ANNALES
POLONICI MATHEMATICI
LVIII.2 (1993)

Some results on stability and on characterization of K-convexity of set-valued functions

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Abstract. We present a stability theorem of Ulam–Hyers type for K-convex set-valued functions, and prove that a set-valued function is K-convex if and only if it is K-midconvex and K-quasiconvex.

1. Introduction. In this paper we study two different problems:

(i) stability of the K-convexity of a set-valued function;
(ii) characterization of K-convex set-valued functions.

The first problem has been studied for functions: in 1941 D. H. Hyers [5] proved that the property of additivity is stable, i.e. if a function f satisfies

$$|f(x+y)-f(x)-f(y)| \le \varepsilon\,, \tag{1.1}$$

where ε is a given positive number, then there exists an additive function g such that

$$|f(x)-g(x)| \le \varepsilon\,. \tag{1.2}$$

In 1952 D. H. Hyers and S. M. Ulam [6] stated that the property of convexity is stable, that is, for every function $f : D \to \mathbb{R}$, where D is a convex subset of $\mathbb{R}^n$, satisfying the inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon\,, \tag{1.3}$$

for all $x, y \in D$, $t \in [0,1]$ and some $\varepsilon > 0$, there exists a convex function $g : D \to \mathbb{R}$ and a constant k_n, depending only on the dimension of the domain, such that

$$g(x) \le f(x) \le g(x)+k_n\varepsilon\,, \quad \forall x \in D\,. \tag{1.4}$$

1991 *Mathematics Subject Classification*: 26B25, 54C60.
Key words and phrases: set-valued functions, K-convex (K-midconvex, K-quasiconvex) set-valued functions, Ulam–Hyers stability.

In 1984 P. W. Cholewa [3] gave a different proof of the theorem of Hyers and Ulam.

Later, in 1988, K. Nikodem $[10]_1$ showed that the property of quasiconvexity of a function is also stable.

For the second problem, in 1989 K. Nikodem $[10]_2$ obtained the following characterization for convex functions defined on an open subset of $\mathbb{R}^n$:

$$f \text{ is convex} \Leftrightarrow f \text{ is midconvex and quasiconvex.} \tag{1.5}$$

Next Z. Kominek [7] and F. A. Behringer [2] showed that (1.5) is also true for functions defined on any convex subset of a real vector space, not necessarily open.

In Section 3 of our note we prove (cf. Theorem 1) that if D is a convex subset of $\mathbb{R}^n$, K a convex cone in $\mathbb{R}^m$ and B the closed unit ball of $\mathbb{R}^m$, then for every set-valued function $F : D \to n(\mathbb{R}^m)$ (cf. (2.1)) satisfying

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K + \varepsilon B \tag*{$(1.3)_1$}$$

for all $x, y \in D$, $t \in [0,1]$ and some $\varepsilon > 0$, there exists a convex set-valued function $G : D \to n(\mathbb{R}^m)$ such that

$$F(x) \subset G(x) \subset F(x) + K + j_{n+m}\varepsilon B\,, \qquad \forall x \in D\,, \tag*{$(1.4)_1$}$$

where the constant j_{n+m} depends only on the dimension of $\mathbb{R}^{n+m}$.

In Section 4 we prove (cf. Corollary 1) that if D is a convex subset of a real vector space, K a closed convex cone of a real topological vector space Y, $t \in (0,1)$ and $F : D \to C(Y)$ (cf. (2.2)) a set-valued function, then, under some assumption on Y (cf. Remark 1),

$$F \text{ is } K\text{-convex} \Leftrightarrow F \text{ is } K\text{-}t\text{-convex and } K\text{-quasiconvex.} \tag*{$(1.5)_1$}$$

This result contains the mentioned theorems proved in $[10]_2$, in [7] and in [2].

Finally, we want to observe that our Theorem 3 is a generalization to set-valued functions of a result of N. Kuhn [8] stating that t-convex (single-valued) functions are midconvex.

2. Let X be a real vector space and Y be a real topological vector space (satisfying the T_0 separation axiom). For $\alpha, \beta \in \mathbb{R}$ and $S, T \subset Y$ we put $\alpha S + \beta T = \{y \in Y : y = \alpha s + \beta t,\ s \in S,\ t \in T\}$. We define

$$n(Y) = \{S \subset Y : S \neq \emptyset\}, \tag{2.1}$$

$$C(Y) = \{S \subset Y : S \text{ compact, } S \neq \emptyset\}, \tag{2.2}$$

$$BC(Y) = \{S \subset Y : S \text{ bounded, convex, } S \neq \emptyset\}\,. \tag{2.3}$$

We assume that D is a nonempty convex subset of X and K is a convex cone in Y. For fixed $t \in (0,1)$, we say that a set-valued function $F : D \to n(Y)$ is *K-t-convex* if

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K$$

for all $x, y \in D$. If $t = 1/2$, F is called *K-midconvex*.

We say that F is *K-quasiconvex* if for every convex set $A \subset Y$ the lower inverse image of $A - K$, i.e. the set

$$F^{-}(A-K) = \{x \in D : F(x) \cap (A-K) \neq \emptyset\},$$

is convex (cf. $[10]_3$, (2.5)).

In the case that Y is a normed space, let B be the closed unit ball in Y and ε a nonnegative number. We say that $F : D \to n(Y)$ is *ε-K-convex* if

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K + \varepsilon B \tag{2.4}$$

for all $x, y \in D$ and $t \in [0,1]$. We recall that F is *K-convex* if it satisfies (2.4) with $\varepsilon = 0$. If F satisfies (2.4) with $\varepsilon = 0$ and $K = \{0\}$ it is said to be *convex*.

$F : D \to n(Y)$ is said to be *weakly K-upper bounded* on a set $A \subset D$ iff

(2.5) there exists a bounded set $B \subset Y$ such that $A \subset F^{-}(B-K)$.

Finally, we denote by

$$\operatorname{Gr} F = \{(x,y) \in X \times Y : x \in D,\ y \in F(x)\}$$

the *graph* of the set-valued function F.

3. In this section we present, for ε-K-convex set-valued functions, a theorem analogous to the stability theorem for functions proved by D. H. Hyers and S. M. Ulam in [6] and by P. W. Cholewa in [3].

Using a method similar to Cholewa's [3] we first prove

LEMMA 1. *Let X be a real vector space, Y a normed space, D a convex subset of X and K a convex cone in Y. If a set-valued function $F : D \to n(Y)$ is ε-K-convex, then for all $p \in \mathbb{N}$, $x_0, \ldots, x_p \in D$ and $t_0, \ldots, t_p \in [0,1]$ with $t_0 + \ldots + t_p = 1$, we have*

$$t_0F(x_0) + \ldots + t_pF(x_p) \subset F(t_0x_0 + \ldots + t_px_p) + K + j_p\varepsilon B \tag{3.1}$$

where $j_p = \min\{k_p, h_p\}$, $k_p = (p^2+3p)/(2p+2)$, and $h_p = m \in \mathbb{N}$ is such that $2^{m-1} \le p < 2^m$.

Proof. For $p = 1$ the inclusion (3.1) is clear because $j_1 = k_1 = h_1 = 1$. Now fix $p > 1$ and assume that (3.1) holds for all natural $n < p$. Take $x_0, \ldots, x_p \in D$ and $t_0, \ldots, t_p \in [0,1]$ with $t_0 + \ldots + t_p = 1$. Without loss of generality we may assume that $t_0 \ge 1/(p+1)$. Let $t = t_1 + \ldots + t_p$ and $t'_i = t_i/t$ for $i = 1, \ldots, p$; then $t \le p/(p+1)$. Thus

$$
\begin{aligned}
(3.2) \quad t_0F(x_0) + \ldots + t_pF(x_p) &= t_0F(x_0) + t[t'_1F(x_1) + \ldots + t'_pF(x_p)] \\
&\subset t_0F(x_0) + t[F(t'_1x_1 + \ldots + t'_px_p) + K + k_{p-1}\varepsilon B] \\
&\subset F(t_0x_0 + \ldots + t_px_p) + K + \varepsilon B + \frac{p}{p+1}k_{p-1}\varepsilon B \\
&= F(t_0x_0 + \ldots + t_px_p) + K + k_p\varepsilon B\,.
\end{aligned}
$$

Now, let $m \in \mathbb{N}$ be such that $2^{m-1} \le p < 2^m$. Put $r = [p/2]$; then $r < 2^{m-1}$ and $p - r - 1 < 2^{m-1}$. Setting $a = t_0 + \ldots + t_r$ and $b = t_{r+1} + \ldots + t_p$, we have

$$
\begin{aligned}
(3.3) \quad & t_0F(x_0) + \ldots + t_pF(x_p) \\
&= a\left[\frac{t_0}{a}F(x_0) + \ldots + \frac{t_r}{a}F(x_r)\right] + b\left[\frac{t_{r+1}}{b}F(x_{r+1}) + \ldots + \frac{t_p}{b}F(x_p)\right] \\
&\subset aF\left(\frac{t_0}{a}x_0 + \ldots + \frac{t_r}{a}x_r\right) + bF\left(\frac{t_{r+1}}{b}x_{r+1} + \ldots + \frac{t_p}{b}x_p\right) \\
&\quad + K + ah_r\varepsilon B + bh_{p-r-1}\varepsilon B \\
&\subset F(t_0x_0 + \ldots + t_px_p) + K + (1 + ah_r + bh_{p-r-1})\varepsilon B \\
&\subset F(t_0x_0 + \ldots + t_px_p) + K + [1 + a(m-1) + b(m-1)]\varepsilon B \\
&= F(t_0x_0 + \ldots + t_px_p) + K + h_p\varepsilon B\,.
\end{aligned}
$$

From (3.2) and (3.3) we obtain the assertion.

THEOREM 1. *Let D be a convex subset of $\mathbb{R}^n$ and K be a convex cone in $\mathbb{R}^m$. If a set-valued function $F : D \to n(\mathbb{R}^m)$ is ε-K-convex, then there exists a convex set-valued function $G : D \to n(\mathbb{R}^m)$ such that*

$$F(x) \subset G(x) \subset F(x) + K + j_{n+m}\varepsilon B$$

for all $x \in D$.

Proof. Let W be the convex hull of the graph of F. We define $G : D \to n(\mathbb{R}^m)$ by

$$G(x) = \{y \in \mathbb{R}^m : (x, y) \in W\}\,, \qquad x \in D\,.$$

Then G is convex because $\mathrm{Gr}\, G = W$ is convex. Moreover, $F(x) \subset G(x)$ for all $x \in D$. To prove the second inclusion fix an $x \in D$ and take an arbitrary $y \in G(x)$. Then $(x, y) \in W$. By the Carathéodory Theorem (cf. [12], Theorem 17.1) we have

$$(x, y) = \sum_{i=0}^{n+m} t_i(x_i, y_i)\,,$$

with some $(x_i, y_i) \in \operatorname{Gr} F$ and $t_0, \ldots, t_{n+m} \in [0,1]$, $t_0 + \ldots + t_{n+m} = 1$. Hence, using Lemma 1, we get

$$y = \sum_{i=0}^{n+m} t_i y_i \in \sum_{i=0}^{n+m} t_i F(x_i) \subset F(x) + K + j_{n+m}\varepsilon B\,.$$

Since this holds for all $y \in G(x)$, the proof is complete.

4. In this section we give two necessary and sufficient conditions for a set-valued function to be K-convex. We first need the following lemma which is an analogue of a result obtained for functions by C. T. Ng and K. Nikodem (cf. [9], Lemma 6).

LEMMA 2. *Let K be a closed convex cone in a real topological vector space Y. If $F : [0,1] \to C(Y)$ is K-midconvex on $[0,1]$ and K-convex on $(0,1)$, then it is K-convex on $[0,1]$.*

Proof. Fix $x, y \in [0,1]$ and $t \in (0,1)$, and put $z = tx + (1-t)y$. Let $u = (x+z)/2$ and $v = (y+z)/2$. Then $u, v \in (0,1)$ and $z = tu + (1-t)v$. Since F is K-convex on $(0,1)$ we get

$$tF(u) + (1-t)F(v) \subset F(z) + K\,. \tag{4.1}$$

On the other hand, by the K-midconvexity of F on $[0,1]$,

$$\frac{F(x)+F(z)}{2} \subset F(u) + K \quad \text{and} \quad \frac{F(y)+F(z)}{2} \subset F(v) + K\,. \tag{4.2}$$

Therefore, by (4.2) and (4.1),

$$\begin{aligned} tF(x) + (1-t)F(y) + F(z) &\subset t(F(x)+F(z)) + (1-t)(F(y)+F(z)) \\ &\subset 2tF(u) + 2(1-t)F(v) + K \\ &\subset 2F(z) + K \subset F(z) + F(z) + K\,. \end{aligned}$$

The set $F(z) + K$ is convex and closed, and $F(z)$ is bounded; so the law of cancellation (cf. [11]) yields the assertion.

THEOREM 2. *Let X be a real vector space, Y a real topological vector space, D a convex subset of X and K a closed convex cone in Y. Moreover, assume that there exists a family $(B_n)_n$, $B_n \in BC(Y)$* (cf. (2.3)), *such that*

$$Y = \bigcup_{n\in\mathbb{N}} (B_n - K)\,. \tag{4.3}$$

Then a set-valued function $F : D \to C(Y)$ is K-convex if and only if it is K-midconvex and K-quasiconvex.

P r o o f. The necessity is trivial (cf. $[10]_3$, Theorem 2.9). Now suppose F is K-midconvex and K-quasiconvex. Fix $x, y \in D$, and define $H : [0,1] \to C(Y)$ by

$$H(t) = F(tx + (1-t)y)\,, \qquad \forall t \in [0,1]\,. \tag{4.4}$$

Clearly H is K-quasiconvex; therefore for all $n \in \mathbb{N}$, the set

$$H^-(B_n - K) = \{t \in [0,1] : H(t) \cap (B_n - K) \neq \emptyset\} \tag{4.5}$$

is an interval in $\mathbb{R}$. In view of (4.3) we have

$$\bigcup_{n\in\mathbb{N}} H^-(B_n - K) = [0,1]\,,$$

and so we can find a natural number p such that

$$\operatorname{int} H^-(B_p - K) \neq \emptyset\,. \tag{4.6}$$

By the K-midconvexity of F it follows that H is K-midconvex on $[0,1]$, and (cf. (4.5) and (4.6)) H is weakly K-upper bounded (cf. (2.5)) on $H^-(B_p–K)$, which has nonempty interior; then using Corollary 3.3 of $[10]_3$ we deduce that H is K-continuous on $(0,1)$. Consequently, H is K-convex on $(0,1)$ (cf. $[10]_3$, Theorem 3.1 or [1], Theorem 4.2) and so it follows by Lemma 2 that H is K-convex on $[0,1]$. Therefore, by (4.4),

$$\begin{aligned} tF(x) + (1-t)F(y) &= tH(1) + (1-t)H(0) \subset H(t) + K \\ &= F(tx + (1-t)y) + K\,, \end{aligned}$$

which proves the K-convexity of F.

R e m a r k 1. The assumption (4.3) is trivially satisfied if Y is a normed space. It is also fulfilled if there exists an order unit in Y, i.e. an element $e \in Y$ such that for every $y \in Y$ we can find an $n \in \mathbb{N}$ with $y \in ne - K$ (then we can assume $B_n = \{ne\}$). In particular, if $\operatorname{int} K \neq \emptyset$, then every element of $\operatorname{int} K$ is an order unit in Y.

Theorem 3. *Let X be a real vector space, Y be a real topological vector space, D a convex subset of X and K a closed convex cone in Y. Let t be a fixed number in $(0,1)$. If a set-valued function $F : D \to C(Y)$ is K-t-convex, then it is K-midconvex.*

P r o o f. Observe first that $F(x) + K$ is convex for all $x \in D$ because

$$tF(x) + (1-t)F(x) \subset F(x) + K$$

and $F(x) + K$ is closed.

Let $x, y \in D$; using the K-t-convexity of F we get

$$\begin{aligned}
&t(1-t)F(x) + t(1-t)F(y) + [1-2t(1-t)]F\left(\frac{x+y}{2}\right)\\
&\quad\subset t\left[(1-t)F(x) + tF\left(\frac{x+y}{2}\right)\right] + (1-t)\left[tF(y) + (1-t)F\left(\frac{x+y}{2}\right)\right]\\
&\quad\subset tF\left((1-t)x + t\frac{x+y}{2}\right) + (1-t)F\left(ty + (1-t)\frac{x+y}{2}\right) + K\\
&\quad\subset F\left(\frac{x+y}{2}\right) + K\\
&\quad\subset 2t(1-t)F\left(\frac{x+y}{2}\right) + [1-2t(1-t)]F\left(\frac{x+y}{2}\right) + K\,.
\end{aligned}$$

Since the set $2t(1-t)F\left(\frac{x+y}{2}\right) + K$ is convex and closed and the set $[1-2t(1-t)]F\left(\frac{x+y}{2}\right)$ is bounded, by the law of cancellation we obtain

$$t(1-t)F(x) + t(1-t)F(y) \subset 2t(1-t)F\left(\frac{x+y}{2}\right) + K\,.$$

Hence

$$\frac{1}{2}[F(x) + F(y)] \subset F\left(\frac{x+y}{2}\right) + K\,,$$

which was to be proved.

Remark 2. In the case of real (single-valued) functions the above result is a consequence of the theorem of N. Kuhn [8]. The idea of the presented proof is taken from Lemma 1 of [4].

As an immediate consequence of Theorems 2 and 3 we obtain the following

Corollary 1. *Let X be a real vector space, Y a real topological vector space, D a convex subset of X, K a closed convex cone in Y and t a fixed number in $(0,1)$. Moreover, assume that there exists a family $(B_n)_n$, $B_n \in BC(Y)$, such that*

$$Y = \bigcup_{n\in\mathbb{N}} (B_n - K)\,.$$

Then a set-valued function $F : D \to C(Y)$ is K-convex if and only if it is K-t-convex and K-quasiconvex.

Remark 3. Observe that, in the case where $K = \{0\}$, it is sufficient to require that the values of the set-valued function in Lemma 2, Theorem 2, Theorem 3 and Corollary 1 are closed and bounded (and not necessarily compact). The corresponding proofs are similar to those given above.

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Reçu par la Rédaction le 28.10.1992
Révisé le 18.1.1993