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Existence of solution of the nonlinear Dirichlet problem for differential-functional equations of elliptic type

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Abstract. Consider a nonlinear differential-functional equation

$$\mathcal{A}u + f(x, u(x), u) = 0\,, \tag{1}$$

where

$$\mathcal{A}u := \sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}\,,$$

$x = (x_1, \ldots, x_m) \in G \subset \mathbb{R}^m$, G is a bounded domain with $C^{2+\alpha}$ $(0 < \alpha < 1)$ boundary, the operator $\mathcal{A}$ is strongly uniformly elliptic in G and u is a real $L^p(\overline{G})$ function.

For the equation (1) we consider the Dirichlet problem with the boundary condition

$$u(x) = h(x) \quad \text{for } x \in \partial G\,. \tag{2}$$

We use Chaplygin's method [5] to prove that problem (1), (2) has at least one regular solution in a suitable class of functions.

Using the method of upper and lower functions, coupled with the monotone iterative technique, H. Amman [3], D. H. Sattinger [13] (see also O. Diekmann and N. M. Temme [6], G. S. Ladde, V. Lakshmikantham, A. S. Vatsala [8], J. Smoller [15]) and I. P. Mysovskikh [11] obtained similar results for nonlinear differential equations of elliptic type.

A special case of (1) is the integro-differential equation

$$\mathcal{A}u + f\Big(x, u(x), \int_G u(x)\,dx\Big) = 0\,.$$

Interesting results about existence and uniqueness of solutions for this equation were obtained by H. Ugowski [17].

1. Notation, definitions and assumptions.

By $C^{l+\alpha}(\overline{G})$ $(l = 0, 1, 2, \ldots; 0 < \alpha \le 1)$ we denote the space of functions $f \in C^l(\overline{G})$ whose

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derivatives of order l are Hölder continuous with finite norm

$$|f|_{l+\alpha}=\sup_{\substack{x\in G\\|k|\le l}}|D^kf(x)|+\sup_{\substack{x,y\in G\\|k|=l,x\ne y}}\frac{|D^kf(x)-D^kf(y)|}{\|x-y\|^\alpha},$$

where $\|x\|^2=\sum_{i=1}^m x_i^2$.

By $H^{m,p}(G)$ ($p\ge 1$) we denote the Sobolev space (see [1]) defined in the following way: $H^{m,p}(G)$ is the space of all functions f having weak derivatives $D^\beta f\in L^p(G)$ for all $|\beta|\le m$ with finite norm

$$\|f\|_{m,p}=\Big(\sum_{|k|\le m}\int_G |D^kf(x)|^p\,dx\Big)^{1/p}.$$

We assume that the operator $\mathcal{A}$ (see the abstract) is strongly uniformly elliptic, i.e., there is a $\mu>0$ such that

$$\sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j\ge\mu\|\xi\|^2$$

for all $x\in G$ and $\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^m$.

Moreover, we assume that $a_{ij}\in C^{0+\alpha}(\overline{G})$ and $a_{ij}=a_{ji}$ ($i,j=1,\dots,m$). The boundary ∂G is assumed to be of class $C^{2+\alpha}$, i.e., a finite union of $C^{2+\alpha}$ surfaces.

We assume that $h\in C^{2+\alpha}(\partial G)$, i.e., there is an $\widetilde{h}\in C^{2+\alpha}(\overline{G})$ such that $\widetilde{h}(x)=h(x)$ for all $x\in\partial G$.

A function u is called *regular* in $\overline{G}$ if $u\in C^0(\overline{G})\cap C^2(G)$.

Functions u and v regular in $\overline{G}$ and satisfying the systems of inequalities

$$\begin{cases}\mathcal{A}u+f(x,u(x),u)\ge 0 & \text{for } x\in G,\\ u(x)\le h(x) & \text{for } x\in\partial G,\end{cases}\tag{3}$$

$$\begin{cases}\mathcal{A}v+f(x,v(x),v)\le 0 & \text{for } x\in G,\\ v(x)\ge h(x) & \text{for } x\in\partial G,\end{cases}\tag{4}$$

are called a *lower* and an *upper functions* for problem (1), (2) in $\overline{G}$, respectively.

Assumption A. We assume that there exists at least one pair u_0,v_0 of lower and upper functions for problem (1), (2) in $\overline{G}$ such that

$$u_0(x)\le v_0(x)\quad\text{for } x\in\overline{G}.$$

Let u_0,v_0 be lower and upper functions for problem (1), (2) in $\overline{G}$. Define

$$K=\{(x,y,s):x\in\overline{G},\ y\in[m_0,M_0],\ s\in\langle u_0,v_0\rangle\},$$

where

$$m_0=\min_{x\in\overline{G}}u_0(x),\qquad M_0=\max_{x\in\overline{G}}v_0(x)$$

and $\langle u_0, v_0\rangle$ is the segment

$$\langle u_0, v_0\rangle := \{s \in L^p(G) : u_0(x) \le s(x) \le v_0(x) \text{ for } x \in G\}\,.$$

We assume that $f : \mathbb{R}^m \times \mathbb{R} \times L^p \ni (x, y, s) \mapsto f(x.y, s) \in \mathbb{R}$ satisfies in K the following assumptions:

(a) $f(\cdot, y, s) \in C^{0+\alpha}(\overline{G})$ for $y \in [m_0, M_0], s \in \langle u_0, v_0\rangle$,
(b) $f(x, \cdot, \cdot)$ is continuous for $x \in \overline{G}$,
(c) the derivative $\partial f/\partial y$ exists and is continuous, and

$$\left|\frac{\partial f}{\partial y}(x, y, s)\right| \le c_0 \quad \text{in } K$$

where $c_0 > 0$ is a constant,
(d) f is increasing with respect to s.

2. Main results. Throughout this paper we assume all assumptions of the first section to hold.

Theorem 1. *The problem* (1), (2) *has at least one regular solution* u *such that*

$$u_0(x) \le u(x) \le v_0(x) \quad \textit{for } x \in \overline{G}\,.$$

Before going into the proof of the theorem we establish some lemmas and make a few remarks.

From assumption (c) it follows that for $k > c_0$,

$$\frac{\partial f}{\partial y} + k > 0 \quad \text{in } K\,. \tag{5}$$

Let β be a sufficiently regular function defined on $\overline{G}$. Denote by $\mathcal{P}$ the operator $\mathcal{P} : \beta \mapsto \gamma = \mathcal{P}\beta$, where γ is the (supposedly unique) solution of the boundary value problem

$$\begin{cases} (\mathcal{A} - k\mathcal{I})\gamma = -[f(x, \beta(x), \beta) + k\beta(x)] & \text{in } G\,, \\ \gamma(x) = h(x) & \text{on } \partial G\,. \end{cases} \tag{6}$$

The operator $\mathcal{P}$ is the composition of the nonlinear operator $\mathcal{F} : \beta \mapsto \delta$, where

$$\mathcal{F}\beta(x) := -[f(x, \beta(x), \beta) + k\beta(x)] = \delta(x) \tag{7}$$

and the linear operator $\mathcal{G} : \delta \mapsto \gamma$, where γ is the (supposedly unique) solution of the linear problem

$$\begin{cases} (\mathcal{A} - k\mathcal{I})\gamma = \delta(x) & \text{in } G\,, \\ \gamma(x) = h(x) & \text{on } \partial G\,. \end{cases} \tag{8}$$

$\mathcal{F}$ is the *Nemytskiĭ operator*. It is sometimes also called the *superposition operator, composition operator*, or *substitution operator*. More information about it can be found in [4].

LEMMA 1. (i) $\mathcal{F}$ *maps* $C^{0+\alpha}(\overline{G})$ *into* $C^{0+\alpha}(\overline{G})$ *and is a bounded and continuous operator between these spaces.*

(ii) $\mathcal{P}$ *maps* $C^{0+\alpha}(\overline{G})$ *into* $C^{0+\alpha}(\overline{G})$ *and is compact.*

Proof. Assumption (c) implies that f satisfies the Lipschitz condition with respect to y. Therefore arguing as in [8, 7] we get (i).

Since the operator $\mathcal{A}$ is strongly uniformly elliptic, $a_{ij} \in C^{0+\alpha}(\overline{G})$, the domain G is bounded, $\partial G \in C^{2+\alpha}$, $h \in C^{2+\alpha}(\partial G)$ and $\delta \in C^{0+\alpha}(\overline{G})$, by the Schauder theorem [14] (see [9]) problem (8) has a unique solution $\gamma \in C^{2+\alpha}(\overline{G})$ such that

$$|\gamma|_{2+\alpha} \le c_1(|\delta|_{0+\alpha} + |\widetilde{h}|_{2+\alpha}), \tag{9}$$

where $c_1 > 0$ is independent of δ and h.

We define a constant operator $\mathcal{G}_1 : C^{0+\alpha}(\overline{G}) \to C^{2+\alpha}(\overline{G})$ by denoting, for every $h \in C^{0+\alpha}(\overline{G})$, by $\mathcal{G}_1(h)$ the unique solution of problem (8) with $\delta(x) = 0$ in $\overline{G}$.

Similarly, we define a linear operator $\mathcal{G}_2 : C^{0+\alpha}(\overline{G}) \to C^{2+\alpha}(\overline{G})$ by denoting, for every $\delta \in C^{0+\alpha}(\overline{G})$, by $\mathcal{G}_2(\delta)$ the unique solution of problem (8) with $h(x) = 0$ on ∂G.

It is easy to see that $\mathcal{G}(\delta) = \mathcal{G}_1(h) + \mathcal{G}_2(\delta)$. It follows from (9) that $\mathcal{G}_2$ is continuous. Consequently, since $\mathcal{G}_1$ is constant with respect to δ, $\mathcal{G}$ is continuous. Thus the operator

$$\mathcal{G} \circ \mathcal{F} : C^{0+\alpha}(\overline{G}) \to C^{2+\alpha}(\overline{G})$$

is bounded and continuous.

Since $\partial G \in C^{2+\alpha}$, the identity operator

$$\mathcal{I} : C^{2+\alpha}(\overline{G}) \to C^{0+\alpha}(\overline{G})$$

is compact (see [19]). Hence the operator

$$\mathcal{P} = \mathcal{I} \circ \mathcal{G} \circ \mathcal{F} : C^{0+\alpha}(\overline{G}) \to C^{0+\alpha}(\overline{G})$$

is compact. This completes the proof of Lemma 1.

LEMMA 2. (i) $\mathcal{F}$ *induces a bounded and continuous operator* $L^p(G) \to L^p(G)$.

(ii) $\mathcal{P}$ *induces a compact operator* $L^p(G) \to L^p(G)$.

Proof. Recall that G is bounded and f satisfies assumptions (a)–(c). Assumption (c) implies that f satisfies the Lipschitz condition with respect to y. Therefore arguing as in [18, 7] (see also [16]) we conclude that $\mathcal{F}$ maps $L^p(G)$ into $L^p(G)$. Hence the nonlinear operator $\mathcal{F}$ is bounded and continuous.

If $\delta \in L^p(G)$, then using the Agmon–Douglis–Nirenberg theorem [2] (see [9]) and repeating the same arguments as in the proof of Lemma 1, we can show that problem (8) has a unique weak solution $\gamma \in H^{2,p}(G)$, which satisfies

$$\|u\|_{2,p} \le c_2(\|\delta\|_{L^p} + \|\widetilde{h}\|_{2,p})\,, \tag{10}$$

where $c_2 > 0$ and c_2 does not depend on δ and h. Hence

$$\mathcal{G} : L^p(G) \to H^{2,p}(G)\,.$$

By (10) and using a similar argument to the proof of Lemma 1 one can show that the operator $\mathcal{G}$ is continuous. Thus $\mathcal{G}\circ\mathcal{F} : L^p(G) \to H^{2,p}(G)$ is bounded and continuous. Since the identity operator $\mathcal{I} : H^{2,p}(G) \to L^p(G)$ is compact (see [20]), the composition $\mathcal{P} = \mathcal{I}\circ\mathcal{G}\circ\mathcal{F} : L^p(G) \to L^p(G)$ is compact. This completes the proof of Lemma 2.

LEMMA 3. (i) *Let β_1 and β_2 be any regular functions such that $\beta_1, \beta_2 \in K$. Then the operator $\mathcal{P}$ is increasing*, *i.e.*, *$\beta_1(x) < \beta_2(x)$ in G implies $\mathcal{P}\beta_1(x) < \mathcal{P}\beta_2(x)$ in G.*

(ii) *If β is an upper* (*resp. a lower*) *function for problem* (1), (2) *in $\overline{G}$, then $\mathcal{P}\beta(x) < \beta(x)$* (*resp. $\mathcal{P}\beta(x) > \beta(x)$*) *in G.*

Proof. (i) Let $\beta_1(x) < \beta_2(x)$ in G. Setting $\gamma_1 = \mathcal{P}\beta_1$ and $\gamma_2 = \mathcal{P}\beta_2$ from (8) it follows that

$$\begin{cases} (\mathcal{A} - k\mathcal{I})(\gamma_2 - \gamma_1) = -[f(x, \beta_2(x), \beta_2) - f(x, \beta_1(x), \beta_1)] \\ \qquad\qquad\qquad\qquad -k[\beta_2(x) - \beta_1(x)] \quad \text{in } G, \\ \gamma_2(x) - \gamma_1(x) = 0 \quad \text{on } \partial G. \end{cases} \tag{11}$$

From this, by the monotonicity of f with respect to s we get

$$\begin{aligned} &(\mathcal{A} - k\mathcal{I})(\gamma_2 - \gamma_1) \\ &\qquad \le -[f(x, \beta_2(x), \beta_1) - f(x, \beta_1(x), \beta_1)] - k(\beta_2(x) - \beta_1(x)) \\ &\qquad = -[f_y(x, \beta_1(x) + \theta(\beta_2(x) - \beta_1(x)), \beta_1) + k](\beta_2(x) - \beta_1(x))\,, \end{aligned}$$

where $0 < \theta < 1$. Consequently, by (5) we have

$$\begin{cases} (\mathcal{A} - k\mathcal{I})(\gamma_2 - \gamma_1) \le 0 & \text{in } G\,, \\ \gamma_2(x) - \gamma_1(x) = 0 & \text{on } \partial G\,. \end{cases} \tag{12}$$

By the strong maximum principle [12], either $\gamma_2(x) - \gamma_1(x) \equiv 0$ or $\gamma_2(x) - \gamma_1(x) > 0$ in G.

We claim that $\gamma_2(x) - \gamma_1(x) > 0$. Indeed, suppose for a contradiction that $\gamma_2(x) - \gamma_1(x) \equiv 0$; then by (11), $\beta_2(x) - \beta_1(x) \equiv 0$ in G, contrary to our assumption that $\beta_1(x) < \beta_2(x)$.

(ii) Putting $\gamma = \mathcal{P}\beta$ and using (6) and (4) we get

$$\begin{aligned} (\mathcal{A} - k\mathcal{I})(\gamma - \beta) &= (\mathcal{A} - k\mathcal{I})\gamma - (\mathcal{A} - k\mathcal{I})\beta \\ &= -[f(x, \beta(x), \beta) + k\beta(x)] - \mathcal{A}\beta + k\beta(x) \\ &= -[\mathcal{A}\beta + f(x, \beta(x), \beta)] \ge 0 \quad \text{in } G \end{aligned}$$

and

$$\gamma(x) - \beta(x) = h(x) - \beta(x) \le 0 \quad \text{on } \partial G\,.$$

Hence, by the strong maximum principle, either $\gamma(x) - \beta(x) \equiv 0$ or $\gamma(x) - \beta(x) > 0$ in G. Since β is not a solution of (1) (when β is a solution of (1) then Theorem 1 holds), the case $\gamma(x) - \beta(x) \equiv 0$ cannot occur. Hence $\gamma(x) < \beta(x)$ in G and the proof of Lemma 3 is complete.

Proof of Theorem 1. Let $\mathcal{P}$ be defined as before. By induction, we define two sequences of functions $\{u_n\}$ and $\{v_n\}$ by setting

$$u_1 = \mathcal{P}u_0, \quad u_n = \mathcal{P}u_{n-1}, \quad n = 1, 2, \ldots,$$
$$v_1 = \mathcal{P}v_0, \quad v_n = \mathcal{P}v_{n-1}, \quad n = 1, 2, \ldots$$

Now we show that $\{u_n\}$ is increasing (resp. $\{v_n\}$ is decreasing) and converges to a solution of problem (1), (2) in $\overline{G}$. Since u_0 and v_0 are regular, by Lemma 1 we see that $u_n, v_n \in C^{2+\alpha}(\overline{G})$. Since v_0 is an upper function for problem (1), (2) in G, by Lemma 3, we obtain

$$v_1(x) = \mathcal{P}v_0(x) < v_0(x) \quad \text{ in } G\,.$$

Consequently, by monotonicity of $\mathcal{P}$ we get

$$v_n(x) = \mathcal{P}v_{n-1}(x) < v_{n-1}(x) \quad \text{ in } G,\ n = 1, 2, \ldots$$

Arguing as above we get $u_{n-1}(x) < u_n(x)$ in $\overline{G}, n = 1, 2, \ldots$ Since the operator $\mathcal{P}$ is monotone, by Assumption A it follows that

$$u_1(x) = \mathcal{P}u_0(x) \le \mathcal{P}v_0(x) = v_1(x) \quad \text{in } \overline{G}$$

and consequently $u_n(x) \le v_n(x)$ in $\overline{G}$, $n = 1, 2, \ldots$ Therefore we get

$$u_0(x) < u_1(x) < \ldots < u_n(x) < \ldots < v_n(x) < \ldots < v_1(x) < v_0(x) \text{ in } \overline{G}. \tag{13}$$

By virtue of (13) we can set

$$\overline{v}(x) = \lim_{n\to\infty} v_n(x) \quad \text{for each } x \in \overline{G} \tag{14}$$

and we see that $u_0(x) \le \overline{v}(x) \le v_0(x)$ for $x \in \overline{G}$. Analogously we can define

$$\underline{u}(x) = \lim_{n\to\infty} u_n(x) \quad \text{for each } x \in \overline{G}\,, \tag{15}$$

which satisfies $u_0(x) \le \underline{u}(x) \le v_0(x)$ for $x \in \overline{G}$.

To complete the proof we must show that $\underline{u}$ and $\overline{v}$ are regular solutions of problem (1), (2) in $\overline{G}$.

If we could prove that the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $C^{0+\alpha}(\overline{G})$, then since the operator $\mathcal{P}$ is compact and monotone, the sequences $\{\mathcal{P}u_n\}$ and $\{\mathcal{P}v_n\}$ would be convergent in $C^{0+\alpha}(\overline{G})$.

Since it is not possible to prove that for any elliptic operator $\mathcal{A}$ the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $C^{0+\alpha}(\overline{G})$, we must find another way.

The inequality (13) implies that $\{u_n\}$ and $\{v_n\}$ are bounded in $L^p(G)$. Since $\mathcal{P}$ is increasing and compact in $L^p(G)$ (see Lemma 2), the sequences $\{\mathcal{P}u_n\}$ and $\{\mathcal{P}v_n\}$ are converging in $L^p(G)$. It is easy to see that

$$\underline{u} = \lim_{n\to\infty} \mathcal{P}u_n = \lim_{n\to\infty} \mathcal{P}^2 u_{n-1} = \mathcal{P}\underline{u} \in L^p(G)$$

and

$$\overline{v} = \lim_{n\to\infty} \mathcal{P}v_n = \lim_{n\to\infty} \mathcal{P}^2 v_{n-1} = \mathcal{P}\overline{v} \in L^p(G).$$

Since $\underline{u}, \overline{v} \in L^p(G)$ and

$$\mathcal{G} \circ \mathcal{F}\underline{u} = \underline{u}\,, \tag{16}$$

$$\mathcal{G} \circ \mathcal{F}\overline{v} = \overline{v}\,, \tag{17}$$

by the Agmon–Douglis–Nirenberg theorem we obtain

$$\underline{u}, \overline{v} \in H^{2,p}(G)\,. \tag{18}$$

Now using the well known fact that for $p > m$ the Sobolev space $H^{2,p}(G)$ is continuously imbedded in $C^{0+\alpha}(\overline{G}), 0 < \alpha < 1$ (see [9]), and by (18) we get

$$\underline{u}, \overline{v} \in C^{0+\alpha}(\overline{G})\,. \tag{19}$$

Applying now the Schauder theorem to the equalities (16), (17) and by (19) we get

$$\underline{u}, \overline{v} \in C^{2+\alpha}(\overline{G})\,.$$

Hence $\underline{u}$ and $\overline{v}$ are regular solutions of problem (1), (2) in $\overline{G}$. Moreover, since the sequences $\{u_n\}, \{v_n\}$ are monotone, by (13)–(15) we see that

$$u_0(x) \le \underline{u}(x) \le \overline{v}(x) \le v_0(x) \quad \text{for } x \in \overline{G}\,. \tag{20}$$

In general $\underline{u}(x) \ne \overline{v}(x)$.

R e m a r k 1. The solutions $\underline{u}$ and $\overline{v}$ are minimal and maximal solutions of problem (1), (2) in the set K, i.e., if w is any solution of problem (1), (2) such that $u_0(x) \le w(x) \le v_0(x)$, then $\underline{u}(x) \le w(x) \le \overline{v}(x)$ in $\overline{G}$.

Indeed, if w is such a solution, then $w = \mathcal{P}w$. Hence, by monotonicity of $\mathcal{P}$ we have

$$w(x) = \mathcal{P}w(x) \le \mathcal{P}v_0(x) = v_1(x) \quad \text{in } \overline{G}\,.$$

By induction we get $w(x) \le v_n(x)$ in $\overline{G}$, so $w(x) \le \lim_{n\to\infty} v_n(x) = \overline{v}(x)$ in $\overline{G}$.

Arguing as above we obtain $\underline{u}(x) = \lim_{n\to\infty} u_n(x) \le w(x)$ in $\overline{G}$.

R e m a r k 2. Uniqueness of solution for a system of differential-functional equations of elliptic type has been studied by M. Malec [10]. He gave some criterion for uniqueness under stronger assumptions.

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