Some results on stability and on characterization of $K$-convexity of set-valued functions

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Abstract. We present a stability theorem of Ulam–Hyers type for $K$-convex set-valued functions, and prove that a set-valued function is $K$-convex if and only if it is $K$-midconvex and $K$-quasiconvex.

1. Introduction. In this paper we study two different problems:
(i) stability of the $K$-convexity of a set-valued function;
(ii) characterization of $K$-convex set-valued functions.

The first problem has been studied for functions: in 1941 D. H. Hyers [5] proved that the property of additivity is stable, i.e. if a function $f$ satisfies
\[(1.1) \quad |f(x + y) - f(x) - f(y)| \leq \varepsilon ,\]
where $\varepsilon$ is a given positive number, then there exists an additive function $g$ such that
\[(1.2) \quad |f(x) - g(x)| \leq \varepsilon .\]

In 1952 D. H. Hyers and S. M. Ulam [6] stated that the property of convexity is stable, that is, for every function $f : D \to \mathbb{R}$, where $D$ is a convex subset of $\mathbb{R}^n$, satisfying the inequality
\[(1.3) \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon ,\]
for all $x, y \in D$, $t \in [0,1]$ and some $\varepsilon > 0$, there exists a convex function $g : D \to \mathbb{R}$ and a constant $k_n$, depending only on the dimension of the domain, such that
\[(1.4) \quad g(x) \leq f(x) \leq g(x) + k_n \varepsilon , \quad \forall x \in D .\]

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Later, in 1988, K. Nikodem [10] showed that the property of quasiconvexity of a function is also stable.

For the second problem, in 1989 K. Nikodem [10] obtained the following characterization for convex functions defined on an open subset of \( \mathbb{R}^n \):

\[
f \text{is convex} \iff f \text{ is midconvex and quasiconvex.}
\]

Next Z. Kominek [7] and F. A. Behringer [2] showed that (1.5) is also true for functions defined on any convex subset of a real vector space, not necessarily open.

In Section 3 of our note we prove (cf. Theorem 1) that if \( D \) is a convex subset of \( \mathbb{R}^n \), \( K \) a convex cone in \( \mathbb{R}^m \) and \( B \) the closed unit ball of \( \mathbb{R}^m \), then for every set-valued function \( F : D \to n(\mathbb{R}^m) \) (cf. (2.1)) satisfying

\[
(1.3)_1 \quad tF(x) + (1 - t)F(y) \subset F(tx + (1 - t)y) + K + \varepsilon B
\]

for all \( x, y \in D, \ t \in [0, 1] \) and some \( \varepsilon > 0 \), there exists a convex set-valued function \( G : D \to n(\mathbb{R}^m) \) such that

\[
(1.4)_1 \quad F(x) \subset G(x) \subset F(x) + K + j_{n+m}\varepsilon B, \quad \forall x \in D,
\]

where the constant \( j_{n+m} \) depends only on the dimension of \( \mathbb{R}^{n+m} \).

In Section 4 we prove (cf. Corollary 1) that if \( D \) is a convex subset of a real vector space, \( K \) a closed convex cone of a real topological vector space \( Y, \ t \in (0, 1) \) and \( F : D \to C(Y) \) (cf. (2.2)) a set-valued function, then, under some assumption on \( Y \) (cf. Remark 1),

\[
(1.5)_1 \quad F \text{ is } K\text{-convex} \iff F \text{ is } K-t\text{-convex and } K\text{-quasiconvex.}
\]

This result contains the mentioned theorems proved in [10]2, in [7] and in [2].

Finally, we want to observe that our Theorem 3 is a generalization to set-valued functions of a result of N. Kuhn [8] stating that \( t\)-convex (single-valued) functions are midconvex.

2. Let \( X \) be a real vector space and \( Y \) be a real topological vector space (satisfying the \( T_0 \) separation axiom). For \( \alpha, \beta \in \mathbb{R} \) and \( S, T \subset Y \) we put \( \alpha S + \beta T = \{ y \in Y : y = as + bt, \ s \in S, \ t \in T \} \). We define

\[
(2.1) \quad n(Y) = \{ S \subset Y : S \neq \emptyset \},
(2.2) \quad C(Y) = \{ S \subset Y : S \text{ compact}, S \neq \emptyset \},
(2.3) \quad BC(Y) = \{ S \subset Y : S \text{ bounded, convex}, S \neq \emptyset \}.
\]
We assume that $D$ is a nonempty convex subset of $X$ and $K$ is a convex cone in $Y$. For fixed $t \in (0, 1)$, we say that a set-valued function $F : D \to n(Y)$ is $K$-convex if

$$tF(x) + (1 - t)F(y) \subset F(tx + (1 - t)y) + K$$

for all $x, y \in D$. If $t = 1/2$, $F$ is called $K$-midconvex.

We say that $F$ is $K$-quasiconvex if for every convex set $A \subset Y$ the lower inverse image of $A - K$, i.e. the set

$$F^{-}(A - K) = \{ x \in D : F(x) \cap (A - K) \neq \emptyset \},$$

is convex (cf. [10], (2.5)).

In the case that $Y$ is a normed space, let $B$ be the closed unit ball in $Y$ and $\varepsilon$ a nonnegative number. We say that $F : D \to n(Y)$ is $\varepsilon$-$K$-convex if

$$tF(x) + (1 - t)F(y) \subset F(tx + (1 - t)y) + K + \varepsilon B$$

for all $x, y \in D$ and $t \in [0, 1]$. We recall that $F$ is $K$-convex if it satisfies (2.4) with $\varepsilon = 0$. If $F$ satisfies (2.4) with $\varepsilon = 0$ and $K = \{0\}$ it is said to be convex.

$F : D \to n(Y)$ is said to be weakly $K$-upper bounded on a set $A \subset D$ iff

$$\text{Gr } F = \{(x, y) \in X \times Y : x \in D, y \in F(x)\}$$

the graph of the set-valued function $F$.

3. In this section we present, for $\varepsilon$-$K$-convex set-valued functions, a theorem analogous to the stability theorem for functions proved by D. H. Hyers and S. M. Ulam in [6] and by P. W. Cholewa in [3].

Using a method similar to Cholewa’s [3] we first prove

**Lemma 1.** Let $X$ be a real vector space, $Y$ a normed space, $D$ a convex subset of $X$ and $K$ a convex cone in $Y$. If a set-valued function $F : D \to n(Y)$ is $\varepsilon$-$K$-convex, then for all $p \in \mathbb{N}$, $x_0, \ldots, x_p \in D$ and $t_0, \ldots, t_p \in [0, 1]$ with $t_0 + \ldots + t_p = 1$, we have

$$(3.1) \quad t_0F(x_0) + \ldots + t_pF(x_p) \subset F(t_0x_0 + \ldots + t_px_p) + K + j_peB$$

where $j_p = \min\{k_p, h_p\}$, $k_p = (p^2 + 3p)/(2p + 2)$, and $h_p = m \in \mathbb{N}$ is such that $2^{m-1} \leq p < 2^m$.

**Proof.** For $p = 1$ the inclusion (3.1) is clear because $j_1 = k_1 = h_1 = 1$. Now fix $p > 1$ and assume that (3.1) holds for all natural $n < p$. Take $x_0, \ldots, x_p \in D$ and $t_0, \ldots, t_p \in [0, 1]$ with $t_0 + \ldots + t_p = 1$. Without loss
of generality we may assume that \( t_0 \geq 1/(p+1) \). Let \( t = t_1 + \ldots + t_p \) and \( t'_i = t_i/t \) for \( i = 1, \ldots, p \); then \( t \leq p/(p+1) \). Thus

\[
(3.2) \quad t_0 F(x_0) + \ldots + t_p F(x_p) = t_0 F(x_0) + t[t'_1 F(x_1) + \ldots + t'_p F(x_p)]
\]

\[
\quad \subset t_0 F(x_0) + t[F(t'_1 x_1 + \ldots + t'_p x_p) + K + k_{p-1} \varepsilon B]
\]

\[
\quad \subset F(t_0 x_0 + \ldots + t_p x_p) + K + \varepsilon B + \frac{p}{p+1} k_{p-1} \varepsilon B
\]

Now, let \( m \in \mathbb{N} \) be such that \( 2^{m-1} \leq p < 2^m \). Put \( r = \lfloor p/2 \rfloor \); then \( r < 2^{m-1} \) and \( p - r - 1 < 2^{m-1} \). Setting \( a = t_0 + \ldots + t_r \) and \( b = t_{r+1} + \ldots + t_p \), we have

\[
(3.3) \quad t_0 F(x_0) + \ldots + t_p F(x_p)
\]

\[
\quad = a \left[ \frac{t_0}{a} F(x_0) + \ldots + \frac{t_r}{a} F(x_r) \right] + b \left[ \frac{t_{r+1}}{b} F(x_{r+1}) + \ldots + \frac{t_p}{b} F(x_p) \right]
\]

\[
\subset a F \left( \frac{t_0}{a} x_0 + \ldots + \frac{t_r}{a} x_r \right) + b F \left( \frac{t_{r+1}}{b} x_{r+1} + \ldots + \frac{t_p}{b} x_p \right)
\]

\[
\quad \quad + K + a h_r \varepsilon B + bh_{p-r-1} \varepsilon B
\]

\[
\subset F(t_0 x_0 + \ldots + t_p x_p) + K + [1 + a h_r + bh_{p-r-1}] \varepsilon B
\]

\[
\subset F(t_0 x_0 + \ldots + t_p x_p) + K + [1 + a(m - 1) + b(m - 1)] \varepsilon B
\]

\[
= F(t_0 x_0 + \ldots + t_p x_p) + K + h_p \varepsilon B.
\]

From (3.2) and (3.3) we obtain the assertion.

**Theorem 1.** Let \( D \) be a convex subset of \( \mathbb{R}^n \) and \( K \) be a convex cone in \( \mathbb{R}^m \). If a set-valued function \( F : D \to n(\mathbb{R}^m) \) is \( \varepsilon \)-\( K \)-convex, then there exists a convex set-valued function \( G : D \to n(\mathbb{R}^m) \) such that

\[
F(x) \subset G(x) \subset F(x) + K + j_{n+m} \varepsilon B
\]

for all \( x \in D \).

**Proof.** Let \( W \) be the convex hull of the graph of \( F \). We define \( G : D \to n(\mathbb{R}^m) \) by

\[
G(x) = \{ y \in \mathbb{R}^m : (x, y) \in W \}, \quad x \in D.
\]

Then \( G \) is convex because \( \text{Gr} \ G = W \) is convex. Moreover, \( F(x) \subset G(x) \) for all \( x \in D \). To prove the second inclusion fix an \( x \in D \) and take an arbitrary \( y \in G(x) \). Then \( (x, y) \in W \). By the Carathéodory Theorem (cf. [12], Theorem 17.1) we have

\[
(x, y) = \sum_{i=0}^{n+m} t_i (x_i, y_i),
\]
with some \((x_i, y_i) \in \text{Gr} F\) and \(t_0, \ldots, t_{n+m} \in [0, 1]\), \(t_0 + \ldots + t_{n+m} = 1\). Hence, using Lemma 1, we get
\[
y = \sum_{i=0}^{n+m} t_i y_i \in \sum_{i=0}^{n+m} t_i F(x_i) \subset F(x) + K + j_{n+m} \varepsilon B.
\]
Since this holds for all \(y \in G(x)\), the proof is complete.

4. In this section we give two necessary and sufficient conditions for a set-valued function to be \(K\)-convex. We first need the following lemma which is an analogue of a result obtained for functions by C. T. Ng and K. Nikodem (cf. [9], Lemma 6).

**Lemma 2.** Let \(K\) be a closed convex cone in a real topological vector space \(Y\). If \(F : [0, 1] \to C(Y)\) is \(K\)-midconvex on \([0, 1]\) and \(K\)-convex on \((0, 1)\), then it is \(K\)-convex on \([0, 1]\).

**Proof.** Fix \(x, y \in [0, 1]\) and \(t \in (0, 1)\), and put \(z = tx + (1-t)y\). Let \(u = (x+z)/2\) and \(v = (y+z)/2\). Then \(u, v \in (0, 1)\) and \(z = tu + (1-t)v\). Since \(F\) is \(K\)-convex on \((0, 1)\) we get
\[
tF(u) + (1-t)F(v) \subset F(z) + K.
\]
On the other hand, by the \(K\)-midconvexity of \(F\) on \([0, 1]\),
\[
\frac{F(x) + F(z)}{2} \subset F(u) + K \quad \text{and} \quad \frac{F(y) + F(z)}{2} \subset F(v) + K.
\]
Therefore, by (4.2) and (4.1),
\[
tF(x) + (1-t)F(y) + F(z) \subset t(F(x) + F(z)) + (1-t)(F(y) + F(z))
\subset 2tF(u) + 2(1-t)F(v) + K
\subset 2F(z) + K \subset F(z) + F(z) + K.
\]
The set \(F(z) + K\) is convex and closed, and \(F(z)\) is bounded; so the law of cancellation (cf. [11]) yields the assertion.

**Theorem 2.** Let \(X\) be a real vector space, \(Y\) a real topological vector space, \(D\) a convex subset of \(X\) and \(K\) a closed convex cone in \(Y\). Moreover, assume that there exists a family \((B_n)_n\), \(B_n \in BC(Y)\) (cf. (2.3)), such that
\[
Y = \bigcup_{n \in \mathbb{N}} (B_n - K).
\]
Then a set-valued function \(F : D \to C(Y)\) is \(K\)-convex if and only if it is \(K\)-midconvex and \(K\)-quasiconvex.
Proof. The necessity is trivial (cf. [10], Theorem 2.9). Now suppose $F$ is $K$-midconvex and $K$-quasiconvex. Fix $x, y \in D$, and define $H : [0, 1] \to C(Y)$ by

$$H(t) = F(tx + (1 - t)y), \quad \forall t \in [0, 1].$$

Clearly $H$ is $K$-quasiconvex; therefore for all $n \in \mathbb{N}$, the set

$$H^-(B_n - K) = \{ t \in [0, 1] : H(t) \cap (B_n - K) \neq \emptyset \}$$

is an interval in $\mathbb{R}$. In view of (4.3) we have

$$\bigcup_{n \in \mathbb{N}} H^-(B_n - K) = [0, 1],$$

and so we can find a natural number $p$ such that

$$\text{int} H^-(B_p - K) \neq \emptyset.$$

By the $K$-midconvexity of $F$ it follows that $H$ is $K$-midconvex on $[0, 1]$, and (cf. (4.5) and (4.6)) $H$ is weakly $K$-upper bounded (cf. (2.5)) on $H^-(B_p - K)$, which has nonempty interior; then using Corollary 3.3 of [10] we deduce that $H$ is $K$-continuous on $(0, 1)$. Consequently, $H$ is $K$-convex on $(0, 1)$ (cf. [10], Theorem 3.1 or [1], Theorem 4.2) and so it follows by Lemma 2 that $H$ is $K$-convex on $[0, 1]$. Therefore, by (4.4),

$$tF(x) + (1 - t)F(y) = tH(1) + (1 - t)H(0) \subset H(t) + K = F(tx + (1 - t)y) + K,$$

which proves the $K$-convexity of $F$.

Remark 1. The assumption (4.3) is trivially satisfied if $Y$ is a normed space. It is also fulfilled if there exists an order unit in $Y$, i.e. an element $e \in Y$ such that for every $y \in Y$ we can find an $n \in \mathbb{N}$ with $y \in ne - K$ (then we can assume $B_n = \{ ne \}$). In particular, if $\text{int} K \neq \emptyset$, then every element of $\text{int} K$ is an order unit in $Y$.

Theorem 3. Let $X$ be a real vector space, $Y$ be a real topological vector space, $D$ a convex subset of $X$ and $K$ a closed convex cone in $Y$. Let $t$ be a fixed number in $(0, 1)$. If a set-valued function $F : D \to C(Y)$ is $K$-t-convex, then it is $K$-midconvex.

Proof. Observe first that $F(x) + K$ is convex for all $x \in D$ because

$$tF(x) + (1 - t)F(x) \subset F(x) + K$$

and $F(x) + K$ is closed.
Let $x, y \in D$; using the $K$-$t$-convexity of $F$ we get
\[
t(1-t)F(x) + t(1-t)F(y) + [1 - 2t(1-t)]F\left(\frac{x + y}{2}\right)
\]
\[
\subset t \left[ (1-t)F(x) + tF\left(\frac{x + y}{2}\right) \right] + (1-t) \left[ tF(y) + (1-t)F\left(\frac{x + y}{2}\right) \right]
\]
\[
\subset tF\left( (1-t)x + t\frac{x + y}{2} \right) + (1-t)F\left( ty + (1-t)\frac{x + y}{2} \right) + K
\]
\[
\subset F\left(\frac{x + y}{2}\right) + K
\]
\[
\subset 2t(1-t)F\left(\frac{x + y}{2}\right) + [1 - 2t(1-t)]F\left(\frac{x + y}{2}\right) + K.
\]
Since the set $2t(1-t)F\left(\frac{x + y}{2}\right) + K$ is convex and closed and the set $[1 - 2t(1-t)]F\left(\frac{x + y}{2}\right)$ is bounded, by the law of cancellation we obtain
\[
t(1-t)F(x) + t(1-t)F(y) \subset 2t(1-t)F\left(\frac{x + y}{2}\right) + K.
\]
Hence
\[
\frac{1}{2} [F(x) + F(y)] \subset F\left(\frac{x + y}{2}\right) + K,
\]
which was to be proved.

Remark 2. In the case of real (single-valued) functions the above result is a consequence of the theorem of N. Kuhn [8]. The idea of the presented proof is taken from Lemma 1 of [4].

As an immediate consequence of Theorems 2 and 3 we obtain the following

**Corollary 1.** Let $X$ be a real vector space, $Y$ a real topological vector space, $D$ a convex subset of $X$, $K$ a closed convex cone in $Y$ and $t$ a fixed number in $(0,1)$. Moreover, assume that there exists a family $(B_n)_n$, $B_n \in BC(Y)$, such that
\[
Y = \bigcup_{n \in \mathbb{N}} (B_n - K).
\]
Then a set-valued function $F : D \to C(Y)$ is $K$-convex if and only if it is $K$-$t$-convex and $K$-quasiconvex.

Remark 3. Observe that, in the case where $K = \{0\}$, it is sufficient to require that the values of the set-valued function in Lemma 2, Theorem 2, Theorem 3 and Corollary 1 are closed and bounded (and not necessarily compact). The corresponding proofs are similar to those given above.
References


