Asymptotic properties of Markov operators
defined by Volterra type integrals

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Abstract. New sufficient conditions for asymptotic stability of Markov operators are given. These criteria are applied to a class of Volterra type integral operators with advanced argument.

Introduction. We shall study asymptotic properties of the iterates \((P^n)\) of the operator

\[
Pf(x) = \int_0^{\lambda(x)} K(x,y) f(y) \, dy
\]

where

\[
K(x,y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y))
\]

and \(Q, \lambda, -H\) are given nonnegative and nondecreasing functions defined on the half line \(\mathbb{R}_+ = [0, \infty)\). The precise assumptions concerning the kernel \(K\) will be formulated in Section 2.

Operators of the form (0.1), (0.2) appear in mathematical models of the cell cycle [5], [10], [11], [12] and in a model of the electrical activity of neurons [7].

In the special case when \(H(x) = e^{-x}\), a sufficient condition for asymptotic stability of the sequence \((P^n)\) was recently given in [2]. It has the form

\[
\liminf_{x \to \infty} (Q(\lambda(x)) - Q(x)) > 1.
\]

In the general situation, with arbitrary \(H\), condition (0.3) was replaced in [7] by

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Since $Q(x)$ and $\lambda(x)$ converge to $+\infty$ as $x \to +\infty$, inequality (0.4) is much more restrictive than (0.3). In particular, (0.4) is not satisfied in some cases important for applications. The purpose of the present paper is to formulate a sufficient condition of the form (0.3) for asymptotic stability of $(P^n)$ valid for a large class of functions.

The organization of the paper is as follows. Section 1 contains some auxiliary definitions and theorems from the theory of Markov operators. Our results in this area are based on special properties of integral and Harris operators [1]. In particular, our Theorem 1.2 extends a recent result of J. Malczak [8]. In Section 2 we discuss the asymptotic properties of the iterates of the operator $P$ given by formulas (0.1), (0.2).

1. Markov operators. Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. Denote by $D = D(X, \mathcal{A}, \mu)$ the subset of $L^1 = L^1(X, \mathcal{A}, \mu)$ which contains all (normalized) densities, i.e.

$$D = \{ f \in L^1 : f \geq 0, \|f\| = 1 \}$$

where $\| \cdot \|$ stands for the norm in $L^1$. A linear mapping $P : L^1 \to L^1$ is called a Markov operator if $P(D) \subset D$.

Let a Markov operator $P$ be given. A density $f$ is called stationary (or invariant) if $Pf = f$. The operator $P$ is called asymptotically stable if there is a density $f_*$ such that

$$(1.1) \quad \lim_{n \to \infty} \|P^n f - f_*\| = 0 \quad \text{for } f \in D.$$  

Of course, a density $f_*$ satisfying condition (1.1) is necessarily stationary and unique.

In order to present a simple criterion for the existence of a stationary density we must recall the notion of Banach limits [4]. A Banach limit $L$ is a linear functional defined on the space $l^\infty$ of bounded sequences $(a_n) = (a_1, a_2, \ldots)$ of real numbers which satisfies the following conditions:

(i) $L(a_n) \geq 0$ if $a_i \geq 0$ ($i = 1, 2, \ldots$),
(ii) $L(a_1, a_2, \ldots) = L(a_2, a_3, \ldots)$,
(iii) $L(1, 1, \ldots) = 1$.

If $(a_n)$ is convergent then $L(a_n) = \lim_{n \to \infty} a_n$, and if $\limsup a_n \leq c$ then $L(a_n) \leq c$.

**Theorem 1.1.** Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be a Markov operator and $L$ a Banach limit. Assume that there exists a set $A \in \mathcal{A}$, $\mu(A) < \infty$, a
number \( \delta > 0 \) and a density \( f \) such that

\[
L \left( \int \frac{P^n f}{(X \setminus A) \cup E} \, d\mu \right) < 1 \quad \text{for } E \subset A \text{ with } \mu(E) < \delta.
\]

Then \( P \) admits a stationary density.

The proof of this result was given by J. Soca la [9]. It should be noted, however, that in Soca la’s statement a stronger form of condition (1.2) was used. Namely, the functional \( L \) was replaced by \( \lim \sup \). The above formulation was proposed by T. Komorowski and J. Tyrcha [3].

Now consider an operator \( P \) of the form

\[
Pf(x) = \int_X k(x, y) f(y) \, d\mu(y)
\]

where \( k : X \times X \to \mathbb{R} \) is a stochastic kernel, i.e. \( k \) is jointly measurable on \( X \times X \) and satisfies

\[
k(x, y) \geq 0 \quad \text{for } (x, y) \in X \times X,
\]

\[
\int_X k(x, y) \, d\mu(x) = 1 \quad \text{for } y \in X.
\]

From (1.4) it follows immediately that \( P \) is a Markov operator; it is called an integral Markov operator.

For integral Markov operators the existence of an invariant density and a simple transitivity condition imply asymptotic stability. To formulate this criterion precisely recall that in the theory of Markov operators the support of an \( f \in L^1(X, A, \mu) \) is defined up to a set of measure zero by the formula

\[
\text{supp } f = \{ x \in X : f(x) \neq 0 \}.
\]

We say that a Markov operator \( P \) overlaps supports if for every \( f, g \in D \) there is a positive integer \( n_0 = n_0(f, g) \) such that

\[
\mu(\text{supp } P^{n_0} f \cap \text{supp } P^{n_0} g) > 0.
\]

Observe that condition (1.5) implies that

\[
\mu(\text{supp } P^n f \cap \text{supp } P^n g) > 0 \quad \text{for } n \geq n_0(f, g).
\]

In fact,

\[
\text{supp } P^n f \cap \text{supp } P^n g \supset \text{supp } P^{n-n_0}(\min\{P^{n_0} f, P^{n_0} g\}).
\]

**Theorem 1.2.** An integral Markov operator which overlaps supports and has a stationary density \( f_* > 0 \) a.e. is asymptotically stable.

**Proof.** Define a new measure space \( (X, A, \overline{\mu}) \) with \( d\overline{\mu} = f_* d\mu \) and consider the operator

\[
\overline{P}f = (1/f_*) P(f \cdot f_*).
\]
Observe that for every \( f \in L^1(\mu) \) the product \( f \cdot f^* \) belongs to \( L^1(\mu) \). It is evident that \( P \) is an integral operator on \( L^1(\mu) \) and that
\[
P1_X = 1_X.
\]
(Here and in the sequel \( 1_E \) denotes the characteristic function of the subset \( E \) of \( X \).) Now we are going to use a well known decomposition property of integral Markov operators satisfying \( P1_X \leq 1_X \) (see [1], Ch. VIII). The space \( X \) may be written in the form
\[
X = X_1 \cup X_2, \quad X_1 = \bigcup_i W_i
\]
where the family \( \{W_i\} \) is at most countable. The sets \( X_1, X_2 \) and \( W_i \) are measurable, disjoint \( (X_1 \cap X_2 = \emptyset, W_i \cap W_j = \emptyset \) for \( i \neq j \) and have the following properties:

(i) For every \( f \in L^1(\mu) \) with \( \text{supp} \ f \subset X_2 \) and for every \( g \in L^\infty(\mu) \),
\[
\lim_{n \to \infty} \int_X g \cdot P^nf \, d\mu = 0.
\]
(ii) For every \( i \) there is a \( j \) such that \( P1_{W_i} = 1_{W_j} \).
(iii) Every set \( W_i \) is either cyclic or wandering. In the first case \( P^k1_{W_i} = 1_{W_i} \) for a positive integer \( k \); in the second, all sets \( W_{in} (n = 0, 1, \ldots) \) defined by \( 1_{W_{in}} = P^n1_{W_i} \) are distinct and hence disjoint.
(iv) For every cyclic \( W_i \) with period \( k \geq 2 \) and for every \( f \in L^1(\mu) \) vanishing outside \( W_i \),
\[
\lim_{n \to \infty} \| P^nf - \left( \int_{W_i} f \, d\pi(W_i) \right) 1_{W_{in}} \|_{L^1(\mu)} = 0.
\]
We shall show that in our case the decomposition formula (1.8) reduces to \( X = W_1 \). In fact, \( \pi(X_2) \leq \pi(X) = 1 \) and we may take \( f = 1_{X_2}, g = 1_X \) in (1.9). Since \( P \) preserves the integral with respect to \( \pi \) this gives \( \pi(X_2) = 0 \). Assume that \( W_i \) is wandering. Then
\[
\text{supp} \ P^n(f \cdot 1_{W_i}) \cap \text{supp} \ P^n(f \cdot 1_{W_{ii}}) = \text{supp} \ P^n1_{W_i} \cap \text{supp} \ P^n1_{W_{ii}} = W_{in} \cap W_{i,n+1} = \emptyset
\]
for every \( n \), which contradicts (1.5) and shows that there are no wandering sets. Assume now that \( W_i \) is cyclic with period \( k \geq 2 \). Then, as previously,
\[
\text{supp} \ P^k(f \cdot 1_{W_i}) \cap \text{supp} \ P^{kn}(f \cdot 1_{W_{ii}}) = W_{i,kn} \cap W_{i,kn+1} = W_i \cap W_{i+1} = \emptyset
\]
for every \( n \). Consequently, each \( W_i \) is cyclic with period \( k = 1 \). Assume that there are two such sets, say \( W_1 \) and \( W_2 \). Then
\[
\text{supp} \ P^n(f \cdot 1_{W_1}) \cap \text{supp} \ P^n(f \cdot 1_{W_2}) = W_1 \cap W_2 = \emptyset
\]
for every $n$, which again contradicts (1.5). Thus there is exactly one cyclic set with cycle length $k = 1$. We denote this set by $W_1$. According to (1.10) with $\overline{\mu}(W_1) = \overline{\mu}(X) = 1$, $k = 1$, we obtain
\begin{equation}
(1.11) \lim_{n \to \infty} \left\| \overline{\mathcal{P}}^n f - \left( \int f \, d\overline{\mu} \right) 1_X \right\|_{L^1(\overline{\mu})} = 0
\end{equation}
for every $f \in L^1(\overline{\mu})$. Evidently, for every $f \in D(\mu)$ we have $f/f_* \in L^1(\overline{\mu})$ and
\begin{align*}
\| P^n f - f_* \|_{L^1(\mu)} &= \left\| f_* \overline{\mathcal{P}}^n (f/f_*) - f_* \int_X (f/f_*) \, d\overline{\mu} \right\|_{L^1(\mu)} \\
&= \left\| \overline{\mathcal{P}}^n (f/f_*) - \left( \int_X (f/f_*) \, d\overline{\mu} \right) 1_X \right\|_{L^1(\overline{\mu})}.
\end{align*}
From this and (1.11) we get (1.1). 

Corollary 1.1. Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be an integral Markov operator which has a positive stationary density $f_*$ ($f_* > 0 \text{ a.e.}$). Assume, moreover, that there exists a set $A \in \mathcal{A}$, $\mu(A) > 0$, with the following property. For every $f \in D$ there is a positive integer $n_0 = n_0(f)$ such that
\[ P^{n_0} f(x) > 0 \quad \text{for a.e. } x \in A. \]
Then $P$ is asymptotically stable.

Theorems 1.1 and 1.2 do not match well. In fact, the invariant density existing by Theorem 1.1 need not be positive on the whole space $X$, which is an important assumption in Theorem 1.2. This situation may be improved by studying $P$ restricted to the support of the invariant density.

Let a Markov operator $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be given. It is well known that for all nonnegative $f, f_* \in L^1(X)$ the inclusion supp $f \subset$ supp $f_*$ implies supp $Pf \subset$ supp $Pf_*$. In particular, if $f_* = Pf_*$ and supp $f_* = C$ then
\[ \text{supp } f \subset C \quad \text{implies} \quad \text{supp } Pf \subset C. \]
This property allows us to consider $P$ on the space $L^1(C)$ of all functions from $L^1(X)$ with supports contained in $C$. We will denote $P$ restricted to $L^1(C)$ by $P_C$.

Theorem 1.3. Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be a Markov operator having an invariant density $f_*$. Assume that the operator $P_C$ with $C = \text{supp } f_*$ is asymptotically stable. Assume, moreover, that there is a $\delta > 0$ such that
\begin{equation}
(1.12) \sup_{\mathcal{C}} \int_C P^n f \, d\mu \geq \delta \quad \text{for } f \in D(X).
\end{equation}
Then $P : L^1(X) \to L^1(X)$ is also asymptotically stable.
Proof. According to the lower bound function theorem (see [6], Ch. 5) in order to prove (1.1) it is sufficient to find a nonnegative $h \in L^1(X)$, $\|h\| > 0$, such that

$$\lim_{n \to \infty} \| (P^n f - h)^- \| = 0 \quad \text{for } f \in D(X)$$

where $\| \cdot \|$ stands for the norm in $L^1(X)$. Define $h = \frac{1}{2} \delta f_*$ and fix an $f \in D(X)$. According to (1.12) there is an integer $m$ such that

$$\eta := \int_C P^m f \, d\mu \geq \frac{1}{2} \delta.$$  

For $n \geq m$ we have

$$P^n f = P^{n-m} (1_{X \setminus C} P^m f) + P^{n-m} (1_C P^m f).$$

Since $P_C$ is asymptotically stable with invariant density $f_*$ we also have

$$\lim_{n \to \infty} \| P^{n-m} (1_C P^m f) - \eta f_* \| = 0.$$  

From (1.14) and the inequality $h \leq \eta f_*$ it follows that

$$\| (P^n f - h)^- \| \leq \| P^{n-m} (1_C P^m f) - \eta f_* \|$$

for $n \geq m$. This and (1.15) imply (1.13).

Using Theorems 1.2 and 1.3 it is easy to derive the following

**Corollary 1.2.** Let $P : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{A}, \mu)$ be an integral Markov operator which overlaps supports and has an invariant density $f_*$. Set $C = \text{supp } f_*$. If there is a $\delta > 0$ such that (1.12) is satisfied, then $P$ is asymptotically stable.

Proof. According to Theorem 1.3 it is enough to prove that the operator $P_C$ is asymptotically stable. Evidently,

$$P_C f(x) = \int_C k(x, y) f(y) \, d\mu(y)$$

for every $f \in L^1(C)$ and

$$0 = \int_C f_*(y) \, d\mu(y) - \int_C P_C f_*(x) \, d\mu(x)$$

$$= \int_C \left( 1 - \int_C k(x, y) \, d\mu(x) \right) f_*(y) \, d\mu(y),$$

whence

$$\int_C k(x, y) \, d\mu(x) = 1 \quad \text{for a.e. } y \in C.$$  

This shows that $P_C$ is an integral Markov operator. Thus we can apply Theorem 1.2 to $P_C$ and its asymptotical stability follows.
2. Volterra operators. In this section we shall consider the integral operator $P$ defined by (0.1) and (0.2) under the following general assumptions (K1) and (K2):

(K1) $H : [0, \infty) \to [0, \infty)$ is nonincreasing, absolutely continuous and $H(0) = 1$, $\lim_{x \to \infty} H(x) = 0$.

(K2) $Q : [0, \infty) \to [0, \infty)$ and $\lambda : [0, \infty) \to [0, \infty)$ are nondecreasing, absolutely continuous and $Q(0) = \lambda(0) = 0$, $\lim_{x \to \infty} Q(x) = \lim_{x \to \infty} \lambda(x) = \infty$.

The above conditions (K1) and (K2) are assumed in the whole of this section and will not be repeated in the statements of the theorems. Moreover, all measure-theoretic notions refer to the standard Lebesgue measure $m$ on $[0, \infty)$.

We start with the following lemma from [7].

Lemma 2.1. If $W : [0, \infty) \to [0, \infty)$ is measurable and $f \in D$, then

\[
\int_0^\infty W(Q(\lambda(x))) Pf(x) \, dx = \int_0^\infty f(y) \, dy \int_0^\infty W(x + Q(y)) h(x) \, dx
\]

where

\[ h(x) = -H'(x). \]

Using Theorem 1.1 we prove the following theorem concerning the existence of a stationary density for $P$.

Theorem 2.1. If there exists an $\alpha \in (0, 1]$ such that

\[
\int_0^\infty x^\alpha h(x) \, dx < \liminf_{x \to \infty} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha),
\]

then the operator $P$ given by formulas (0.1), (0.2) has a stationary density.

Proof. Evidently, $P$ is an integral Markov operator defined on $L^1([0, \infty))$. Define

\[ \sigma = \int_0^\infty x^\alpha h(x) \, dx. \]

Using (2.2) we can find positive numbers $\varepsilon, \rho$ and $x_0$ such that

\[ \sigma + \varepsilon < \rho < (Q(\lambda(x)))^\alpha - Q(x)^\alpha \quad \text{for } x \geq x_0. \]

We are going to show that for every $f \in D$ there exists an integer $n_0(f)$ such that

\[
\int_0^{x_0} \sum_{k=1}^n \frac{1}{n} P^k f(x) \, dx \geq \frac{\varepsilon}{2M} \quad \text{for } n \geq n_0(f)
\]
where
\begin{equation}
M := \sup_{[0,x_0]} (|Q(\lambda(x))| \leq Q(x) - \varrho |).
\end{equation}

Using (2.1) with \(W(x) = x^\alpha\) and \(f \in D\) we have
\begin{equation}
\int_0^\infty (Q(\lambda(x)))^\alpha Pf(x) dx = \int_0^\infty f(y) dy \int_0^\infty (x + Q(y))^\alpha h(x) dx
\leq \int_0^\infty f(y) dy \int_0^\infty (x^\alpha + Q(y)^\alpha) h(x) dx
= \sigma + \int_0^\infty f(y) Q(y)^\alpha dy.
\end{equation}

Fix \(f \in D\) such that
\begin{equation}
\int_0^\infty Q(x)^\alpha f(x) dx < \infty
\end{equation}
and define
\begin{equation}
f_n = \frac{1}{n} \sum_{k=1}^n P^k f \quad \text{for } n = 1, 2, \ldots
\end{equation}

From (2.3), (2.6) and (2.7) it follows that
\begin{equation*}
\int_0^\infty (Q(\lambda(x)))^\alpha Pf_n(x) dx \leq \sigma + \int_0^\infty Q(x)^\alpha f_n(x) dx
\end{equation*}
and that the integral on the right hand side is finite for every \(n\). Hence
\begin{equation*}
\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x) dx \leq \sigma + \frac{1}{n} \int_0^\infty (Q(\lambda(x)))^\alpha Pf(x) dx.
\end{equation*}

Since \(\sigma < \varrho - \varepsilon\), there exists a positive integer \(n_0(f)\) such that
\begin{equation*}
\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x) dx \leq \varrho - \varepsilon \quad \text{for } n \geq n_0(f).
\end{equation*}

On the other hand, taking into account (2.3) we have
\begin{equation*}
\int_0^\infty ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x) dx
\geq \int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x) dx + \varrho \int_{x_0}^\infty f_n(x) dx.
\end{equation*}
Consequently,
\[
\int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha) f_n(x) \, dx \leq \varrho - \varepsilon - \varrho \int_{x_0}^\infty f_n(x) \, dx
\]
\[
= \varrho \int_0^{x_0} f_n(x) \, dx - \varepsilon
\]
for \( n \geq n_0(f) \), which together with (2.5) gives
\[
-M \int_0^{x_0} f_n(x) \, dx \leq \int_0^{x_0} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha - \varrho) f_n(x) \, dx \leq -\varepsilon
\]
for \( n \geq n_0(f) \). This implies (2.4) and even a stronger inequality with the right hand side \( \varepsilon/M \). The above argument was valid for \( f \) satisfying (2.7).

To get (2.4) for every \( f \in D \) it is enough to observe that the set of all \( f \in D \) such that (2.7) holds is dense in \( D \).

Now we are going to show that there exists a \( \delta > 0 \) such that
\[
\int_E Pf(x) \, dx \leq \frac{\varepsilon}{4M} \quad \text{for } f \in D \text{ and } E \subset [0, x_0], m(E) \leq \delta.
\]
In fact, since \( h \) is integrable we can find a \( \gamma > 0 \) such that
\[
\int_F h(x) \, dx \leq \frac{\varepsilon}{4M} \quad \text{for } m(F) \leq \gamma.
\]
Further, since \( Q \circ \lambda \) is absolutely continuous, there exists a \( \delta > 0 \) such that
\[
m(Q(\lambda(E))) \leq \gamma \quad \text{for } E \subset [0, x_0], m(E) \leq \delta.
\]
Now let \( E \subset [0, x_0] \) be measurable and \( m(E) \leq \delta \). Setting \( W = 1_{Q(\lambda(E))} \) we have \( 1_E \leq W \circ Q \circ \lambda \), which according to (2.1) gives
\[
\int_E Pf(x) \, dx = \int_0^\infty 1_E(x) Pf(x) \, dx \leq \int_0^\infty W(Q(\lambda(x))) Pf(x) \, dx
\]
\[
= \int_0^\infty f(y) \, dy \int_0^\infty W(x + Q(y)) h(x) \, dx
\]
\[
= \int_0^\infty f(y) \, dy \int_{Q(\lambda(E)) - Q(y)} h(x) \, dx \leq \frac{\varepsilon}{4M}
\]
for every \( f \in D \).

In order to verify condition (1.2) fix an \( f \in D \) and a positive integer \( n \geq n_0(f) \) such that (2.4) is satisfied. Fix a measurable set \( E \subset [0, x_0] \) with
\( m(E) \leq \delta \). Then
\[
\int_E f_n(x) \, dx = \int_E P \left( \frac{1}{n} \sum_{k=0}^{n-1} P^k f \right) \, dx \leq \frac{\varepsilon}{4M}
\]
and using (2.4) we obtain
\[
\int f_n(x) \, dx = 1 - \int_0^{x_0} f_n(x) \, dx + \int_E f_n(x) \, dx \leq 1 - \frac{\varepsilon}{2M} + \frac{\varepsilon}{4M} = 1 - \frac{\varepsilon}{4M}
\]
for \( n \geq n_0(f) \). Now let \( L_0 \) be a Banach limit and let
\[
L(a_k) = L_0 \left( \frac{1}{k} \sum_{i=1}^{k} a_i \right)
\]
for every bounded sequence \( (a_k) \) of real numbers. Evidently, \( L \) is also a Banach limit and
\[
L \left( \int_{(x_0, \infty) \cup E} P^m f(x) \, dx \right) = L_0 \left( \int_{(x_0, \infty) \cup E} f_n(x) \, dx \right) \leq 1 - \frac{\varepsilon}{4M},
\]
which shows that (1.2) with \( A = [0,x_0] \) is satisfied.

Now we use Corollary 1.2 to find a sufficient condition for the asymptotic stability of the operator \( P \) given by (0.1), (0.2).

**Theorem 2.2.** If there exists a positive number \( \alpha \leq 1 \) such that (2.2) holds, and a nonnegative number \( c \) such that
\[
h(x) > 0 \quad \text{for a.e. } x \geq c,
\]
then the operator \( P \) given by (0.1), (0.2) is asymptotically stable.

**Proof.** According to Theorem 2.1 the operator \( P \) has a stationary density \( f_* \). Define \( C = \text{supp} f_* \) and fix positive numbers \( \varepsilon, \varrho, x_0 \) such that (2.3) holds. Further, choose a positive number \( a \) such that
\[
\lambda(a) \geq x_0, \quad Q(\lambda(a)) \geq c + Q(x_0),
\]
and define
\[
A = \{ x \geq a : (Q \circ \lambda)'(x) > 0 \}.
\]
Since \( Q \circ \lambda \) is absolutely continuous and \( \lim_{x \to \infty} Q(\lambda(x)) = \infty \), the set \( A \) is unbounded (\( \text{ess sup } A = \infty \)). Finally, define the number \( M \) by (2.5).
If \( x \in A \), then

\[
    f_*(x) = Pf_*(x) = (Q \circ \lambda)'(x) \int_0^{\lambda(x)} h(Q(\lambda(x))) - Q(y) \, f_*(y) \, dy
\]

\[
    \geq (Q \circ \lambda)'(x) \int_0^{\lambda_0} h(Q(\lambda(x))) - Q(y) \, f_*(y) \, dy
\]

and

\[
    (Q \circ \lambda)'(x) > 0, \quad h(Q(\lambda(x))) - Q(y) > 0 \quad \text{for } y \in [0, x_0].
\]

From (2.4) with \( f = f_* \) it follows that

\[
    \int_0^{\lambda_0} f_*(y) \, dy > 0.
\]

This shows that \( f_*(x) > 0 \) for \( x \in A \) and that \( A \subseteq C \). Using (2.4) it is also easy to show that

\[
    \sup_n \int_C P^n f(x) \, dx \geq \frac{\varepsilon}{2M} \int_0^\infty h(u) \, du \quad \text{for } f \in D.
\]

In fact, according to (2.4) for every density \( f \) there is a positive integer \( k \) such that

\[
    \int_0^{\lambda_0} P^k f(x) \, dx \geq \frac{\varepsilon}{2M}
\]

and consequently,

\[
    \int_C P^{k+1} f(x) \, dx \geq \int_A P^{k+1} f(x) \, dx
\]

\[
    = \int_A (Q \circ \lambda)'(x) \int_0^{\lambda(x)} h(Q(\lambda(x))) - Q(y) \, P^k f(y) \, dy
\]

\[
    \geq \int_A (Q \circ \lambda)'(x) \int_0^{\lambda_0} h(Q(\lambda(x))) - Q(y) \, P^k f(y) \, dy
\]

\[
    = \int_0^{\lambda_0} P^k f(y) \, dy \int_a^\infty (Q \circ \lambda)'(x) h(Q(\lambda(x))) - Q(y) \, dx
\]

\[
    \geq \int_0^{\lambda_0} P^k f(y) \, dy \int_a^\infty h(u) \, du \geq \frac{\varepsilon}{2M} \int_0^\infty h(u) \, du.
\]

Finally, observe that for every density \( f \) there exists a positive number
\[ b = b(f) \] such that
\[ Pf(x) > 0 \quad \text{for} \quad x \in [b, \infty) \cap A. \]

To show this choose \( b_0 > 0 \) such that \( \int_0^{b_0} f(y) \, dy > 0 \), and \( b > 0 \) such that \( \lambda(b) \geq b_0, \quad Q(\lambda(b)) \geq c + Q(b_0) \). For \( x \in [b, \infty) \cap A \) we then have
\[ Pf(x) \geq (Q \circ \lambda)'(x) \int_0^{b_0} h(Q(\lambda(x)) - Q(y)) f(y) \, dy > 0. \]

Setting \( d = d(f, g) = \max\{b(f), b(g)\} \) we obtain
\[ m(\text{supp} \, Pf \cap \text{supp} \, Pg) \geq m([d, \infty) \cap A) > 0 \quad \text{for} \quad f, g \in D. \]

Thus all the requirements of Corollary 1.2 are satisfied and the proof is complete.

The following example shows that assumption (2.9) in the statement of Theorem 2.2 is essential.

**Example 2.1.** Let \( h : [0, \infty) \to [0, \infty) \) be an integrable function such that
\[ \int_0^\infty h(x) \, dx = 1 \quad \text{and} \quad h(x) = 0 \quad \text{for} \quad x \geq \sqrt{c} - c \]
where \( c \in (0, 1) \) is a constant. Consider the operator \( P : L^1 \to L^1 \) given by the formula
\[ Pf(x) = \begin{cases} 
\frac{1}{2\sqrt{x}} \int_0^{\sqrt{x}} h(\sqrt{x} - y) f(y) \, dy & \text{for} \quad x \in (0, 1), \\
2 \int_0^{2x-1} h(2x - y - 1) f(y) \, dy & \text{for} \quad x \geq 1.
\end{cases} \tag{2.11} \]

In this case \( Q(x) = x, \)
\[ \lambda(x) = \begin{cases} 
\sqrt{x} & \text{for} \quad x \in [0, 1], \\
2x - 1 & \text{for} \quad x > 1,
\end{cases} \quad H(x) = 1 - \int_0^x h(t) \, dt, \]
and evidently the assumptions (K1) and (K2) are satisfied. Moreover, for every \( \alpha \in (0, 1), \)
\[ \int_0^\infty x^\alpha h(x) \, dx < 1 < \infty = \lim_{x \to \infty} ((Q(\lambda(x)))^\alpha - Q(x)^\alpha). \]

According to Theorem 2.1 the operator \( P \) has a stationary density. Using (2.11) it is easy to verify the following property of \( P \). If \( \text{supp} \, f \subset [1, \infty) \) then \( \text{supp} \, Pf \subset [1, \infty) \) and if \( \text{supp} \, f \subset (0, c) \) then \( \text{supp} \, Pf \subset (0, c) \). Since \( c < 1 \), condition (1.1) cannot be satisfied with an \( f_* \) independent on \( f \). Thus \( P \) is not asymptotically stable.
In the previous results concerning the operator (0.1), (0.2) an important role was played by condition (2.2). Thus a natural question arises: What could we say about the behaviour of \((P^n f)\) when (2.2) is not satisfied? A partial answer to this question may be given by showing that if an opposite condition to (2.2) is satisfied then the operator \(P\) is sweeping [2].

We say that a Markov operator \(P : L^1([0, \infty)) \to L^1([0, \infty))\) is sweeping if

\[
\lim_{n \to \infty} \int_0^r P^n f(x) \, dx = 0 \quad \text{for every } f \in D \text{ and } r \geq 0.
\]

**Theorem 2.3.** Assume that

\[
\sup_{x \geq x_0} ((Q(\lambda(x)))^\beta - Q(x)^\beta) < \int_0^\infty x^\beta h(x) \, dx < \infty
\]

for an \(x_0 \geq 0\) and \(\beta \geq 1\) and that

\[
\int_{Q(\lambda(x_0))}^{\infty} h(x) \, dx > 0.
\]

Then the operator \(P\) given by (0.1), (0.2) is sweeping.

**Proof.** Define

\[
z_0 = (Q(\lambda(x_0)))^\beta, \quad w(z) = \begin{cases} e^{-z z_0} & \text{for } z \in [0, z_0], \\
e^{-z z} & \text{for } z > z_0,
\end{cases}
\]

and

\[
V(x) = w((Q(\lambda(x)))^\beta)
\]

where \(\varepsilon > 0\) will be chosen later. We shall show that there exists a nonnegative constant \(\gamma < 1\) such that

\[
\int_0^\infty V(x) P f(x) \, dx \leq \gamma \int_0^\infty V(x) f(x) \, dx \quad \text{for } f \in D.
\]

Since \(V(x)\) admits a positive minimum on every compact set this inequality implies (2.12) (see also [2]).

According to (2.13) there exists a number \(\varrho\) such that

\[
\sup_{x \geq x_0} ((Q(\lambda(x)))^\beta - Q(x)^\beta) < \varrho < \int_0^\infty x^\beta h(x) \, dx.
\]

Define

\[
I(y) = \int_0^\infty \frac{w(x + Q(y))^\beta}{V(y)} h(x) \, dx \quad \text{for } y \geq 0.
\]
If \( y \leq x_0 \), then \( V(y) = w(z_0) \) and

\[
I(y) \leq \int_0^\infty \frac{w(x^\beta)}{V(y)} h(x) \, dx = \int_0^\infty \frac{w(x^\beta)}{w(z_0)} h(x) \, dx + \int_{Q(\lambda(x_0))}^\infty \frac{w(x^\beta)}{w(z_0)} h(x) \, dx
\]

\[
= \int_0^\infty h(x) \, dx + \int_{Q(\lambda(x_0))}^\infty h(x)e^{-\varepsilon(x^\beta-z_0)} \, dx
\]

\[
= 1 - \int_{Q(\lambda(x_0))}^\infty h(x)(1 - e^{-\varepsilon(x^\beta-z_0)}) \, dx =: \gamma_1(\varepsilon) < 1.
\]

If \( y > x_0 \), then \((Q(\lambda(y)))^\beta - Q(y)^\beta < \varrho\) and, since \( w(z) \leq e^{-\varepsilon z} \) for \( z \geq 0 \),

\[
\frac{w((x+Q(y))^{\beta})}{V(y)} \leq \frac{e^{-\varepsilon(x+Q(y))^{\beta}}}{e^{-\varepsilon(Q(\lambda(y)))^{\beta}}} \leq \frac{e^{-\varepsilon(x^{\beta})}}{e^{-\varepsilon(\varrho+Q(y))^{\beta}}} \leq e^{-\varepsilon(x^{\beta})};
\]

consequently,

\[
I(y) \leq \int_0^\infty h(x)e^{-\varepsilon(x^{\beta}-\varrho)} \, dx =: \gamma_2(\varepsilon).
\]

From Lemma 2.1 it follows that

\[
\int_0^\infty V(x) Pf(x) \, dx = \int_0^\infty f(y) \, dy \int_0^\infty w((x+Q(y))^{\beta}) h(x) \, dx
\]

\[
= \int_0^\infty f(y)V(y)I(y) \, dy
\]

\[
\leq \gamma_1(\varepsilon) \int_0^{x_0} V(y)f(y) \, dy + \gamma_2(\varepsilon) \int_{x_0}^\infty V(y)f(y) \, dy
\]

for every density \( f \). Since \( \gamma_1(\varepsilon) < 1 \) for every \( \varepsilon > 0 \), in order to show (2.14) with a constant \( \gamma < 1 \) it is enough to prove that there exists and \( \varepsilon > 0 \) such that \( \gamma_2(\varepsilon) < 1 \). But the function \( \gamma_2 \) is differentiable on \([0, \infty)\) and

\[
\gamma_2'(\varepsilon) = -\int_0^\infty h(x)(x^\beta - \varrho)e^{-\varepsilon(x^\beta-\varrho)} \, dx,
\]

whence

\[
\gamma_2'(0) = \varrho - \int_0^\infty x^\beta h(x) \, dx < 0.
\]

Consequently, for sufficiently small \( \varepsilon > 0 \) we have \( \gamma_2(\varepsilon) < \gamma_2(0) = 1 \), which completes the proof. \( \blacksquare \)
References


