

Qualitative behavior of a class of second order nonlinear differential equations on the halfline

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Abstract. A differential equation of the form $(q(t)k(u)u')' = F(t, u)u'$ is considered and solutions u with $u(0) = 0$ are studied on the halfline $[0, \infty)$. Theorems about the existence, uniqueness, boundedness and dependence of solutions on a parameter are given.

1. Introduction. In [6] the differential equation

$$(1) \quad (k(u)u')' = f(t)u',$$

where $k, f \in C^0(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, was considered and the author gave sufficient conditions for the existence and uniqueness of nonnegative solutions u such that $u(0) = 0$, $u(t) > 0$ for $t > 0$. This problem is connected with the description of the mathematical model of the infiltration of water. For more details see e.g. [3]–[5]. Special cases of (1) were considered in [1], [2], [4] and [5]. In [7] the differential equation

$$(2) \quad (q(t)k(u)u')' = f(t)h(u)u',$$

where $q, k, f, h \in C^0(\mathbb{R}_+)$, was considered and sufficient conditions for the existence, uniqueness and boundedness of nonnegative solutions of (2) on \mathbb{R}_+ were given. Simultaneously the dependence of solutions of the differential equation

$$(3) \quad (q(t)k(u)u')' = \lambda f(t)h(u)u', \quad \lambda > 0,$$

on the positive parameter λ was studied.

In this paper we consider the differential equation

$$(4) \quad (q(t)k(u)u')' = F(t, u)u',$$

1991 *Mathematics Subject Classification*: 34A34, 34C11, 34B15, 45G10.

Key words and phrases: nonlinear differential equation, nonnegative solution, non-positive solution, existence and uniqueness of solutions, bounded solution, dependence of solutions on a parameter, boundary value problem.

where $q \in C^0(\mathbb{R}_+)$, $k \in C^0(\mathbb{R})$, $F \in C^0(\mathbb{R}_+ \times \mathbb{R})$, which is a generalization of (2), and study solutions u of this equation on \mathbb{R}_+ such that $u(0) = 0$. We give assumptions under which any nontrivial solution of (4) is either strictly increasing or strictly decreasing on \mathbb{R}_+ and discuss the existence, uniqueness and boundedness of solutions. In the last section we consider the dependence of solutions of the differential equation

$$(4_\lambda) \quad (q(t)k(u)u')' = \lambda F(t, u)u', \quad \lambda > 0,$$

on the positive parameter λ and study the boundary value problem (4_λ) , $\lim_{t \rightarrow \infty} u(t, \lambda) = a$ ($a \in \mathbb{R} - \{0\}$). Our theorems imply all results of [6] and [7]. The proofs of existence theorems are based on the Tikhonov–Schauder fixed point theorem, on the iterative method and on the monotone behavior of some operators. The proofs of uniqueness are different from the ones in [6] and [7].

2. Notations, lemmas. Consider the differential equation (4), where q, k, F satisfy the following assumptions ($\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$):

$$(H_1) \quad q \in C^0(\mathbb{R}_+), \quad q(t) > 0 \text{ for } t \in (0, \infty), \quad \int_0^\infty \frac{ds}{q(s)} < \infty;$$

$$(H_2) \quad k \in C^0(\mathbb{R}), \quad k(u)u > 0 \text{ for } u \in \mathbb{R}, u \neq 0;$$

$$(H_3) \quad \int_0^\infty \frac{k(s)}{s} ds < \infty, \quad \int_0^0 \frac{k(s)}{s} ds < \infty,$$

$$\int_{-\infty}^\infty \frac{k(s)}{s} ds = \infty, \quad \int_{-\infty}^\infty \frac{k(s)}{s} ds = \infty;$$

$$(H_4) \quad F \in C^0(\mathbb{R}_+ \times \mathbb{R}), \quad f_1(t)h_1(u) \leq F(t, u) \leq f_2(t)h_2(u)$$

for $(t, u) \in \mathbb{R}_+ \times \mathbb{R}$, where

$f_i \in C^0(\mathbb{R}_+)$, $0 < f_i(t)$ for $t \in \mathbb{R}_+$, f_i is decreasing on \mathbb{R}_+ ,

$h_i \in C^0(\mathbb{R})$, $h_i(u)u \geq 0$ for $u \in \mathbb{R}$ and

$H_i(u) := \int_0^u h_i(x) dx$ is strictly increasing on \mathbb{R}_+ ,

$T_i(u) := \int_u^0 h_i(x) dx$ is strictly increasing on \mathbb{R}_- ($i = 1, 2$);

$$(H_5) \quad \int_0^\infty \frac{k(x)}{H_i(x)} dx < \infty, \quad \int_0^0 \frac{k(x)}{T_i(x)} dx < \infty,$$

$$\int_{-\infty}^\infty \frac{k(x)}{H_i(x)} dx = \infty, \quad \int_{-\infty}^\infty \frac{k(x)}{T_i(x)} dx = \infty \quad (i = 1, 2).$$

By a *solution* of (4) we mean a function $u \in C^0(\mathbb{R}_+)$ which is continuously differentiable on $(0, \infty)$ and such that $u(0)=0$, $\lim_{t \rightarrow 0^+} q(t)k(u(t))u'(t) = 0$, $q(t)k(u(t))u'(t)$ is continuously differentiable on $(0, \infty)$ and (4) is satisfied on $(0, \infty)$.

Remark 1. It follows from (H_2) and (H_4) that $k(0) = 0$, $h_i(0) = 0$ ($i = 1, 2$), and $F(t, 0) = 0$ for $t \in \mathbb{R}_+$. Therefore $u = 0$ is a solution of (4).

LEMMA 1. Let $u(t)$ be a nontrivial solution of (4). Then either $u'(t) > 0$ on $(0, \infty)$ or $u'(t) < 0$ on $(0, \infty)$.

Proof. Let u be a solution of (4), $u \neq 0$. Then

$$(5) \quad q(t)k(u(t))u'(t) = \int_0^t F(s, u(s))u'(s) ds \quad \text{for } t \in (0, \infty).$$

Assume there exist $0 < t_1 < t_2$ such that $u'(t_1) = u'(t_2) = 0$, $u'(t) \neq 0$ for $t \in (t_1, t_2)$. Suppose $u'(t) > 0$ on (t_1, t_2) . If $u(t) > 0$ (resp. < 0) on (t_1, t_2) , then

$$\begin{aligned} \int_{t_1}^{t_2} f_1(s)h_1(u(s))u'(s) ds &\leq \int_{t_1}^{t_2} F(s, u(s))u'(s) ds \quad (= 0) \\ &\leq \int_{t_1}^{t_2} f_2(s)h_2(u(s))u'(s) ds, \end{aligned}$$

which contradicts

$$\begin{aligned} \int_{t_1}^{t_2} f_1(s)h_1(u(s))u'(s) ds &\geq f_1(t_2) \int_{u(t_1)}^{u(t_2)} h_1(s) ds > 0 \\ \left(\text{resp. } \int_{t_1}^{t_2} f_2(s)h_2(u(s))u'(s) ds &\leq f_2(t_2) \int_{u(t_1)}^{u(t_2)} h_2(s) ds < 0 \right). \end{aligned}$$

Therefore $u(\xi) = 0$ for a $\xi \in (t_1, t_2)$ and then $u(t) < 0$ on $[t_1, \xi)$ and

$$\int_{t_1}^{\xi} F(s, u(s))u'(s) ds \leq \int_{t_1}^{\xi} f_2(s)h_2(u(s))u'(s) ds \leq f_2(\xi) \int_{u(t_1)}^0 h_2(s) ds < 0,$$

which contradicts $0 = \int_{t_1}^{\xi} F(s, u(s))u'(s) ds$. We can easily check that $u'(t) < 0$ on (t_1, t_2) is also impossible.

Assume $u'(\tau) = 0$ for a $\tau \in (0, \infty)$ and $u'(t) \neq 0$ for $t \in (0, \tau)$. If $u'(t) > 0$ (resp. $u'(t) < 0$) on $(0, \tau)$, then $u(t) > 0$ (resp. $u(t) < 0$) on this interval and

$$\int_0^{\tau} F(s, u(s))u'(s) ds \geq \int_0^{\tau} f_1(s)h_1(u(s))u'(s) ds \geq f_1(\tau) \int_0^{u(\tau)} h_1(s) ds > 0$$

$$\left(\text{resp. } \int_0^\tau F(s, u(s))u'(s) ds \geq \int_0^\tau f_2(s)h_2(u(s))u'(s)ds \right. \\ \left. \geq f_2(\tau) \int_0^{u(\tau)} h_2(s)ds > 0 \right),$$

which contradicts $0 = \int_0^\tau F(s, u(s))u'(s) ds$.

This proves $u'(t) \neq 0$ on $(0, \infty)$ and since $u \in C^1((0, \infty))$ we have either $u'(t) > 0$ on $(0, \infty)$ or $u'(t) < 0$ on $(0, \infty)$. ■

Remark 2. It follows from Lemma 1 that $u \in \mathcal{A}_+ \cup \mathcal{A}_-$ for any nontrivial solution u of (4), where

$$\mathcal{A}_+ = \{u \in C^0(\mathbb{R}_+) : u(0) = 0, u \text{ is strictly increasing on } \mathbb{R}_+\}, \\ \mathcal{A}_- = \{u \in C^0(\mathbb{R}_+) : u(0) = 0, u \text{ is strictly decreasing on } \mathbb{R}_+\}.$$

Set

$$W_+(u) = \int_0^u k(s) ds \quad \text{for } u \in \mathbb{R}_+, \quad W_-(u) = \int_0^u k(s) ds \quad \text{for } u \in \mathbb{R}_-.$$

Obviously, W_+ is strictly increasing on \mathbb{R}_+ , and W_- is strictly decreasing on \mathbb{R}_- .

LEMMA 2. *If u is a solution of (4), $u \neq 0$, then u is a solution of the integral equation*

$$(6_+) \quad u(t) = W_+^{-1} \left(\int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \right)$$

or the integral equation

$$(6_-) \quad u(t) = W_-^{-1} \left(- \int_0^t \frac{1}{q(s)} \int_{u(s)}^0 F(u^{-1}(x), x) dx ds \right)$$

in the set \mathcal{A}_+ or \mathcal{A}_- , respectively. Conversely, if $u \in \mathcal{A}_\varepsilon$, $\varepsilon \in \{+, -\}$, is a solution of (6_ε) , then u is a solution of (4) and $u \neq 0$. Here W_ε^{-1} and u^{-1} denote the inverse functions to W_ε and u , respectively.

Proof. Let $u \neq 0$ be a solution of (4). Then $u \in \mathcal{A}_+ \cup \mathcal{A}_-$ by Remark 2 and (5) holds. If $u \in \mathcal{A}_+$ (resp. $u \in \mathcal{A}_-$), then

$$W_+(u(t)) = \int_0^t \frac{1}{q(s)} \int_0^s F(x, u(x))u'(x) dx ds \\ = \int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds$$

$$\left(\text{resp. } W_-(u(t)) = \int_0^t \frac{1}{q(s)} \int_0^s F(x, u(x)) u'(x) dx ds \right. \\ \left. = - \int_0^t \frac{1}{q(s)} \int_{u(s)}^0 F(u^{-1}(x), x) dx ds \right)$$

for $t \in \mathbb{R}_+$ and u is a solution of (6_+) or (6_-) in \mathcal{A}_+ or \mathcal{A}_- , respectively.

Conversely, let $u \in \mathcal{A}_\varepsilon$, $\varepsilon \in \{+, -\}$, be a solution of (6_ε) . Then $u(0) = 0$, $u \in C^0(\mathbb{R}_+)$, u is continuously differentiable on $(0, \infty)$,

$$(7) \quad \lim_{t \rightarrow 0_+} q(t)k(u(t))u'(t) = \lim_{t \rightarrow 0_+} \int_0^{u(t)} F(u^{-1}(s), s) ds = 0$$

and $(q(t)k(u(t))u'(t))' = F(t, u(t))u'(t)$ for $t > 0$. Consequently, u is a solution of (4) and $u \neq 0$. ■

Remark 3. It follows from Lemma 2 that solving (4) is equivalent to solving (6_+) and (6_-) in \mathcal{A}_+ and \mathcal{A}_- , respectively.

Set

$$K_j(u) = \int_0^u \frac{k(s)}{H_j(s)} ds \quad \text{for } u \in \mathbb{R}_+, \\ V_j(u) = \int_u^0 \frac{k(s)}{T_j(s)} ds \quad \text{for } u \in \mathbb{R}_-, \quad j = 1, 2,$$

and

$$k_1(t) = \int_0^t \frac{f_1(s)}{q(s)} ds, \quad k_2(t) = f_2(0) \int_0^t \frac{ds}{q(s)}, \\ l_1(t) = f_1(0) \int_0^t \frac{ds}{q(s)}, \quad l_2(t) = \int_0^t \frac{f_2(s)}{q(s)} ds, \\ \underline{\varphi}_+(t) = K_1^{-1}(k_1(t)), \quad \bar{\varphi}_+(t) = K_2^{-1}(k_2(t)), \\ \underline{\varphi}_-(t) = V_1^{-1}(l_1(t)), \quad \bar{\varphi}_-(t) = V_2^{-1}(l_2(t)) \quad \text{for } t \in \mathbb{R}_+.$$

Obviously, $\lim_{u \rightarrow \infty} K_j(u) = \infty$, $\lim_{u \rightarrow -\infty} V_j(u) = \infty$ ($j = 1, 2$) by (H_5) .

LEMMA 3. If $u \in \mathcal{A}_\varepsilon$ is a solution of (6_ε) , $\varepsilon \in \{+, -\}$, then

$$(8_\varepsilon) \quad \underline{\varphi}_\varepsilon(t) \leq u(t) \leq \bar{\varphi}_\varepsilon(t) \quad \text{for } t \in \mathbb{R}_+$$

and

$$(9_+) \quad u(t_2) - u(t_1) \geq f_1(t_2)H_1(\underline{\varphi}_+(t_1)) \int_{t_1}^{t_2} \frac{dt}{q(t)} \\ \times [\max\{k(x) : \underline{\varphi}_+(t_1) \leq x \leq \bar{\varphi}_+(t_2)\}]^{-1} \quad \text{for } u \in \mathcal{A}_+ \text{ and } 0 < t_1 < t_2,$$

$$(9_-) \quad u(t_1) - u(t_2) \geq -f_2(t_2)(T_2(\bar{\varphi}_-(t_1))) \int_{t_1}^{t_2} \frac{dt}{q(t)} \\ \times [\max\{-k(x) : \underline{\varphi}_-(t_2) \leq x \leq \bar{\varphi}_-(t_1)\}]^{-1} \quad \text{for } u \in \mathcal{A}_- \text{ and } 0 < t_1 < t_2.$$

Proof. Let $u \in \mathcal{A}_+$ be a solution of (6₊). Then (cf. (H₄))

$$f_1(t) \int_0^{u(t)} h_1(s) ds \leq q(t)k(u(t))u'(t) \quad \left(= \int_0^{u(t)} F(u^{-1}(s), s) ds \right) \\ \leq f_2(0) \int_0^{u(t)} h_2(s) ds,$$

hence

$$(10) \quad \frac{f_1(t)}{q(t)} \leq \left(\int_0^{u(t)} \frac{k(s)}{H_1(s)} ds \right)', \\ \frac{f_2(0)}{q(t)} \geq \left(\int_0^{u(t)} \frac{k(s)}{H_2(s)} ds \right)' \quad \text{for } t \in (0, \infty),$$

and integrating (10) from 0 to t , we obtain

$$\int_0^t \frac{f_1(s)}{q(s)} ds \leq \int_0^{u(t)} \frac{k(s)}{H_1(s)} ds = K_1(u(t)), \\ f_2(0) \int_0^t \frac{ds}{q(s)} \geq \int_0^{u(t)} \frac{k(s)}{H_2(s)} ds = K_2(u(t)).$$

Consequently,

$$\underline{\varphi}_+(t) \leq u(t) \leq \bar{\varphi}_+(t) \quad \text{for } t \in \mathbb{R}_+.$$

Let $0 < t_1 < t_2$. Then

$$W_+(u(t_2)) - W_+(u(t_1)) = \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \\ \geq \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} f_1(u^{-1}(x))h_1(x) dx ds$$

$$\begin{aligned} &\geq f_1(t_2) \int_{t_1}^{t_2} \frac{ds}{q(s)} \int_0^{\varphi_+(t_1)} h_1(s) ds \\ &= f_1(t_2) H_1(\varphi_+(t_1)) \int_{t_1}^{t_2} \frac{ds}{q(s)}, \end{aligned}$$

and since

$$u(t_2) - u(t_1) = \frac{1}{k(\xi)} (W_+(u(t_2)) - W_+(u(t_1))),$$

where $\xi \in (u(t_1), u(t_2)) \subset (\varphi_+(t_1), \bar{\varphi}_+(t_2))$, we see that (9₊) is true.

Let $u \in \mathcal{A}_-$ be a solution of (6₋). Then (cf. (H₄))

$$\begin{aligned} -f_2(t) \int_{u(t)}^0 h_2(s) ds &\leq q(t)k(u(t))u'(t) \quad \left(= \int_0^{u(t)} F(u^{-1}(s), s) ds \right) \\ &\leq -f_1(0) \int_{u(t)}^0 h_1(s) ds, \end{aligned}$$

hence

$$\begin{aligned} \frac{f_2(t)}{q(t)} &\leq \left(\int_{u(t)}^0 \frac{k(s)}{T_2(s)} ds \right)', \\ \frac{f_1(0)}{q(t)} &\geq \left(\int_{u(t)}^0 \frac{k(s)}{T_1(s)} ds \right)' \quad \text{for } t \in (0, \infty), \end{aligned}$$

and integrating the last inequalities from 0 to t , we obtain

$$\begin{aligned} \int_0^t \frac{f_2(s)}{q(s)} ds &\leq \int_{u(t)}^0 \frac{k(s)}{T_2(s)} ds = V_2(u(t)), \\ f_1(0) \int_0^t \frac{ds}{q(s)} &\geq \int_{u(t)}^0 \frac{k(s)}{T_1(s)} ds = V_1(u(t)). \end{aligned}$$

Consequently,

$$\varphi_-(t) \leq u(t) \leq \bar{\varphi}_-(t) \quad \text{for } t \in \mathbb{R}_+.$$

Let $0 < t_1 < t_2$. Then

$$\begin{aligned} W_-(u(t_2)) - W_-(u(t_1)) &= - \int_{t_1}^{t_2} \frac{1}{q(s)} \int_{u(s)}^0 F(u^{-1}(x), x) dx ds \\ &\geq - \int_{t_1}^{t_2} \frac{1}{q(s)} \int_{u(s)}^0 f_2(u^{-1}(x)) h_2(x) dx ds \end{aligned}$$

$$\begin{aligned}
&\geq -f_2(t_2) \int_{t_1}^{t_2} \frac{ds}{q(s)} \int_{\bar{\varphi}_-(t_1)}^0 h_2(s) ds \\
&= -f_2(t_2) T_2(\bar{\varphi}_-(t_1)) \int_{t_1}^{t_2} \frac{ds}{q(s)},
\end{aligned}$$

and since

$$u(t_1) - u(t_2) = -(1/k(\xi))(W_-(u(t_2)) - W_-(u(t_1))),$$

where $\xi \in (u(t_2), u(t_1)) \subset (\underline{\varphi}_-(t_2), \bar{\varphi}_-(t_1))$, we see that (9_-) is true. ■

Set $\mathcal{K}_\varepsilon = \{u \in \mathcal{A}_\varepsilon : \varphi_\varepsilon(t) \leq u(t) \leq \bar{\varphi}_\varepsilon(t) \text{ for } t \in \mathbb{R}_+, u \text{ satisfies } (9_\varepsilon)\}$, and define $T_\varepsilon : \mathcal{K}_\varepsilon \rightarrow C^0(\mathbb{R}_+)$ by

$$(T_\varepsilon u)(t) = W_\varepsilon^{-1} \left(\int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \right) \quad \varepsilon \in \{+, -\}.$$

LEMMA 4. $T_\varepsilon : \mathcal{K}_\varepsilon \rightarrow \mathcal{K}_\varepsilon$ for each $\varepsilon \in \{+, -\}$.

Proof. We will prove $T_+ : \mathcal{K}_+ \rightarrow \mathcal{K}_+$ (the proof of $T_- : \mathcal{K}_- \rightarrow \mathcal{K}_-$ is very similar and will be omitted). Let $u \in \mathcal{K}_+$. Setting

$$\begin{aligned}
\alpha(t) &= \int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds - W_+(\underline{\varphi}_+(t)), \\
\beta(t) &= \int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds - W_+(\bar{\varphi}_+(t))
\end{aligned}$$

for $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
\alpha'(t) &= \frac{1}{q(t)} \int_0^{u(t)} F(u^{-1}(x), x) dx - k(\underline{\varphi}_+(t)) \underline{\varphi}'_+(t) \\
&= \frac{1}{q(t)} \left[\int_0^{u(t)} F(u^{-1}(x), x) dx - f_1(t) H_1(\underline{\varphi}_+(t)) \right] \\
&\geq \frac{1}{q(t)} \left[\int_0^{u(t)} f_1(u^{-1}(x)) h_1(x) dx - f_1(t) \int_0^{\underline{\varphi}_+(t)} h_1(x) dx \right] \\
&\geq \frac{1}{q(t)} \int_0^{u(t)} (f_1(u^{-1}(x)) - f_1(t)) h_1(x) dx \geq 0,
\end{aligned}$$

$$\begin{aligned}
 \beta'(t) &= \frac{1}{q(t)} \int_0^{u(t)} F(u^{-1}(x), x) dx - k(\bar{\varphi}_+(t))\bar{\varphi}'_+(t) \\
 &= \frac{1}{q(t)} \left[\int_0^{u(t)} F(u^{-1}(x), x) dx - f_2(0)H_2(\bar{\varphi}_+(t)) \right] \\
 &\leq \frac{1}{q(t)} \left[\int_0^{u(t)} f_2(u^{-1}(x))h_2(x)dx - f_2(0) \int_0^{\bar{\varphi}_+(t)} h_2(x) dx \right] \\
 &\leq \frac{1}{q(t)} \int_0^{u(t)} (f_2(u^{-1}(x)) - f_2(0))h_2(x) dx \leq 0
 \end{aligned}$$

for $t \in (0, \infty)$. Since $\alpha(0) = \beta(0) = 0$ and $\alpha'(t) \geq 0, \beta'(t) \leq 0$ on $(0, \infty)$, we have $\alpha(t) \geq 0, \beta(t) \leq 0$ and consequently,

$$(11) \quad \underline{\varphi}_+(t) \leq (T_+u)(t) \leq \bar{\varphi}_+(t) \quad \text{for } t \in \mathbb{R}_+ \text{ and } u \in \mathcal{K}_+.$$

Let $0 < t_1 < t_2$ and let $u \in \mathcal{K}_+$. Then

$$\begin{aligned}
 W_+((T_+u)(t_2)) - W_+((T_+u)(t_1)) &= \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \\
 &\geq \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} f_1(u^{-1}(x))h_1(x) dx ds \\
 &\geq \int_{t_1}^{t_2} \frac{f_1(s)}{q(s)} H_1(u(s)) ds \\
 &\geq f_1(t_2)H_1(\underline{\varphi}_+(t_1)) \int_{t_1}^{t_2} \frac{ds}{q(s)}
 \end{aligned}$$

and

$$\begin{aligned}
 W_+((T_+u)(t_2)) - W_+((T_+u)(t_1)) &= k(\xi)[(T_+u)(t_2) - (T_+u)(t_1)] \\
 &\leq [(T_+u)(t_2) - (T_+u)(t_1)] \max\{k(u) : \underline{\varphi}_+(t_1) \leq u \leq \bar{\varphi}_+(t_2)\}
 \end{aligned}$$

(here $\xi \in ((T_+u)(t_1), (T_+u)(t_2)) \subset (\underline{\varphi}_+(t_1), \bar{\varphi}_+(t_2))$), hence

$$\begin{aligned}
 (12) \quad (T_+u)(t_2) - (T_+u)(t_1) \\
 \geq f_1(t_2)H_1(\underline{\varphi}_+(t_1)) \int_{t_1}^{t_2} \frac{ds}{q(s)} [\max\{k(u) : \underline{\varphi}_+(t_1) \leq u \leq \bar{\varphi}_+(t_2)\}]^{-1}.
 \end{aligned}$$

From (11) and (12) it follows that $T_+u \in \mathcal{K}_+$, therefore $T_+ : \mathcal{K}_+ \rightarrow \mathcal{K}_+$. ■

3. Existence theorems

THEOREM 1. *Let assumptions (H₁)–(H₅) be satisfied. Then a solution $u \in \mathcal{A}_\varepsilon$ of (4) exists for each $\varepsilon \in \{+, -\}$.*

PROOF. By Lemma 2, $u \in \mathcal{A}_\varepsilon$ is a solution of (4) if and only if u is a fixed point of the operator T_ε . We will prove that under assumptions (H₁)–(H₅) a fixed point of T_+ exists. The existence of a fixed point of T_- can be proved similarly.

Let \mathbf{X} be the Fréchet space of C^0 -functions on \mathbb{R}_+ with the topology of uniform convergence on compact subintervals of \mathbb{R}_+ . Then \mathcal{K}_+ is a bounded closed convex subset of \mathbf{X} and $T_+ : \mathcal{K}_+ \rightarrow \mathcal{K}_+$ (see Lemma 4) is a continuous operator. It follows from the inequalities ($0 \leq t_1 < t_2$, $u \in \mathcal{K}_+$)

$$\begin{aligned} 0 &\leq W_+((T_+u)(t_2)) - W_+((T_+u)(t_1)) = \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \\ &\leq \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^{u(s)} f_2(u^{-1}(x))h_2(x) dx ds \leq f_2(0)H_2(\bar{\varphi}_+(t_2)) \int_{t_1}^{t_2} \frac{ds}{q(s)} \end{aligned}$$

and from the Arzelà–Ascoli theorem that $T_+(\mathcal{K}_+)$ is a relatively compact subset of \mathbf{X} . According to the Tikhonov–Schauder fixed point theorem there exists a fixed point u_+ of T_+ . ■

THEOREM 2. *Let assumptions (H₁)–(H₅) be satisfied and suppose that*

(H₆) $F(\cdot, u)\text{sign}(u)$ *is decreasing on \mathbb{R}_+ for each fixed $u \in \mathbb{R}$.*

Then there exist solutions $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon$ of (4) for each $\varepsilon \in \{+, -\}$ such that $u_\varepsilon(t) \leq v_\varepsilon(t)$ for $t \in \mathbb{R}_+$. Moreover,

$$(13) \quad u_\varepsilon(t) \leq u(t) \leq v_\varepsilon(t), \quad t \in \mathbb{R}_+, \quad \varepsilon \in \{+, -\},$$

for any solution $u \in \mathcal{A}_\varepsilon$ of (4) and

$$(14) \quad u(t) \neq v(t), \quad t > 0,$$

for any two different solutions u, v of (4).

PROOF. Let assumptions (H₁)–(H₆) be satisfied and let $u \in \mathcal{A}_+$ be a solution of (4). Define sequences $\{u_n\} \subset \mathcal{A}_+$, $\{v_n\} \subset \mathcal{A}_+$ by the recurrence formulas

$$\begin{aligned} u^{[0]} &= \underline{\varphi}_+, & u^{[n+1]} &= T_+(u^{[n]}), \\ v^{[0]} &= \bar{\varphi}_+, & v^{[n+1]} &= T_+(v^{[n]}), \end{aligned}$$

for $n \in \mathbb{N}$. Then $u^{[0]}(t) \leq u(t) \leq v^{[0]}(t)$ on \mathbb{R}_+ by Lemma 3 and $u^{[0]}(t) \leq u^{[1]}(t) \leq v^{[0]}(t)$, $u^{[0]}(t) \leq v^{[1]}(t) \leq v^{[0]}(t)$ on \mathbb{R}_+ by Lemma 4. Since $\alpha, \beta \in$

\mathcal{A}_+ , $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{R}_+$ implies

$$\begin{aligned} (\mathbb{T}_+\alpha)(t) &= W_+^{-1} \left(\int_0^t \frac{1}{q(s)} \int_0^{\alpha(s)} F(\alpha^{-1}(x), x) dx ds \right) \\ &\leq W_+^{-1} \left(\int_0^t \frac{1}{q(s)} \int_0^{\beta(s)} F(\beta^{-1}(x), x) dx ds \right) = (\mathbb{T}_+\beta)(t) \end{aligned}$$

on \mathbb{R}_+ and $\mathbb{T}_+ : \mathcal{K}_+ \rightarrow \mathcal{K}_+$ by Lemma 4, we deduce

$$\begin{aligned} u^{[0]}(t) &\leq u^{[1]}(t) \leq \dots \leq u^{[n]}(t) \\ &\leq \dots \leq u(t) \leq \dots \leq v^{[n]}(t) \leq \dots \leq v^{[1]}(t) \leq v^{[0]}(t) \end{aligned}$$

for $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Therefore the limits $\lim_{n \rightarrow \infty} u^{[n]}(t) =: u_+(t)$, $\lim_{n \rightarrow \infty} v^{[n]}(t) =: v_+(t)$ exist for all $t \geq 0$, $u_+(t) \leq u(t) \leq v_+(t)$ on \mathbb{R}_+ and using the Lebesgue dominated convergence theorem we see that $u_+, v_+ \in \mathcal{K}_+$ are solutions of (6₊) and thus also solutions of (4) by Lemma 2.

Let $u, v \in \mathcal{A}_+$ be different solutions of (4). First, suppose there exists a $t_1 > 0$ such that $u(t) < v(t)$ for $t \in (0, t_1)$ and $u(t_1) = v(t_1)$. Then

$$\begin{aligned} 0 &= W_+(v(t_1)) - W_+(u(t_1)) \\ &= \int_0^{t_1} \frac{1}{q(s)} \left[\int_0^{v(s)} F(v^{-1}(x), x) dx - \int_0^{u(s)} F(u^{-1}(x), x) dx \right] ds \\ &\geq \int_0^{t_1} \frac{1}{q(s)} \left[\int_0^{u(s)} (F(v^{-1}(x), x) - F(u^{-1}(x), x)) dx \right] ds \end{aligned}$$

and since (cf. (H₆)) $F(v^{-1}(x), x) - F(u^{-1}(x), x) \geq 0$ for $x \in [0, u(t_1)]$ we have necessarily $F(v^{-1}(x), x) - F(u^{-1}(x), x) = 0$ for $x \in [0, u(t_1)]$. Then

$$W_+(v(t_1)) - W_+(u(t_1)) = \int_0^{t_1} \frac{1}{q(s)} \int_{u(s)}^{v(s)} F(v^{-1}(x), x) dx ds > 0,$$

which contradicts $W_+(v(t_1)) - W_+(u(t_1)) = 0$.

Secondly, assume there exist $0 < t_1 < t_2$ such that $u(t_1) = v(t_1)$, $u(t_2) = v(t_2)$, $u(t) \neq v(t)$ for $t \in (t_1, t_2)$, for example assume

$$u(t) > v(t) \quad \text{for } t \in (t_1, t_2).$$

Then $u'(t_1) \geq v'(t_1)$, $u'(t_2) \leq v'(t_2)$ and

$$\begin{aligned} (15) \quad 0 &\leq q(t_1)k(u(t_1))(u'(t_1) - v'(t_1)) - q(t_2)k(u(t_2))(u'(t_2) - v'(t_2)) \\ &= \int_{u(t_2)}^{u(t_1)} [F(u^{-1}(x), x) - F(v^{-1}(x), x)] dx. \end{aligned}$$

On the other hand, since $u(t_2) > u(t_1)$ and (cf. (H₆)) $F(u^{-1}(x), x) - F(v^{-1}(x), x) \geq 0$ on $[u(t_1), u(t_2)]$ we have

$$\int_{u(t_2)}^{u(t_1)} [F(u^{-1}(x), x) - F(v^{-1}(x), x)] dx \leq 0,$$

therefore in virtue of (15), $u'(t_1) = v'(t_1)$, $u'(t_2) = v'(t_2)$ and $F(u^{-1}(x), x) - F(v^{-1}(x), x) = 0$ for $x \in [u(t_1), u(t_2)]$. Then

$$(16) \quad 0 = (W_+(u(t_2)) - W_+(u(t_1))) - (W_+(v(t_2)) - W_+(v(t_1))) \\ = \int_{t_1}^{t_2} \frac{1}{q(s)} \int_{v(s)}^{u(s)} F(u^{-1}(x), x) dx ds$$

and since $H_1(u(t)) - H_1(v(t)) > 0$ on (t_1, t_2) ,

$$\int_{t_1}^{t_2} \frac{1}{q(s)} \int_{v(s)}^{u(s)} F(u^{-1}(x), x) dx ds \geq f_1(t_2) \int_{t_1}^{t_2} \frac{1}{q(s)} \int_{v(s)}^{u(s)} h_1(x) dx ds \\ = f_1(t_2) \int_{t_1}^{t_2} \frac{1}{q(s)} (H_1(u(s)) - H_1(v(s))) ds > 0,$$

which contradicts (16). Hence $u(t) \neq v(t)$ for $t \in (0, \infty)$. For $\varepsilon = -$ the proof is similar. ■

4. Bounded solutions

THEOREM 3. *Let assumptions (H₁)–(H₅) be satisfied. Then*

- (i) *any nontrivial solution of (4) is bounded (on \mathbb{R}_+) if and only if $\int_0^\infty ds/q(s) < \infty$,*
- (ii) *any nontrivial solution of (4) is unbounded (on \mathbb{R}_+) if and only if $\int_0^\infty ds/q(s) = \infty$.*

Proof. We will prove our theorem for $\varepsilon = +$ (the case $\varepsilon = -$ is similar). First observe that either $\int_0^\infty ds/q(s) < \infty$ or $\int_0^\infty ds/q(s) = \infty$.

- (a) If $\int_0^\infty ds/q(s) < \infty$ then by Lemma 3 any solution of (4) is bounded.
- (b) Let $\int_0^\infty ds/q(s) = \infty$ and $u \in \mathcal{A}_+$ be a solution of (4). Then $u \neq 0$,

$$W_+(u(t)) \\ = \int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \geq \int_0^t \frac{1}{q(s)} \int_0^{u(s)} f_1(u^{-1}(x)) h_1(x) dx ds$$

and since $0 \neq (\int_0^{u(s)} f_1(u^{-1}(x)) h_1(x) dx)' \geq 0$ on \mathbb{R}_+ we see $\lim_{t \rightarrow \infty} W_+(u(t)) = \infty$, hence $\lim_{t \rightarrow \infty} u(t) = \infty$.

Let $u \in \mathcal{A}_+$ be a solution of (4). Since u is strictly increasing on \mathbb{R}_+ (by Lemma 1), u is either bounded or unbounded on \mathbb{R}_+ . Accordingly, either $\int_0^\infty ds/q(s) < \infty$ or $\int_0^\infty ds/q(s) = \infty$ by (b) and (a) above. ■

5. Uniqueness theorems. Let assumptions (H_1) – (H_5) be satisfied. By Theorem 1 equation (4) has at least one solution in \mathcal{A}_+ and at least one solution in \mathcal{A}_- . Since $u = 0$ is also a solution of (4), we see that under assumptions (H_1) – (H_5) equation (4) admits at least three different solutions. In the next theorems we will give sufficient conditions for the existence of just three different solutions of (4).

THEOREM 4. *Let assumptions (H_1) – (H_5) be satisfied. Moreover, assume that*

(H7) *There exists an $\varepsilon > 0$ such that:*

- (i) $|F(t_1, u) - F(t_2, u)| \leq L|t_1 - t_2| \min\{|h_1(u)|, |h_2(u)|\}$ for $(t_i, u) \in [0, \varepsilon] \times [-\varepsilon, \varepsilon]$ ($i = 1, 2$), where $L > 0$ is a constant,
- (ii) *the modulus of continuity $\gamma(t)$ ($:= \sup\{|q(t_1) - q(t_2)| : t_1, t_2 \in [0, \varepsilon], |t_1 - t_2| \leq t\}$) of q on $[0, \varepsilon]$ satisfies*

$$\limsup_{t \rightarrow 0^+} \gamma(t)/t < \infty;$$

(H8) *The function $F(t, u)$ locally satisfies a Lipschitz condition on $(0, \infty) \times (\mathbb{R} - \{0\})$ with respect to t .*

Then equation (4) admits just three different solutions.

PROOF. Assume $u_1, u_2 \in \mathcal{A}_+$ are solutions of (4) and assume on the contrary $u_1 \neq u_2$. First, we will prove $u_1(t) = u_2(t)$ on an interval $[0, b]$, $b > 0$. Setting $A_j = \lim_{t \rightarrow \infty} u_j(t)$ and $w_j = u_j^{-1}$ ($j = 1, 2$), we see $0 < A_j \leq \infty$, $w_j : [0, A_j] \rightarrow \mathbb{R}_+$ are continuous strictly increasing functions and

$$w'_j(x) = k(x)q(w_j(x)) \left[\int_0^x F(w_j(s), s) ds \right]^{-1} \quad \text{for } x \in (0, A_j), \quad j = 1, 2.$$

Hence

$$w_j(x) = \int_0^x k(s)q(w_j(s)) \left[\int_0^s F(w_j(t), t) dt \right]^{-1} ds \quad \text{for } x \in (0, A_j), \quad j = 1, 2,$$

and thus for $x \in [0, \min(A_1, A_2))$ we have

$$(17) \quad w_1(x) - w_2(x) = \int_0^x (q(w_1(s)) - q(w_2(s)))k(s) \left[\int_0^s F(w_2(t), t) dt \right]^{-1} ds \\ + \int_0^x \frac{q(w_1(s))k(s)}{\int_0^s F(w_1(t), t) dt \int_0^s F(w_2(t), t) dt} \int_0^s (F(w_2(t), t) - F(w_1(t), t)) dt ds.$$

Let $\varepsilon > 0$ be as in assumption (H₇) and set $a = \min\{u_1(\varepsilon), u_2(\varepsilon)\}$, $X(x) = \max\{|w_1(s) - w_2(s)| : 0 \leq s \leq x\}$ for $x \in [0, a]$. Suppose $X(x) > 0$ for $x \in (0, a]$. Then (cf. (H₇))

$$|q(w_1(x)) - q(w_2(x))| \leq \gamma(X(x)) \quad \text{for } x \in [0, a]$$

and using (H₇) and (17) we have

$$\begin{aligned} |w_1(x) - w_2(x)| &\leq \int_0^x \frac{\gamma(X(s))k(s)}{\int_0^s f_1(w_2(t))h_1(t)dt} ds \\ +L \int_0^x \frac{q(w_1(s))k(s)}{\int_0^s f_1(w_1(t))h_1(t)dt \int_0^s f_1(w_2(t))h_1(t)dt} \int_0^s |w_2(t) - w_1(t)|h_1(t) dt ds \\ &\leq \frac{\gamma(X(x))}{f_1(\varepsilon)} K_1(x) + LX(x) \frac{K_1(x)}{f_1^2(\varepsilon)} \max\{q(t) : 0 \leq t \leq \varepsilon\} \end{aligned}$$

for $0 \leq x \leq a$. Hence

$$X(x) \leq (B\gamma(X(x)) + CX(x))K_1(x), \quad x \in [0, a],$$

where $B = 1/f_1(\varepsilon)$, $C = B^2L \max\{q(t) : 0 \leq t \leq \varepsilon\}$, and therefore

$$\frac{\gamma(X(x))}{X(x)} K_1(x) \geq (1/B)(1 - CK_1(x)), \quad x \in (0, a].$$

Then (cf. (H₇))

$$\lim_{x \rightarrow 0^+} \frac{\gamma(X(x))}{X(x)} K_1(x) = 0,$$

which contradicts $\lim_{x \rightarrow 0^+} (1/B)(1 - CK_1(x)) = 1/B > 0$. This proves $u_1(t) = u_2(t)$ for $t \in [0, b]$ with a positive number b .

Secondly, assume $[0, c]$ is the maximal interval where $u_1(t) = u_2(t)$. Define $Y(t) = \max\{|u_2(s) - u_1(s)| : c \leq s \leq t\}$, $\alpha(t) = \min\{u_1(t), u_2(t)\}$, $\beta(t) = \max\{u_1(t), u_2(t)\}$ for $t \geq c$. Then $Y(c) = 0$, $\alpha(c) = \beta(c)$, $0 \leq \beta(t) - \alpha(t) \leq Y(t) > 0$ for $t > c$. From the equality ($t \geq c$)

$$\begin{aligned} W_+(u_2(t)) - W_+(u_1(t)) \\ = \int_c^t \frac{1}{q(s)} \left[\int_{u_1(c)}^{u_2(s)} F(u_2^{-1}(x), x) dx - \int_{u_2(c)}^{u_1(s)} F(u_1^{-1}(x), x) dx \right] ds \end{aligned}$$

it follows that

$$\begin{aligned} W_+(u_2(t)) - W_+(u_1(t)) &= \int_c^t \frac{1}{q(s)} \left[\int_{u_1(c)}^{u_2(s)} (F(u_2^{-1}(x), x) - F(u_1^{-1}(x), x)) dx \right. \\ &\quad \left. + \int_{u_2(s)}^{u_1(s)} F(u_1^{-1}(x), x) dx \right] ds \quad \text{for } t \in [c, \infty). \end{aligned}$$

By assumption (H₈) there exist positive numbers ε_0 and K such that $|F(t_1, u) - F(t_2, u)| \leq K|t_1 - t_2|$ for $(t_j, u) \in [c, c + \varepsilon_0] \times [u_1(c), u_1(c) + \varepsilon_0]$. Set $\varepsilon = \min\{\varepsilon_0, \beta^{-1}(\varepsilon_0 + u_1(c)) - c\}$, $m = \min\{u_1'(x) : c \leq x \leq \alpha^{-1}(\beta(c + \varepsilon))\}$ (> 0 by Lemma 1), $M = \max\{F(u_1^{-1}(x), x) : \alpha(c) \leq x \leq \beta(c + \varepsilon)\}$. Then

$$|F(u_2^{-1}(x), x) - F(u_1^{-1}(x), x)| \leq K|u_2^{-1}(x) - u_1^{-1}(x)| \quad \text{for } x \in [\alpha(c), \alpha(c + \varepsilon)]$$

and for $x \in [c, c + \varepsilon]$ we have

$$\begin{aligned} |u_1^{-1}(u_2(x)) - x| &= |u_1^{-1}(u_2(x)) - u_1^{-1}(u_1(x))| = |(u_1^{-1})'(\xi)| |u_2(x) - u_1(x)| \\ &= (1/|u_1'(\eta)|) |u_2(x) - u_1(x)| \leq (1/m) |u_2(x) - u_1(x)| \leq Y(x)/m, \end{aligned}$$

where $\xi \in (\alpha(x), \beta(x))$ and $\eta = u_1^{-1}(\xi) \in [u_1^{-1}(\alpha(x)), u_1^{-1}(\beta(x))] \subset [c, \alpha^{-1}(\beta(c + \varepsilon))]$. Consequently,

$$|u_1^{-1}(x) - u_2^{-1}(x)| \leq (1/m)Y(u_2^{-1}(x)) \quad \text{for } x \in [u_1(c), u_2(c + \varepsilon)].$$

Therefore ($r = \max\{u_2'(t) : c \leq t \leq c + \varepsilon\}$)

$$\begin{aligned} &|W_+(u_2(t)) - W_+(u_1(t))| \\ &\leq \int_c^t \frac{1}{q(s)} \left[K \int_{u_1(c)}^{u_2(s)} |u_2^{-1}(x) - u_1^{-1}(x)| dx + \int_{\alpha(s)}^{\beta(s)} F(u_1^{-1}(x), x) dx \right] ds \\ &\leq \int_c^t \frac{1}{q(s)} \left[K \int_{u_1(c)}^{u_2(s)} (Y(u_2^{-1}(x))/m) dx + M(\beta(s) - \alpha(s)) \right] ds \\ &\leq \int_c^t \frac{1}{q(s)} \left[(rK/m) \int_c^s Y(x) dx + MY(s) \right] ds \\ &\leq \int_c^t \frac{1}{q(s)} [(K\epsilon r/m) + M] Y(s) ds \leq ((K\epsilon r/m) + M) Y(t) \int_c^t \frac{ds}{q(s)} \end{aligned}$$

for $t \in [c, c + \varepsilon]$. Since ($t \in [c, c + \varepsilon]$)

$$|W_+(u_2(t)) - W_+(u_1(t))| = k(\xi) |u_2(t) - u_1(t)|,$$

where $\xi \in (\alpha(t), \beta(t)) \subset [\alpha(c), \beta(c + \varepsilon)]$, we have

$$|u_2(t) - u_1(t)| \leq [(K\epsilon r/m) + M](Y(t)/p) \int_c^t \frac{ds}{q(s)},$$

with $p = \min\{k(u) : \alpha(c) \leq u \leq \beta(c + \varepsilon)\}$ (> 0), hence

$$Y(t) \leq [(K\epsilon r/m) + M](Y(t)/p) \int_c^t \frac{ds}{q(s)}, \quad t \in [c, c + \varepsilon].$$

Then

$$1 \leq [(K\varepsilon r/m) + M](1/p) \int_c^t \frac{ds}{q(s)},$$

for $c < t \leq c + \varepsilon$, which is impossible. This proves $u_1(t) = u_2(t)$ for $t \in \mathbb{R}_+$. The uniqueness of solutions of (4) in \mathcal{A}_- can be treated quite analogously. ■

THEOREM 5. *Suppose that assumptions (H₁)–(H₇) are satisfied. Then equation (4) admits just three different solutions.*

PROOF. It is sufficient to prove that under assumptions (H₁)–(H₇), $u_\varepsilon = v_\varepsilon$, $\varepsilon = +, -$, where $u_\varepsilon, v_\varepsilon$ are defined in Theorem 2. If for example, $u_+ \neq v_+$ then $u_+(t) < v_+(t)$ on $(0, \infty)$ by Theorem 2. Since assumptions (H₁)–(H₅) and (H₇) imply (see the first part of the proof of Theorem 4) $u_+(t) = v_+(t)$ on an interval $[0, b]$ ($b > 0$), we have a contradiction. ■

6. Dependence of solutions on a parameter. Consider the differential equation

$$(18_\lambda) \quad (q(t)k(u)u')' = \lambda F(t, u)u', \quad \lambda > 0,$$

depending on the positive parameter λ .

THEOREM 6. *Suppose that assumptions (H₁)–(H₆) are satisfied. Then for each $\varepsilon \in \{+, -\}$ there exist solutions $u_\varepsilon(t; \lambda), v_\varepsilon(t; \lambda)$ of (18_λ) such that $u_\varepsilon(t; \lambda) \leq v_\varepsilon(t; \lambda)$ for $t \in \mathbb{R}_+$. Moreover,*

$$u_\varepsilon(t; \lambda) \leq u(t; \lambda) \leq v_\varepsilon(t; \lambda), \quad t \in \mathbb{R}_+,$$

for any solution $u(t; \lambda) \in \mathcal{A}_\varepsilon$ of (18_λ) and

$$(19) \quad \begin{aligned} u_+(t; \lambda_1) &< u_+(t; \lambda_2), & v_+(t; \lambda_1) &< v_+(t; \lambda_2), \\ u_-(t; \lambda_1) &> u_-(t; \lambda_2), & v_-(t; \lambda_1) &> v_-(t; \lambda_2) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $0 < \lambda_1 < \lambda_2$.

PROOF. The first part of the statement follows from Theorem 2. Set

$$(20) \quad \begin{aligned} \underline{\varphi}_+(t; \lambda) &= K_1^{-1}(\lambda k_1(t)), & \bar{\varphi}_+(t; \lambda) &= K_2^{-1}(\lambda k_2(t)), \\ \underline{\varphi}_-(t; \lambda) &= V_1^{-1}(\lambda l_1(t)), & \bar{\varphi}_-(t; \lambda) &= V_2^{-1}(\lambda l_2(t)) \end{aligned}$$

for $t \in \mathbb{R}_+$, $\lambda > 0$. Since (18_λ) can be rewritten in the form

$$\left(\frac{q(t)}{\lambda} k(u)u' \right)' = F(t, u)u', \quad \lambda > 0,$$

we have (see Lemma 3)

$$(21_+) \quad \begin{aligned} u(t_2) - u(t_1) &\geq \lambda f_1(t_2) H_1(\underline{\varphi}_+(t_1; \lambda)) \int_{t_1}^{t_2} \frac{dt}{q(t)} \\ &\quad \times [\max\{k(x) : \underline{\varphi}_+(t_1; \lambda) \leq x \leq \bar{\varphi}_+(t_2; \lambda)\}]^{-1} \end{aligned}$$

for any solution $u \in \mathcal{A}_+$ of (18 $_\lambda$) and $0 < t_1 < t_2$, and

$$(21_-) \quad u(t_1) - u(t_2) \geq -\lambda f_2(t_2) T_2(\bar{\varphi}_-(t_1; \lambda)) \int_{t_1}^{t_2} \frac{dt}{q(t)} \\ \times [\max\{-k(x) : \underline{\varphi}_-(t_2; \lambda) \leq x \leq \bar{\varphi}_-(t_1; \lambda)\}]^{-1}$$

for any solution $u \in \mathcal{A}_-$ of (18 $_\lambda$) and $0 < t_1 < t_2$.

Set $\mathcal{K}_{\lambda, \varepsilon} = \{u \in \mathcal{A}_\varepsilon : \underline{\varphi}_\varepsilon(t; \lambda) \leq u(t) \leq \bar{\varphi}_\varepsilon(t; \lambda) \text{ for } t \in \mathbb{R}_+, u \text{ satisfies (21}_\varepsilon)\}$ and define $T_{\lambda, \varepsilon} : \mathcal{K}_{\lambda, \varepsilon} \rightarrow C^0(\mathbb{R}_+)$ by

$$(T_{\lambda, \varepsilon} u)(t) = W_\varepsilon^{-1} \left(\lambda \int_0^t \frac{1}{q(s)} \int_0^{u(s)} F(u^{-1}(x), x) dx ds \right),$$

where $\varepsilon \in \{+, -\}$, $\lambda > 0$. Then (cf. Lemma 4) $T_{\lambda, \varepsilon} : \mathcal{K}_{\lambda, \varepsilon} \rightarrow \mathcal{K}_{\lambda, \varepsilon}$. Next set

$$u_{\lambda, \varepsilon}^{[0]}(t) = \underline{\varphi}_\varepsilon(t; \lambda), \quad u_{\lambda, \varepsilon}^{[n+1]}(t) = (T_{\lambda, \varepsilon} u_{\lambda, \varepsilon}^{[n]})(t), \\ v_{\lambda, \varepsilon}^{[0]}(t) = \bar{\varphi}_\varepsilon(t; \lambda), \quad v_{\lambda, \varepsilon}^{[n+1]}(t) = (T_{\lambda, \varepsilon} v_{\lambda, \varepsilon}^{[n]})(t)$$

for $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\lambda > 0$ and $\varepsilon \in \{+, -\}$. Then the limits

$$\lim_{n \rightarrow \infty} u_{\lambda, \varepsilon}^{[n]}(t) =: u_\varepsilon(t; \lambda), \quad \lim_{n \rightarrow \infty} v_{\lambda, \varepsilon}^{[n]}(t; \lambda) =: v_\varepsilon(t; \lambda)$$

exist for $t \in \mathbb{R}$, $\lambda > 0$, $\varepsilon \in \{+, -\}$.

Let $0 < t_1 < t_2$ and let for example $\varepsilon = +$ (for the case $\varepsilon = -$, the proof is similar). Then

$$\underline{\varphi}_+(t; \lambda_1) < \underline{\varphi}_+(t; \lambda_2), \quad \bar{\varphi}_+(t; \lambda_1) < \bar{\varphi}_+(t; \lambda_2)$$

and for each $\alpha, \beta \in \mathcal{A}_+$ with $\alpha(t) < \beta(t)$ on $(0, \infty)$ we have

$$(T_{\lambda_1, +} \alpha)(t) = W_+^{-1} \left(\lambda_1 \int_0^t \frac{1}{q(s)} \int_0^{\alpha(s)} F(\alpha^{-1}(x), x) dx ds \right) \\ < W_+^{-1} \left(\lambda_2 \int_0^t \frac{1}{q(s)} \int_0^{\beta(s)} F(\beta^{-1}(x), x) dx ds \right) = (T_{\lambda_2, +} \beta)(t),$$

and therefore

$$u_{\lambda_1, +}^{[n]}(t) < u_{\lambda_2, +}^{[n]}(t), \quad v_{\lambda_1, +}^{[n]}(t) < v_{\lambda_2, +}^{[n]}(t) \quad \text{for } t \in (0, \infty), n \in \mathbb{N}.$$

Hence

$$u_+(t; \lambda_1) \leq u_+(t; \lambda_2), \quad v_+(t; \lambda_1) \leq v_+(t; \lambda_2) \quad \text{for } t \in \mathbb{R}_+.$$

If $r(t_0; \lambda_1) = r(t_0; \lambda_2)$ for a $t_0 > 0$, where r is either u_+ or v_+ , then $(r_j(t) = r(t; \lambda_j)$ on \mathbb{R}_+ , $j = 1, 2)$

$$\begin{aligned} r_1(t_0) &= W_+ \left(\lambda_1 \int_0^{t_0} \frac{1}{q(s)} \int_0^{r_1(s)} F(r_1^{-1}(x), x) dx ds \right) \\ &< W_+ \left(\lambda_2 \int_0^{t_0} \frac{1}{q(s)} \int_0^{r_2(s)} F(r_2^{-1}(x), x) dx ds \right) = r_2(t_0), \end{aligned}$$

which is a contradiction. ■

THEOREM 7. *Let $\int_0^\infty ds/q(s) < \infty$ and let assumptions (H₁)–(H₇) be satisfied. Then for each $a \in \mathbb{R}$, $a \neq 0$, there exists a unique $\lambda_0 > 0$ such that equation (18 _{λ}) for $\lambda = \lambda_0$ has a (necessarily unique) solution $u(t; \lambda_0)$ with*

$$\lim_{t \rightarrow \infty} u(t; \lambda_0) = a.$$

Proof. By Theorem 5 equation (18 _{λ}) has a unique solution $u_+(t; \lambda) \in \mathcal{A}_+$ and a unique solution $u_-(t; \lambda) \in \mathcal{A}_-$ for each $\lambda > 0$ and moreover the finite limits $\lim_{t \rightarrow \infty} u_+(t; \lambda) (> 0)$ and $\lim_{t \rightarrow \infty} u_-(t; \lambda) (< 0)$ exist by Theorem 3. Define

$$g_+(\lambda) = \lim_{t \rightarrow \infty} u_+(t; \lambda), \quad g_-(\lambda) = \lim_{t \rightarrow \infty} u_-(t; \lambda) \quad \text{for } \lambda > 0.$$

Then $g_+ : (0, \infty) \rightarrow (0, \infty)$ and $g_- : (0, \infty) \rightarrow (-\infty, 0)$. In view of Theorem 6, g_+ is increasing on $(0, \infty)$ and g_- is decreasing on $(0, \infty)$. If for example $g_+(\lambda_1) = g_+(\lambda_2)$ for some $0 < \lambda_1 < \lambda_2$, then setting $r_1(t) = u_+(t; \lambda_1)$, $r_2(t) = u_+(t; \lambda_2)$ for $t \in \mathbb{R}_+$ we have $r_1(t) < r_2(t)$ on $(0, \infty)$, $r_1^{-1}(t) > r_2^{-1}(t)$, $F(r_1^{-1}(t), t) < F(r_2^{-1}(t), t)$ for $t \in (0, g_+(\lambda_1))$, hence

$$\begin{aligned} g_+(\lambda_1) &= \int_0^\infty \frac{1}{q(s)} \int_0^{r_1(s)} F(r_1^{-1}(x), x) dx ds \\ &< \int_0^\infty \frac{1}{q(s)} \int_0^{r_2(s)} F(r_2^{-1}(x), x) dx ds = g_+(\lambda_2), \end{aligned}$$

a contradiction. Consequently, g_+ is strictly increasing, and g_- is strictly decreasing. To prove our theorem it is enough to show that g_+ and g_- map $(0, \infty)$ onto $(0, \infty)$ and $(-\infty, 0)$, respectively. We prove for example that g_+ maps $(0, \infty)$ onto itself. First, we see from $\varphi_+(t; \lambda) \leq u_+(t; \lambda) \leq \bar{\varphi}_+(t; \lambda)$, where φ_+ , $\bar{\varphi}_+$ are defined by (20), that $\lim_{\lambda \rightarrow 0^+} g_+(\lambda) = 0$ and $\lim_{\lambda \rightarrow \infty} g_+(\lambda) = \infty$. Secondly, assume, on the contrary, $\lim_{\lambda \rightarrow \lambda_0^-} g_+(\lambda) < \lim_{\lambda \rightarrow \lambda_0^+} g_+(\lambda)$ for a $\lambda_0 > 0$. Setting $v_1(t) = \lim_{\lambda \rightarrow \lambda_0^-} u_+(t; \lambda)$, $v_2(t) = \lim_{\lambda \rightarrow \lambda_0^+} u_+(t; \lambda)$ for $t \geq 0$, we get $v_1 \neq v_2$. Using the Lebesgue dominated convergence theorem as $\lambda \rightarrow \lambda_0^-$ and $\lambda \rightarrow \lambda_0^+$ in the equality $(r_\lambda(t) = u_+(t; \lambda)$ for $(t; \lambda) \in \mathbb{R}_+ \times (0, \infty)$)

$$r_\lambda(t) = W_+ \left(\lambda \int_0^t \frac{1}{q(s)} \int_0^{r_\lambda(s)} F(r_\lambda^{-1}(x), x) dx ds \right)$$

we see that

$$v_j(t) = W_+ \left(\lambda_0 \int_0^t \frac{1}{q(s)} \int_0^{v_j(s)} F(v_j^{-1}(x), x) dx ds \right) \quad \text{for } t \geq 0, j = 1, 2.$$

Therefore v_1, v_2 are solutions of (18 $_\lambda$) for $\lambda = \lambda_0$, and consequently $v_1 = v_2$, a contradiction. ■

References

- [1] F. A. Atkinson and L. A. Peletier, *Similarity profiles of flows through porous media*, Arch. Rational Mech. Anal. 42 (1971), 369–379.
- [2] —, —, *Similarity solutions of the nonlinear diffusion equation*, ibid. 54 (1974), 373–392.
- [3] J. Bear, D. Zaslavsky and S. Irmay, *Physical Principles of Water Percolation and Seepage*, UNESCO, 1968.
- [4] J. Goncerzewicz, H. Marcinkowska, W. Okrański and K. Tabisz, *On the percolation of water from a cylindrical reservoir into the surrounding soil*, Zastos. Mat. 16 (1978), 249–261.
- [5] W. Okrański, *Integral equations methods in the theory of the water percolation*, in: Mathematical Methods in Fluid Mechanics, Proc. Conf. Oberwolfach, 1981, Band 24, P. Lang, Frankfurt/M, 1982, 167–176.
- [6] —, *On a nonlinear ordinary differential equation*. Ann. Polon. Math. 49 (1989), 237–245.
- [7] S. Staněk, *Nonnegative solutions of a class of second order nonlinear differential equations*, ibid. 57 (1992), 71–82.

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Reçu par la Rédaction le 26.3.1992
Révisé le 7.7.1992