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Asymptotic properties of Markov operators defined by Volterra type integrals

by Karol Baron and Andrzej Lasota (Katowice)

Abstract. New sufficient conditions for asymptotic stability of Markov operators are given. These criteria are applied to a class of Volterra type integral operators with advanced argument.

Introduction. We shall study asymptotic properties of the iterates (P^n) of the operator

$$Pf(x) = \int_0^{\lambda(x)} K(x,y)\, f(y)\, dy \tag{0.1}$$

where

$$K(x,y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) \tag{0.2}$$

and $Q, \lambda, -H$ are given nonnegative and nondecreasing functions defined on the half line $\mathbb{R}_+ = [0,\infty)$. The precise assumptions concerning the kernel K will be formulated in Section 2.

Operators of the form (0.1), (0.2) appear in mathematical models of the cell cycle [5], [10], [11], [12] and in a model of the electrical activity of neurons [7].

In the special case when $H(x) = e^{-x}$, a sufficient condition for asymptotic stability of the sequence (P^n) was recently given in [2]. It has the form

$$\liminf_{x\to\infty} (Q(\lambda(x)) - Q(x)) > 1. \tag{0.3}$$

In the general situation, with arbitrary H, condition (0.3) was replaced in [7] by

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$$\liminf_{x\to\infty} \frac{Q(\lambda(x))}{Q(x)} > 1. \tag{0.4}$$

Since $Q(x)$ and $\lambda(x)$ converge to $+\infty$ as $x \to +\infty$, inequality (0.4) is much more restrictive than (0.3). In particular, (0.4) is not satisfied in some cases important for applications. The purpose of the present paper is to formulate a sufficient condition of the form (0.3) for asymptotic stability of (P^n) valid for a large class of functions.

The organization of the paper is as follows. Section 1 contains some auxiliary definitions and theorems from the theory of Markov operators. Our results in this area are based on special properties of integral and Harris operators [1]. In particular, our Theorem 1.2 extends a recent result of J. Malczak [8]. In Section 2 we discuss the asymptotic properties of the iterates of the operator P given by formulas (0.1), (0.2).

1. Markov operators. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Denote by $D = D(X, \mathcal{A}, \mu)$ the subset of $L^1 = L^1(X, \mathcal{A}, \mu)$ which contains all (normalized) densities, i.e.

$$D = \{f \in L^1 : f \geq 0, \|f\| = 1\}$$

where $\|\cdot\|$ stands for the norm in L^1. A linear mapping $P : L^1 \to L^1$ is called a *Markov operator* if $P(D) \subset D$.

Let a Markov operator P be given. A density f is called *stationary* (or *invariant*) if $Pf = f$. The operator P is called *asymptotically stable* if there is a density f_* such that

$$\lim_{n\to\infty} \|P^n f - f_*\| = 0 \quad \text{for } f \in D. \tag{1.1}$$

Of course, a density f_* satisfying condition (1.1) is necessarily stationary and unique.

In order to present a simple criterion for the existence of a stationary density we must recall the notion of Banach limits [4]. A *Banach limit* L is a linear functional defined on the space l^∞ of bounded sequences $(a_n) = (a_1, a_2, \ldots)$ of real numbers which satisfies the following conditions:

(i) $L(a_n) \geq 0$ if $a_i \geq 0$ $(i = 1, 2, \ldots)$,
(ii) $L(a_1, a_2, \ldots) = L(a_2, a_3, \ldots)$,
(iii) $L(1, 1, \ldots) = 1$.

If (a_n) is convergent then $L(a_n) = \lim_{n\to\infty} a_n$, and if $\limsup a_n \leq c$ then $L(a_n) \leq c$.

Theorem 1.1. *Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be a Markov operator and L a Banach limit. Assume that there exists a set $A \in \mathcal{A}$, $\mu(A) < \infty$, a* *number $\delta > 0$ and a density f such that*

$$L\Big(\int\limits_{(X\setminus A)\cup E} P^n f\,d\mu\Big) < 1 \quad \textit{for } E \subset A \textit{ with } \mu(E) < \delta\,. \tag{1.2}$$

Then P admits a stationary density.

The proof of this result was given by J. Socała [9]. It should be noted, however, that in Socała's statement a stronger form of condition (1.2) was used. Namely, the functional L was replaced by lim sup. The above formulation was proposed by T. Komorowski and J. Tyrcha [3].

Now consider an operator P of the form

$$Pf(x) = \int\limits_X k(x,y)\,f(y)\,d\mu(y) \tag{1.3}$$

where $k : X \times X \to \mathbb{R}$ is a *stochastic kernel*, i.e. k is jointly measurable on $X \times X$ and satisfies

$$\begin{aligned} &k(x,y) \ge 0 \quad \text{for } (x,y) \in X \times X\,,\\ &\int\limits_X k(x,y)\,d\mu(x) = 1 \quad \text{for } y \in X\,. \end{aligned} \tag{1.4}$$

From (1.4) it follows immediately that P is a Markov operator; it is called an *integral Markov operator*.

For integral Markov operators the existence of an invariant density and a simple transitivity condition imply asymptotic stability. To formulate this criterion precisely recall that in the theory of Markov operators the *support* of an $f \in L^1(X,\mathcal{A},\mu)$ is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x \in X : f(x) \ne 0\}\,.$$

We say that a Markov operator P *overlaps supports* if for every $f, g \in D$ there is a positive integer $n_0 = n_0(f,g)$ such that

$$\mu(\operatorname{supp} P^{n_0} f \cap \operatorname{supp} P^{n_0} g) > 0\,. \tag{1.5}$$

Observe that condition (1.5) implies that

$$\mu(\operatorname{supp} P^n f \cap \operatorname{supp} P^n g) > 0 \quad \text{for } n \ge n_0(f,g)\,.$$

In fact,

$$\operatorname{supp} P^n f \cap \operatorname{supp} P^n g \supset \operatorname{supp} P^{n-n_0}(\min\{P^{n_0} f, P^{n_0} g\})\,.$$

THEOREM 1.2. *An integral Markov operator which overlaps supports and has a stationary density $f_* > 0$ a.e. is asymptotically stable.*

Proof. Define a new measure space $(X,\mathcal{A},\overline{\mu})$ with $d\overline{\mu} = f_* d\mu$ and consider the operator

$$\overline{P} f = (1/f_*)\,P(f\cdot f_*)\,. \tag{1.6}$$

Observe that for every $f \in L^1(\overline{\mu})$ the product $f \cdot f_*$ belongs to $L^1(\mu)$. It is evident that $\overline{P}$ is an integral operator on $L^1(\overline{\mu})$ and that

$$\overline{P}1_X = 1_X\,. \tag{1.7}$$

(Here and in the sequel 1_E denotes the characteristic function of the subset E of X.) Now we are going to use a well known decomposition property of integral Markov operators satisfying $\overline{P}1_X \le 1_X$ (see [1], Ch. VIII). The space X may be written in the form

$$X = X_1 \cup X_2, \qquad X_1 = \bigcup_i W_i \tag{1.8}$$

where the family $\{W_i\}$ is at most countable. The sets X_1, X_2 and W_i are measurable, disjoint ($X_1 \cap X_2 = \emptyset$, $W_i \cap W_j = \emptyset$ for $i \ne j$) and have the following properties:

(i) For every $f \in L^1(\overline{\mu})$ with $\operatorname{supp} f \subset X_2$ and for every $g \in L^\infty(\overline{\mu})$,

$$\lim_{n\to\infty} \int\limits_X g \cdot \overline{P}^n f \, d\overline{\mu} = 0\,. \tag{1.9}$$

(ii) For every i there is a j such that $\overline{P}1_{W_i} = 1_{W_j}$.

(iii) Every set W_i is either cyclic or wandering. In the first case $\overline{P}^k 1_{W_i} = 1_{W_i}$ for a positive integer k; in the second, all sets W_{in} ($n = 0, 1, \ldots$) defined by $1_{W_{in}} = \overline{P}^n 1_{W_i}$ are distinct and hence disjoint.

(iv) For every cyclic W_i with period k and for every $f \in L^1(\overline{\mu})$ vanishing outside W_i,

$$\lim_{n\to\infty} \Big\| \overline{P}^{nk} f - \Big(\int\limits_{W_i} f \, d\overline{\mu} / \overline{\mu}(W_i) \Big) 1_{W_{in}} \Big\|_{L^1(\overline{\mu})} = 0\,. \tag{1.10}$$

We shall show that in our case the decomposition formula (1.8) reduces to $X = W_1$. In fact, $\overline{\mu}(X_2) \le \overline{\mu}(X) = 1$ and we may take $f = 1_{X_2}, g = 1_X$ in (1.9). Since $\overline{P}$ preserves the integral with respect to $\overline{\mu}$ this gives $\overline{\mu}(X_2) = 0$. Assume that W_i is wandering. Then

$$\begin{aligned} \operatorname{supp} P^n(f_* \cdot 1_{W_i}) \cap \operatorname{supp} P^n(f_* \cdot 1_{W_{i1}}) &= \operatorname{supp} \overline{P}^n 1_{W_i} \cap \operatorname{supp} \overline{P}^n 1_{W_{i1}} \\ &= W_{in} \cap W_{i,n+1} = \emptyset \end{aligned}$$

for every n, which contradicts (1.5) and shows that there are no wandering sets. Assume now that W_i is cyclic with period $k \ge 2$. Then, as previously,

$$\begin{aligned} \operatorname{supp} P^{kn}(f_* \cdot 1_{W_i}) \cap \operatorname{supp} P^{kn}(f_* \cdot 1_{W_{i1}}) &= W_{i,kn} \cap W_{i,kn+1} \\ &= W_i \cap W_{i1} = \emptyset \end{aligned}$$

for every n. Consequently, each W_i is cyclic with period $k=1$. Assume that there are two such sets, say W_1 and W_2. Then

$$\operatorname{supp} P^n(f_* \cdot 1_{W_1}) \cap \operatorname{supp} P^n(f_* \cdot 1_{W_2}) = W_1 \cap W_2 = \emptyset$$

for every n, which again contradicts (1.5). Thus there is exactly one cyclic set with cycle length $k = 1$. We denote this set by W_1. According to (1.10) with $\overline{\mu}(W_1) = \overline{\mu}(X) = 1$, $k = 1$, we obtain

$$\lim_{n\to\infty} \Big\| \overline{P}^n f - \Big(\int_X f \, d\overline{\mu} \Big) 1_X \Big\|_{L^1(\overline{\mu})} = 0 \tag{1.11}$$

for every $f \in L^1(\overline{\mu})$. Evidently, for every $f \in D(\mu)$ we have $f/f_* \in L^1(\overline{\mu})$ and

$$\begin{aligned} \|P^n f - f_*\|_{L^1(\mu)} &= \Big\| f_* \overline{P}^n (f/f_*) - f_* \int_X (f/f_*) \, d\overline{\mu} \Big\|_{L^1(\mu)} \\ &= \Big\| \overline{P}^n (f/f_*) - \Big(\int_X (f/f_*) \, d\overline{\mu} \Big) 1_X \Big\|_{L^1(\overline{\mu})} . \end{aligned}$$

From this and (1.11) we get (1.1). ■

COROLLARY 1.1. *Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be an integral Markov operator which has a positive stationary density f_* ($f_* > 0$ a.e.). Assume*, *moreover*, *that there exists a set $A \in \mathcal{A}$, $\mu(A) > 0$, with the following property. For every $f \in D$ there is a positive integer $n_0 = n_0(f)$ such that*

$$P^{n_0} f(x) > 0 \quad \text{for a.e. } x \in A .$$

Then P is asymptotically stable.

Theorems 1.1 and 1.2 do not match well. In fact, the invariant density existing by Theorem 1.1 need not be positive on the whole space X, which is an important assumption in Theorem 1.2. This situation may be improved by studying P restricted to the support of the invariant density.

Let a Markov operator $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be given. It is well known that for all nonnegative $f, f_* \in L^1(X)$ the inclusion $\operatorname{supp} f \subset \operatorname{supp} f_*$ implies $\operatorname{supp} Pf \subset \operatorname{supp} Pf_*$. In particular, if $f_* = Pf_*$ and $\operatorname{supp} f_* = C$ then

$$\operatorname{supp} f \subset C \quad \text{implies} \quad \operatorname{supp} Pf \subset C .$$

This property allows us to consider P on the space $L^1(C)$ of all functions from $L^1(X)$ with supports contained in C. We will denote P restricted to $L^1(C)$ by P_C.

THEOREM 1.3. *Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be a Markov operator having an invariant density f_*. Assume that the operator P_C with $C = \operatorname{supp} f_*$ is asymptotically stable. Assume*, *moreover*, *that there is a $\delta > 0$ such that*

$$\sup_n \int_C P^n f \, d\mu \ge \delta \quad \text{for } f \in D(X) . \tag{1.12}$$

Then $P : L^1(X) \to L^1(X)$ is also asymptotically stable.

P r o o f. According to the lower bound function theorem (see [6], Ch. 5) in order to prove (1.1) it is sufficient to find a nonnegative $h \in L^1(X)$, $\| h \| > 0$, such that

$$\lim_{n\to\infty} \| (P^n f - h)^- \| = 0 \quad \text{for } f \in D(X) \tag{1.13}$$

where $\| \cdot \|$ stands for the norm in $L^1(X)$. Define $h = \frac{1}{2}\delta f_*$ and fix an $f \in D(X)$. According to (1.12) there is an integer m such that

$$\eta := \int_C P^m f \, d\mu \ge \tfrac{1}{2}\,\delta .$$

For $n \ge m$ we have

$$P^n f = P^{n-m}(1_{X\setminus C} P^m f) + P_C^{n-m}(1_C P^m f) . \tag{1.14}$$

Since P_C is asymptotically stable with invariant density f_* we also have

$$\lim_{n\to\infty} \| P_C^{n-m}(1_C P^m f) - \eta f_* \| = 0 . \tag{1.15}$$

From (1.14) and the inequality $h \le \eta f_*$ it follows that

$$\| (P^n f - h)^- \| \le \| P_C^{n-m}(1_C P^m f) - \eta f_* \|$$

for $n \ge m$. This and (1.15) imply (1.13). ∎

Using Theorems 1.2 and 1.3 it is easy to derive the following

COROLLARY 1.2. *Let* $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ *be an integral Markov operator which overlaps supports and has an invariant density* f_*. *Set* $C = \operatorname{supp} f_*$. *If there is a* $\delta > 0$ *such that* (1.12) *is satisfied*, *then* P *is asymptotically stable.*

P r o o f. According to Theorem 1.3 it is enough to prove that the operator P_C is asymptotically stable. Evidently,

$$P_C f(x) = \int_C k(x,y)\, f(y)\, d\mu(y)$$

for every $f \in L^1(C)$ and

$$\begin{aligned} 0 &= \int_C f_*(y)\, d\mu(y) - \int_C P_C f_*(x)\, d\mu(x) \\ &= \int_C \Big(1 - \int_C k(x,y)\, d\mu(x)\Big) f_*(y)\, d\mu(y) , \end{aligned}$$

whence

$$\int_C k(x,y)\, d\mu(x) = 1 \quad \text{for a.e. } y \in C .$$

This shows that P_C is an integral Markov operator. Thus we can apply Theorem 1.2 to P_C and its asymptotical stability follows. ∎

2. Volterra operators. In this section we shall consider the integral operator P defined by (0.1) and (0.2) under the following general assumptions (K1) and (K2):

(K1) $H : [0,\infty) \to [0,\infty)$ is nonincreasing, absolutely continuous and

$$H(0) = 1, \qquad \lim_{x\to\infty} H(x) = 0\,.$$

(K2) $Q : [0,\infty) \to [0,\infty)$ and $\lambda : [0,\infty) \to [0,\infty)$ are nondecreasing, absolutely continuous and

$$Q(0) = \lambda(0) = 0, \qquad \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty\,.$$

The above conditions (K1) and (K2) are assumed in the whole of this section and will not be repeated in the statements of the theorems. Moreover, all measure-theoretic notions refer to the standard Lebesgue measure m on $[0,\infty)$.

We start with the following lemma from [7].

LEMMA 2.1. *If* $W : [0,\infty) \to [0,\infty)$ *is measurable and* $f \in D$, *then*

$$\int_0^\infty W(Q(\lambda(x)))\,Pf(x)\,dx = \int_0^\infty f(y)\,dy \int_0^\infty W(x+Q(y))\,h(x)\,dx \tag{2.1}$$

where

$$h(x) = -H'(x)\,.$$

Using Theorem 1.1 we prove the following theorem concerning the existence of a stationary density for P.

THEOREM 2.1. *If there exists an* $\alpha \in (0,1]$ *such that*

$$\int_0^\infty x^\alpha h(x)\,dx < \liminf_{x\to\infty}((Q(\lambda(x)))^\alpha - Q(x)^\alpha)\,, \tag{2.2}$$

then the operator P *given by formulas* (0.1), (0.2) *has a stationary density.*

Proof. Evidently, P is an integral Markov operator defined on $L^1([0,\infty))$. Define

$$\sigma = \int_0^\infty x^\alpha h(x)\,dx\,.$$

Using (2.2) we can find positive numbers ε, ϱ and x_0 such that

$$\sigma + \varepsilon < \varrho < (Q(\lambda(x)))^\alpha - Q(x)^\alpha \quad \text{for } x \ge x_0\,. \tag{2.3}$$

We are going to show that for every $f \in D$ there exists an integer $n_0(f)$ such that

$$\int_0^{x_0} \frac{1}{n}\sum_{k=1}^n P^k f(x)\,dx \ge \frac{\varepsilon}{2M} \quad \text{for } n \ge n_0(f) \tag{2.4}$$

where

$$M := \sup_{[0,x_0]} |\,(Q(\lambda(x)))^\alpha - Q(x)^\alpha - \varrho\,|\ . \tag{2.5}$$

Using (2.1) with $W(x) = x^\alpha$ and $f \in D$ we have

$$\begin{aligned}\int_0^\infty (Q(\lambda(x)))^\alpha Pf(x)\,dx &= \int_0^\infty f(y)\,dy \int_0^\infty (x+Q(y))^\alpha h(x)\,dx \\ &\le \int_0^\infty f(y)\,dy \int_0^\infty (x^\alpha + Q(y)^\alpha) h(x)\,dx \\ &= \sigma + \int_0^\infty f(y)\,Q(y)^\alpha\,dy\,.\end{aligned} \tag{2.6}$$

Fix $f \in D$ such that

$$\int_0^\infty Q(x)^\alpha f(x)\,dx < \infty \tag{2.7}$$

and define

$$f_n = \frac{1}{n}\sum_{k=1}^n P^k f \qquad \text{for } n = 1,2,\ldots \tag{2.8}$$

From (2.3), (2.6) and (2.7) it follows that

$$\int_0^\infty (Q(\lambda(x)))^\alpha Pf_n(x)\,dx \le \sigma + \int_0^\infty Q(x)^\alpha f_n(x)\,dx$$

and that the integral on the right hand side is finite for every n. Hence

$$\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x)\,dx \le \sigma + \frac{1}{n}\int_0^\infty (Q(\lambda(x)))^\alpha Pf(x)\,dx\,.$$

Since $\sigma < \varrho - \varepsilon$, there exists a positive integer $n_0(f)$ such that

$$\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x)\,dx \le \varrho - \varepsilon \qquad \text{for } n \ge n_0(f)\,.$$

On the other hand, taking into account (2.3) we have

$$\begin{aligned}&\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x)\,dx \\ &\qquad \ge \int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x)\,dx + \varrho\int_{x_0}^\infty f_n(x)\,dx\,.\end{aligned}$$

Consequently,

$$\int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x)\,dx \le \varrho - \varepsilon - \varrho \int_{x_0}^{\infty} f_n(x)\,dx$$
$$= \varrho \int_0^{x_0} f_n(x)\,dx - \varepsilon$$

for $n \ge n_0(f)$, which together with (2.5) gives

$$-M \int_0^{x_0} f_n(x)\,dx \le \int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha - \varrho) f_n(x)\,dx \le -\varepsilon$$

for $n \ge n_0(f)$. This implies (2.4) and even a stronger inequality with the right hand side ε/M. The above argument was valid for f satisfying (2.7). To get (2.4) for every $f \in D$ it is enough to observe that the set of all $f \in D$ such that (2.7) holds is dense in D.

Now we are going to show that there exists a $\delta > 0$ such that

$$\int_E Pf(x)\,dx \le \frac{\varepsilon}{4M} \quad \text{for } f \in D \text{ and } E \subset [0, x_0], m(E) \le \delta\,.$$

In fact, since h is integrable we can find a $\gamma > 0$ such that

$$\int_F h(x)\,dx \le \frac{\varepsilon}{4M} \quad \text{for } m(F) \le \gamma\,.$$

Further, since $Q \circ \lambda$ is absolutely continuous, there exists a $\delta > 0$ such that

$$m(Q(\lambda(E))) \le \gamma \quad \text{for } E \subset [0, x_0],\ m(E) \le \delta\,.$$

Now let $E \subset [0, x_0]$ be measurable and $m(E) \le \delta$. Setting $W = 1_{Q(\lambda(E))}$ we have $1_E \le W \circ Q \circ \lambda$, which according to (2.1) gives

$$\int_E Pf(x)\,dx = \int_0^\infty 1_E(x) Pf(x)\,dx \le \int_0^\infty W(Q(\lambda(x)))Pf(x)\,dx$$
$$= \int_0^\infty f(y)\,dy \int_0^\infty W(x + Q(y))\,h(x)\,dx$$
$$= \int_0^\infty f(y)\,dy \int_{Q(\lambda(E))-Q(y)} h(x)\,dx \le \frac{\varepsilon}{4M}$$

for every $f \in D$.

In order to verify condition (1.2) fix an $f \in D$ and a positive integer $n \ge n_0(f)$ such that (2.4) is satisfied. Fix a measurable set $E \subset [0, x_0]$ with

$m(E) \le \delta$. Then

$$\int_E f_n(x)\,dx = \int_E P\Big(\frac{1}{n}\sum_{k=0}^{n-1} P^k f\Big)\,dx \le \frac{\varepsilon}{4M}$$

and using (2.4) we obtain

$$\int_{(x_0,\infty)\cup E} f_n(x)\,dx = 1 - \int_0^{x_0} f_n(x)\,dx + \int_E f_n(x)\,dx$$
$$\le 1 - \frac{\varepsilon}{2M} + \frac{\varepsilon}{4M} = 1 - \frac{\varepsilon}{4M}$$

for $n \ge n_0(f)$. Now let L_0 be a Banach limit and let

$$L(a_k) = L_0\Big(\frac{1}{k}\sum_{i=1}^{k} a_i\Big)$$

for every bounded sequence (a_k) of real numbers. Evidently, L is also a Banach limit and

$$L\Big(\int_{(x_0,\infty)\cup E} P^n f(x)\,dx\Big) = L_0\Big(\int_{(x_0,\infty)\cup E} f_n(x)\,dx\Big) \le 1 - \frac{\varepsilon}{4M},$$

which shows that (1.2) with $A = [0, x_0]$ is satisfied. ■

Now we use Corollary 1.2 to find a sufficient condition for the asymptotic stability of the operator P given by (0.1), (0.2).

THEOREM 2.2. *If there exists a positive number $\alpha \le 1$ such that* (2.2) *holds, and a nonnegative number c such that*

$$h(x) > 0 \quad \text{for a.e. } x \ge c, \tag{2.9}$$

then the operator P given by (0.1), (0.2) *is asymptotically stable.*

Proof. According to Theorem 2.1 the operator P has a stationary density f_*. Define $C = \operatorname{supp} f_*$ and fix positive numbers $\varepsilon, \varrho, x_0$ such that (2.3) holds. Further, choose a positive number a such that

$$\lambda(a) \ge x_0, \qquad Q(\lambda(a)) \ge c + Q(x_0),$$

and define

$$A = \{x \ge a : (Q \circ \lambda)'(x) > 0\}.$$

Since $Q \circ \lambda$ is absolutely continuous and $\lim_{x\to\infty} Q(\lambda(x)) = \infty$, the set A is unbounded ($\operatorname{ess\,sup} A = \infty$). Finally, define the number M by (2.5).

If $x \in A$, then

$$f_*(x) = Pf_*(x) = (Q\circ\lambda)'(x) \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y)) f_*(y)\,dy$$
$$\geq (Q\circ\lambda)'(x) \int_0^{x_0} h(Q(\lambda(x)) - Q(y)) f_*(y)\,dy$$

and

$$(Q\circ\lambda)'(x) > 0, \quad h(Q(\lambda(x)) - Q(y)) > 0 \quad \text{for } y \in [0, x_0].$$

From (2.4) with $f = f_*$ it follows that

$$\int_0^{x_0} f_*(y)\,dy > 0.$$

This shows that $f_*(x) > 0$ for $x \in A$ and that $A \subset C$. Using (2.4) it is also easy to show that

$$\sup_n \int_C P^n f(x)\,dx \geq \frac{\varepsilon}{2M} \int_{Q(\lambda(a))}^{\infty} h(u)\,du \quad \text{for } f \in D. \tag{2.10}$$

In fact, according to (2.4) for every density f there is a positive integer k such that

$$\int_0^{x_0} P^k f(x)\,dx \geq \frac{\varepsilon}{2M}$$

and consequently,

$$\begin{aligned}
\int_C P^{k+1} f(x)\,dx &\geq \int_A P^{k+1} f(x)\,dx \\
&= \int_A (Q\circ\lambda)'(x)\,dx \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y)) P^k f(y)\,dy \\
&\geq \int_A (Q\circ\lambda)'(x)\,dx \int_0^{x_0} h(Q(\lambda(x)) - Q(y)) P^k f(y)\,dy \\
&= \int_0^{x_0} P^k f(y)\,dy \int_a^{\infty} (Q\circ\lambda)'(x) h(Q(\lambda(x)) - Q(y))\,dx \\
&\geq \int_0^{x_0} P^k f(y)\,dy \int_{Q(\lambda(a))}^{\infty} h(u)\,du \geq \frac{\varepsilon}{2M} \int_{Q(\lambda(a))}^{\infty} h(u)\,du.
\end{aligned}$$

Finally, observe that for every density f there exists a positive number

$b = b(f)$ such that

$$Pf(x) > 0 \quad \text{for } x \in [b, \infty) \cap A\,.$$

To show this choose $b_0 > 0$ such that $\int_0^{b_0} f(y)\,dy > 0$, and $b > 0$ such that $\lambda(b) \ge b_0$, $Q(\lambda(b)) \ge c + Q(b_0)$. For $x \in [b, \infty) \cap A$ we then have

$$Pf(x) \ge (Q \circ \lambda)'(x) \int_0^{b_0} h(Q(\lambda(x)) - Q(y)) f(y)\,dy > 0\,.$$

Setting $d = d(f, g) = \max\{b(f), b(g)\}$ we obtain

$$m(\operatorname{supp} Pf \cap \operatorname{supp} Pg) \ge m([d, \infty) \cap A) > 0 \quad \text{for } f, g \in D\,.$$

Thus all the requirements of Corollary 1.2 are satisfied and the proof is complete. ∎

The following example shows that assumption (2.9) in the statement of Theorem 2.2 is essential.

EXAMPLE 2.1. Let $h : [0, \infty) \to [0, \infty)$ be an integrable function such that

$$\int_0^\infty h(x)\,dx = 1 \quad \text{and} \quad h(x) = 0 \quad \text{for } x \ge \sqrt{c} - c$$

where $c \in (0, 1)$ is a constant. Consider the operator $P : L^1 \to L^1$ given by the formula

$$Pf(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} \displaystyle\int_0^{\sqrt{x}} h(\sqrt{x} - y) f(y)\,dy & \text{for } x \in (0, 1)\,, \\ 2 \displaystyle\int_0^{2x-1} h(2x - y - 1) f(y)\,dy & \text{for } x \ge 1\,. \end{cases} \tag{2.11}$$

In this case $Q(x) = x$,

$$\lambda(x) = \begin{cases} \sqrt{x} & \text{for } x \in [0, 1]\,, \\ 2x - 1 & \text{for } x > 1\,, \end{cases} \qquad H(x) = 1 - \int_0^x h(t)\,dt\,,$$

and evidently the assumptions (K1) and (K2) are satisfied. Moreover, for every $\alpha \in (0, 1]$,

$$\int_0^\infty x^\alpha h(x)\,dx < 1 < \infty = \lim_{x \to \infty} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha)\,.$$

According to Theorem 2.1 the operator P has a stationary density. Using (2.11) it is easy to verify the following property of P. If $\operatorname{supp} f \subset [1, \infty)$ then $\operatorname{supp} Pf \subset [1, \infty)$ and if $\operatorname{supp} f \subset (0, c)$ then $\operatorname{supp} Pf \subset (0, c)$. Since $c < 1$, condition (1.1) cannot be satisfied with an f_* independent on f. Thus P is not asymptotically stable.

In the previous results concerning the operator (0.1), (0.2) an important role was played by condition (2.2). Thus a natural question arises: What could we say about the behaviour of $(P^n f)$ when (2.2) is not satisfied? A partial answer to this question may be given by showing that if an opposite condition to (2.2) is satisfied then the operator P is sweeping [2].

We say that a Markov operator $P : L^1([0,\infty)) \to L^1([0,\infty))$ is *sweeping* if

$$\lim_{n\to\infty} \int_0^r P^n f(x)\,dx = 0 \quad \text{for every } f \in D \text{ and } r \geq 0\,. \tag{2.12}$$

THEOREM 2.3. *Assume that*

$$\sup_{x\geq x_0} ((Q(\lambda(x)))^\beta - Q(x)^\beta) < \int_0^\infty x^\beta h(x)\,dx < \infty \tag{2.13}$$

for an $x_0 \geq 0$ and $\beta \geq 1$ and that

$$\int_{Q(\lambda(x_0))}^\infty h(x)\,dx > 0\,.$$

Then the operator P given by (0.1), (0.2) *is sweeping.*

Proof. Define

$$z_0 = (Q(\lambda(x_0)))^\beta, \qquad w(z) = \begin{cases} e^{-\varepsilon z_0} & \text{for } z \in [0, z_0]\,, \\ e^{-\varepsilon z} & \text{for } z > z_0\,, \end{cases}$$

and

$$V(x) = w((Q(\lambda(x)))^\beta)$$

where $\varepsilon > 0$ will be chosen later. We shall show that there exists a nonnegative constant $\gamma < 1$ such that

$$\int_0^\infty V(x)Pf(x)\,dx \leq \gamma \int_0^\infty V(x)f(x)\,dx \quad \text{for } f \in D\,. \tag{2.14}$$

Since $V(x)$ admits a positive minimum on every compact set this inequality implies (2.12) (see also [2]).

According to (2.13) there exists a number ϱ such that

$$\sup_{x\geq x_0} ((Q(\lambda(x)))^\beta - Q(x)^\beta) < \varrho < \int_0^\infty x^\beta h(x)\,dx\,.$$

Define

$$I(y) = \int_0^\infty \frac{w((x+Q(y))^\beta)}{V(y)} h(x)\,dx \quad \text{for } y \geq 0\,.$$

If $y \le x_0$, then $V(y) = w(z_0)$ and

$$I(y) \le \int_0^\infty \frac{w(x^\beta)}{V(y)} h(x)\,dx = \int_0^{Q(\lambda(x_0))} \frac{w(x^\beta)}{w(z_0)} h(x)\,dx + \int_{Q(\lambda(x_0))}^\infty \frac{w(x^\beta)}{w(z_0)} h(x)\,dx$$

$$= \int_0^{Q(\lambda(x_0))} h(x)\,dx + \int_{Q(\lambda(x_0))}^\infty h(x) e^{-\varepsilon(x^\beta - z_0)}\,dx$$

$$= 1 - \int_{Q(\lambda(x_0))}^\infty h(x)(1 - e^{-\varepsilon(x^\beta - z_0)})\,dx =: \gamma_1(\varepsilon) < 1\,.$$

If $y > x_0$, then $(Q(\lambda(y)))^\beta - Q(y)^\beta < \varrho$ and, since $w(z) \le e^{-\varepsilon z}$ for $z \ge 0$,

$$\frac{w((x + Q(y))^\beta)}{V(y)} \le \frac{e^{-\varepsilon(x+Q(y))^\beta}}{e^{-\varepsilon(Q(\lambda(y)))^\beta}} \le \frac{e^{-\varepsilon(x+Q(y))^\beta}}{e^{-\varepsilon(\varrho + Q(y)^\beta)}} \le e^{-\varepsilon(x^\beta - \varrho)};$$

consequently,

$$I(y) \le \int_0^\infty h(x) e^{-\varepsilon(x^\beta - \varrho)}\,dx =: \gamma_2(\varepsilon)\,.$$

From Lemma 2.1 it follows that

$$\int_0^\infty V(x) Pf(x)\,dx = \int_0^\infty f(y)\,dy \int_0^\infty w((x + Q(y))^\beta) h(x)\,dx$$

$$= \int_0^\infty f(y) V(y) I(y)\,dy$$

$$\le \gamma_1(\varepsilon) \int_0^{x_0} V(y) f(y)\,dy + \gamma_2(\varepsilon) \int_{x_0}^\infty V(y) f(y)\,dy$$

for every density f. Since $\gamma_1(\varepsilon) < 1$ for every $\varepsilon > 0$, in order to show (2.14) with a constant $\gamma < 1$ it is enough to prove that there exists and $\varepsilon > 0$ such that $\gamma_2(\varepsilon) < 1$. But the function γ_2 is differentiable on $[0, \infty)$ and

$$\gamma_2'(\varepsilon) = -\int_0^\infty h(x)(x^\beta - \varrho) e^{-\varepsilon(x^\beta - \varrho)}\,dx\,,$$

whence

$$\gamma_2'(0) = \varrho - \int_0^\infty x^\beta h(x)\,dx < 0\,.$$

Consequently, for sufficiently small $\varepsilon > 0$ we have $\gamma_2(\varepsilon) < \gamma_2(0) = 1$, which completes the proof. ■

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
40-007 KATOWICE, POLAND

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