) + Mw(t).
\]

Using (2.1), (H1) and (3.4), and reasoning as in the proof of Theorem 1, we can say that (3.3) has a solution \( u \in X \). It is not difficult (using Lemma 4) to prove that this solution is unique. Using the same arguments as in the proof of Theorem 2.1 of [10], it can be proved that \( v \leq u \leq w \). Hence (3.3) is equivalent to

\[
\begin{cases}
-u''(t) = f(t, q(t), u'(t)) + M(t)q(t) - u(t), \\
u(0) = u(2\pi), \quad u'(0) = u'(2\pi).
\end{cases}
\]

Now, define the operator \( T : X \to X \), \( T(q) = u \), where \( u \) is the solution of (3.3).

We shall prove that if \( v \leq q_1 \leq q_2 \leq w \), \( q_1, q_2 \in X \), then \( T(q_1) \leq T(q_2) \). Indeed, let \( u_i = T(q_i), i = 1,2 \). Then

\[
\begin{cases}
-u''_i(t) = f(t, q_i(t), u'_i(t)) + M(t)q_i(t) - u_i(t), \\
u_i(0) = u_i(2\pi), \quad u'_i(0) = u'_i(2\pi).
\end{cases}
\]

If \( u_1 \leq u_2 \) is not true, then there exist \( \varepsilon > 0 \) and \( t_0 \in I \) such that \( u_1(t_0) = u_2(t_0) + \varepsilon \) and \( u_1 \leq u_2 + \varepsilon \) on \( I \).

First, we shall prove that there exists \( (t_1, t_2) \subset I^0 \) such that \( u_1 > u_2 \) and \( u'_1 \leq u'_2 \) on \( (t_1, t_2) \), \( u'_1(t_1) = u'_2(t_1) \) and \( u_1(t_1) - u_2(t_1) \geq u_1(t_2) - u_2(t_2) \).

Indeed, let \( y(t) = u_1(t) - u_2(t) \). If there exists \( [t_1, t_2] \) such that \( y(t) = \varepsilon \) on \( [t_1, t_2] \), then the conclusion holds. Suppose that for any subinterval \( (a, b) \subset I^0 \), there exists \( t \in (a, b) \) such that \( y(t) < \varepsilon \). If \( t_0 = 2\pi \), then \( t_0 = 0 \). Thus \( y(0) = y(2\pi) = \varepsilon \) and \( 0 \leq y'(2\pi) = y'(0) \leq 0 \). If \( t_0 \in I^0 \), then \( y'(t_0) = 0 \). Hence we always have \( y'(t_0) = 0 \).
Since \( y(0) = y(2\pi) \), we can take \( t_0 < 2\pi \). Because \( y(t_0) = \varepsilon \geq y(t) \) and \( y(t) \neq \varepsilon \) in any right neighborhood of \( t_0 \), there exists \( t_2 \in (t_0, 2\pi) \) such that \( y'(t_2) < 0 \) and \( y(t_2) > 0 \). Hence, there exists \( t_1 \in [t_0, t_2) \) such that \( y'(t_1) = 0 \) and \( y(t) < 0 \) for \( t \in (t_1, t_2) \). Consequently, \((t_1, t_2)\) satisfies our requirements.

We consider (3.6) in \((t_1, t_2)\). Since \( y' \leq 0 \) on \((t_1, t_2)\), (H2) and (H3) imply that

\[
- u'(t) + u''(t) = f(t, u(t), u'(t)) - f\left(t, q_2(t), u_2'(t)\right) + M(t)q_1(t) - q_2(t),
\]

\[
-M(t)\left[q_1(t) - u_2'(t)\right] \leq -N(t)\left[u_1'(t) - u_2'(t)\right] - M(t)\left[u_1(t) - u_2(t)\right]
\]

for a.e. \((t_1, t_2)\).

The function \( y = u_1 - u_2 \) satisfies

\[
\begin{align*}
y''(t) &\geq M(t)y(t) + N(t)y'(t) > N(t)y'(t), \\
y(t_1) &\geq y(t_2),
\end{align*}
\]

for a.e. \((t_1, t_2)\).

Solving the differential inequality, we obtain

\[
y'(t_2) \exp \left( - \int_{t_1}^{t_2} y'(s) \, ds \right) > y'(t_1) = 0.
\]

This is a contradiction with \( y'(t_2) \leq 0 \). Therefore, \( u_1 \leq u_2 \) on \( I \).

Now, define sequences \( v_0 = v, v_n = T(v_{n-1}) \), \( w_0 = w \) and \( w_n = T(w_{n-1}) \). Because the solution \( u \) of (3.3) satisfies \( v \leq u \leq w \) on \( I \), using the monotonicity of \( T \) we see that \( v = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = w \). Hence, the limits \( \lim_{n \to \infty} v_n(t) = x(t) \) and \( \lim_{n \to \infty} w_n(t) = z(t) \) exist. Note that \( v_n \) satisfies

\[
\begin{align*}
-v_n''(t) &= f(t, v_{n-1}(t), v_n'(t)) + M(t)\left[v_{n-1}(t) - v_n(t)\right] \equiv \tilde{f}(t, v_n(t), v_n'(t)), \\
v_n(0) &= v_n(2\pi), \quad v_n'(0) = v_n'(2\pi), \quad v(t) \leq v_n(t) \leq w(t),
\end{align*}
\]

and

\[
|\tilde{f}(t, v_n(t), v_n'(t))| \leq g(|v_n'(t)|) + C \equiv \tilde{g}(|v_n'(t)|)
\]

and

\[
\int_{\chi}^{\infty} \frac{s}{\tilde{g}(s) + K} \, ds = \infty.
\]

By Lemma 1, there exists a constant \( N \) depending only on \( g, v \) and \( w \) such that \( |v_n'| \leq N \) on \( I \) for any \( n = 1, 2, \ldots \), that is, \( \{v_n\} \) is a bounded set of \( X \).

Similarly, \( \{w_n\} \) is a bounded set of \( X \). Using the same arguments as in Lemma 3, it follows that \( v_n \xrightarrow{X} x \) and \( w_n \xrightarrow{X} z \), that is,

\[
\lim_{n \to \infty} (v_n(t), v_n'(t), w_n(t), w_n'(t)) = (x(t), x'(t), z(t), z'(t)) \quad \text{uniformly on} \ I.
\]
Writing the integral equations of \(v\) and \(w\) respectively and using standard arguments, we deduce that \(x\) and \(z\) satisfy (P) and \(v \leq x \leq z \leq w\) on \(I\). Now, we know that if \(u \in X\), \(v \leq u \leq w\) and \(u\) solves (P), then \(Tu = u\), so that \(v_n \leq u \leq w_n\) for any \(n = 1, 2, \ldots\) and thus \(x \leq u \leq z\) on \(I\).

This completes the proof of Theorem 3.

Acknowledgements. The authors are thankful to the referee for helpful comments and suggestions.

References


