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Some subclasses of close-to-convex functions

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Abstract. For $\alpha \in [0,1]$ and $\beta \in (-\pi/2, \pi/2)$ we introduce the classes $C_\beta(\alpha)$ defined as follows: a function f regular in $U = \{z : |z| < 1\}$ of the form $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$, $z \in U$, belongs to the class $C_\beta(\alpha)$ if $\mathrm{Re}\{e^{i\beta}(1-\alpha^2 z^2) f'(z)\} > 0$ for $z \in U$. Estimates of the coefficients, distortion theorems and other properties of functions in $C_\beta(\alpha)$ are examined.

1. Denote by $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk in the complex plane $\mathbb{C}$. Let P denote the class of functions p of the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, $z \in U$, which are regular in U and have a positive real part. Denote by Ω the class of functions ω regular in U such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. A regular function f in U is called *subordinate* to a regular function F in U if there exists a function $\omega \in \Omega$ such that $f(z) = F(\omega(z))$, $z \in U$. We write then $f \prec F$ or $f(z) \prec F(z)$, $z \in U$.

Definition 1.1. A function f of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad z \in U, \tag{1.1}$$

regular in U belongs to the class $C_\beta(\alpha)$, $\alpha \in \mathbb{C}$, $\beta \in (-\pi/2, \pi/2)$, if

$$\mathrm{Re}\{e^{i\beta}(1-\alpha^2 z^2) f'(z)\} > 0, \quad z \in U. \tag{1.2}$$

We also set

$$C(\alpha) = \bigcup_{\beta \in (-\pi/2, \pi/2)} C_\beta(\alpha).$$

If $\alpha = |\alpha| e^{i\theta}$, $\theta \in [0, 2\pi)$, and $f \in C_\beta(\alpha)$, $\beta \in (-\pi/2, \pi/2)$, then the function $g(z) = e^{-i\theta} f(e^{i\theta} z)$, $z \in U$, belongs to $C_\beta(|\alpha|)$. Thus we may assume that α is real. By (1.2) it is sufficient to take α from the interval $[0,1]$ because the assumption $|\alpha| > 1$ implies that $C_\beta(\alpha) = \emptyset$ for all $\beta \in (-\pi/2, \pi/2)$.

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Hengartner and Schober [4] established that the inequality

$$\operatorname{Re}\{(1-z^2)f'(z)\} > 0, \quad z \in U, \tag{1.3}$$

characterizes the class of univalent functions f of the form

$$f(z) = a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad a_1 \in \mathbb{C},\ |a_1| = 1,\ z \in U,$$

with the normalization

$$\liminf_{z\to -1} \operatorname{Re} f(z) = \inf_{z\in U} \operatorname{Re} f(z), \quad \limsup_{z\to 1} \operatorname{Re} f(z) = \sup_{z\in U} \operatorname{Re} f(z)$$

which map U onto domains convex in the direction of the imaginary axis. This class was denoted by $C\widetilde{V}_2(i)$. The condition (1.3) implies that $\operatorname{Re} f'(0) = \operatorname{Re} a_1 > 0$.

Following the definition of α-spiral functions (Špaček [10]) and functions close-to-convex with argument β (Goodman and Saff [3]) we introduce in (1.3) the factor $e^{i\beta} = f'(0)$. Therefore for $\beta \in (-\pi/2, \pi/2)$ we distinguish the class β-$CV_2(i)$ of functions f of the form (1.1) regular in U defined by the inequality

$$\operatorname{Re}\{e^{i\beta}(1-z^2)f'(z)\} > 0, \quad z \in U. \tag{1.4}$$

Thus for $\alpha = 1$ and fixed $\beta \in (-\pi/2, \pi/2)$ we have $C_\beta(1) = \beta$-$CV_2(i)$.

Of course, if $f \in \beta$-$CV_2(i)$, $\beta \in (-\pi/2, \pi/2)$, then the function $g(z) = e^{i\beta} f(z)$, $z \in U$, belongs to $C\widetilde{V}_2(i)$. Conversely, if $f \in C\widetilde{V}_2(i)$, then there exists $\beta \in (-\pi/2, \pi/2)$ such that the function $g(z) = e^{-i\beta} f(z)$, $z \in U$, belongs to β-$CV_2(i)$.

For $\alpha = 0$, (1.2) yields a univalence condition found independently by Noshiro [9] and Warschawski [12]. The class of functions that satisfy this condition:

$$\operatorname{Re}\{e^{i\beta} f'(z)\} > 0, \quad z \in U, \tag{1.5}$$

is usually denoted by $P'(\beta)$ and the functions are called of bounded rotation with argument β.

Notice that (1.2) can be written as

$$\operatorname{Re}\{\alpha^2 e^{i\beta}(1-z^2)f'(z) + (1-\alpha^2)e^{i\beta} f'(z)\} > 0.$$

Taking $\gamma = \alpha^2$, $\alpha \in [0,1]$, we see that the left hand side of (1.2) is a convex combination of the left hand sides of (1.4) and (1.5). This method of defining new classes of analytic functions is due to Mocanu [8] who introduced the α-convex functions. This concept was used by many authors. For example, in [1] the classes $H(\alpha)$, with α real, of functions f of the form (1.1) regular in U are defined by the inequality

$$\operatorname{Re}\left\{(1-\alpha)f'(z) + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right\} > 0, \quad z \in U.$$

The class $C_0(\alpha)$ was examined in [6].

2. In this section estimates of the coefficients of functions in $C_\beta(\alpha)$ are obtained.

THEOREM 2.1. *If $f \in C_\beta(\alpha)$, $\alpha \in [0,1]$, $\beta \in (-\pi/2, \pi/2)$, then f is univalent in U.*

Proof. For $\alpha = 0$ this is shown in [9] and [12].

Let now $\alpha \in (0,1]$. The function

$$\varphi_\alpha(z) = \frac{1}{2\alpha} \log \frac{1+\alpha z}{1-\alpha z}, \quad z \in U, \quad \varphi_\alpha(0) = 0,$$

is convex and univalent in U. Moreover, if $f \in C_\beta(\alpha)$, where $\beta \in (-\pi/2, \pi/2)$, then

$$\operatorname{Re}\left\{e^{i\beta} \frac{f'(z)}{\varphi'_\alpha(z)}\right\} = \operatorname{Re}\{e^{i\beta}(1-\alpha^2 z^2) f'(z)\} > 0, \quad z \in U.$$

This means that f is close-to-convex and univalent (see [5]).

THEOREM 2.2. *If $\beta \in (-\pi/2, \pi/2)$, α_1, $\alpha_2 \in [0,1]$ and $\alpha_1 \neq \alpha_2$, then $C_\beta(\alpha_1) \not\subseteq C_\beta(\alpha_2)$ and $C_\beta(\alpha_2) \not\subseteq C_\beta(\alpha_1)$.*

Proof. Let $0 \le \alpha_2 < \alpha_1 \le 1$.

1°. Let f be the solution of the equation

$$e^{i\beta}(1-\alpha_1^2 z^2) f'(z) = \frac{1+z^2}{1-z^2} \cos\beta + i \sin\beta, \quad z \in U, \tag{2.1}$$

where $\beta \in (-\pi/2, \pi/2)$. Of course, $f \in C_\beta(\alpha_1)$ and by (2.1) we have

$$\begin{aligned}\operatorname{Arg}\{e^{i\beta}(1-\alpha_2^2 z^2) f'(z)\} &= \operatorname{Arg}\left\{\left(\frac{1+z^2}{1-z^2}\cos\beta + i\sin\beta\right)\frac{1-\alpha_2^2 z^2}{1-\alpha_1^2 z^2}\right\} \\ &= \operatorname{Arg}\left\{\frac{1+z^2}{1-z^2}\cos\beta + i\sin\beta\right\} + \operatorname{Arg}\frac{1-\alpha_2^2 z^2}{1-\alpha_1^2 z^2},\end{aligned} \tag{2.2}$$

where $z \in U$, $\operatorname{Arg}(e^{i\beta}) = \beta$ and $\operatorname{Arg} 1 = 0$.

Let now $z = e^{it}$, $t \in (0,\pi) \cup (\pi, 2\pi)$. Then

$$\frac{1+z^2}{1-z^2} = i\frac{\cos t}{\sin t}$$

and

$$\frac{1+z^2}{1-z^2}\cos\beta + i\sin\beta = i\left(\frac{\cos t}{\sin t}\cos\beta + \sin\beta\right). \tag{2.3}$$

For fixed $\beta \in (-\pi/2, \pi/2)$ we can choose $t_0 \in (0, \pi/2)$ such that

$$\frac{\cos t_0}{\sin t_0}\cos\beta + \sin\beta > 0. \tag{2.4}$$

Set $z_0 = e^{it_0}$. From (2.3) and (2.4) we have

$$\operatorname{Arg}\left\{\frac{1+z_0^2}{1-z_0^2}\cos\beta + i\sin\beta\right\} = \frac{\pi}{2}\,. \tag{2.5}$$

On the other hand, if $z = e^{it}$, where $t \in (0,\pi)\cup(\pi,2\pi)$, then

$$\frac{1-\alpha_2^2 z^2}{1-\alpha_1^2 z^2} = \frac{1+\alpha_1^2\alpha_2^2 - (\alpha_1^2+\alpha_2^2)\cos 2t}{1-2\alpha_1^2\cos 2t + \alpha_1^4} + i\frac{(\alpha_1^2-\alpha_2^2)\sin 2t}{1-2\alpha_1^2\cos 2t+\alpha_1^4}\,. \tag{2.6}$$

The real part in (2.6) is positive for all $t \in (0,\pi)\cup(\pi,2\pi)$. Moreover, if $\alpha_2 < \alpha_1$ and $t \in (0,\pi/2)$, then the imaginary part in (2.6) is also positive. In particular, this holds for t_0. Therefore (2.6) yields

$$0 < \operatorname{Arg}\frac{1-\alpha_2^2 z_0^2}{1-\alpha_1^2 z_0^2} < \frac{\pi}{2}\,. \tag{2.7}$$

Using (2.5) and (2.7) we conclude that

$$\frac{\pi}{2} < \operatorname{Arg}\left\{\frac{1+z_0^2}{1-z_0^2}\cos\beta + i\sin\beta\right\} + \operatorname{Arg}\frac{1-\alpha_2^2 z_0^2}{1-\alpha_1^2 z_0^2} < \pi\,. \tag{2.8}$$

Let now (z_n), $n \in \mathbb{N}$, where $z_n = r_n e^{it_0}$, $0 < r_n < 1$, be a sequence that converges to z_0. Then there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ inequalities (2.8) are satisfied with z_n in place of z_0. Finally, by (2.2) and (2.8) for $n > n_0$ we have

$$\frac{\pi}{2} < \operatorname{Arg}\{e^{i\beta}(1-\alpha_2^2 z_n^2) f'(z_n)\} < \pi\,.$$

This means that $f \notin C_\beta(\alpha_2)$.

2^{o}. Let now f be the solution of the equation

$$e^{i\beta}(1-\alpha_2^2 z^2) f'(z) = \frac{1-z^2}{1+z^2}\cos\beta + i\sin\beta, \qquad z \in U\,.$$

Obviously, $f \in C_\beta(\alpha_2)$.

If $z = e^{it}$, $t \in (0,\pi/2)\cup(\pi/2,3\pi/2)\cup(3\pi/2,2\pi)$, then

$$\frac{1-z^2}{1+z^2}\cos\beta + i\sin\beta = i\left(-\frac{\sin t}{\cos t}\cos\beta + \sin\beta\right). \tag{2.9}$$

For fixed $\beta \in (-\pi/2,\pi/2)$ we can choose $t_0 \in (0,\pi/2)$ such that

$$-\frac{\sin t}{\cos t}\cos\beta + \sin\beta < 0\,.$$

If we set $z_0 = e^{it_0}$, then from the above and (2.9) we have

$$\operatorname{Arg}\left\{\frac{1-z_0^2}{1+z_0^2}\cos\beta + i\sin\beta\right\} = -\frac{\pi}{2}\,. \tag{2.10}$$

For $\alpha_2 < \alpha_1$ and $t = t_0$ the imaginary part in (2.6) is negative with α_2 in place of α_1 and vice versa. Therefore

$$-\pi < \operatorname{Arg} \frac{1-\alpha_1^2 z_0^2}{1-\alpha_2^2 z_0^2} < -\frac{\pi}{2}\,.$$

Hence and from (2.10) we conclude that

$$-\pi < \operatorname{Arg} \left\{ \frac{1-z_0^2}{1+z_0^2} \cos\beta + i\sin\beta \right\} + \operatorname{Arg} \frac{1-\alpha_1^2 z_0^2}{1-\alpha_2^2 z_0^2} < -\frac{\pi}{2}\,.$$

Thus for $z \in U$ near to z_0 we have

$$-\pi < \operatorname{Arg}\{e^{i\beta}(1-\alpha_1^2 z^2) f'(z)\} < -\frac{\pi}{2}\,.$$

This means that $f \notin C_\beta(\alpha_1)$ and ends the proof.

Now we find coefficient bounds for the class $C_\beta(\alpha)$.

THEOREM 2.3. *If $f \in C_\beta(\alpha)$, $\alpha \in (0,1)$, $\beta \in (-\pi/2, \pi/2)$ and f is of the form* (1.1), *then, for all $k \in \mathbb{N}$,*

$$|a_{2k}| \le \frac{1-\alpha^{2k}}{(1-\alpha^2)k} \cos\beta\,, \tag{2.11}$$

$$|a_{2k+1}| \le \frac{2\cos\beta + (1-2\cos\beta)\alpha^{2k} - \alpha^{2(k+1)}}{(1-\alpha^2)(2k+1)}\,. \tag{2.12}$$

Proof. By (1.2) there exists a function

$$q(z) = \cos\beta + i\sin\beta + \sum_{n=1}^{\infty} q_n z^n, \qquad z \in U\,,$$

such that $\operatorname{Re} q(z) > 0$ for $z \in U$ and

$$e^{i\beta}(1-\alpha^2 z^2) f'(z) = q(z)\,. \tag{2.13}$$

Then for $\beta \in (-\pi/2, \pi/2)$ the function

$$p(z) = \frac{1}{\cos\beta}(q(z) - i\sin\beta) = 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots, \qquad z \in U,$$

belongs to P. Moreover,

$$q_n = p_n \cos\beta, \qquad n \in \mathbb{N}\,. \tag{2.14}$$

Equating coefficients in (2.13) we have

$$\begin{gathered} 2e^{i\beta} a_2 = q_1, \qquad e^{i\beta}(3a_3 - \alpha^2) = q_2, \quad \ldots, \\ e^{i\beta}[(n+1)a_{n+1} - (n-1)\alpha^2 a_{n-1}] = q_n\,. \end{gathered} \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$a_{n+1} = \frac{(n-1)\alpha^2 a_{n-1} + e^{-i\beta} p_n \cos\beta}{n+1}\,. \tag{2.16}$$

If $n=2k-1$, $k\in\mathbb{N}$, then from (2.16) we have

$$a_{2k}=\frac{e^{-i\beta}\cos\beta}{2k}\sum_{j=1}^{k}\alpha^{2(k-j)}p_{2j-1}\,. \tag{2.17}$$

Hence using the known estimates $|p_n|\le 2$, $n\in\mathbb{N}$, we obtain

$$|a_{2k}|\le\frac{\cos\beta}{k}\sum_{j=1}^{k}\alpha^{2(k-j)}=\frac{1-\alpha^{2k}}{(1-\alpha^2)k}\cos\beta\,.$$

If $n=2k$, $k\in\mathbb{N}$, then (2.16) yields

$$a_{2k+1}=\frac{\alpha^{2k}+e^{-i\beta}\cos\beta\sum_{j=1}^{k}\alpha^{2(k-j)}p_{2j}}{2k+1}\,. \tag{2.18}$$

Hence

$$|a_{2k+1}|\le\frac{\alpha^{2k}+2\cos\beta\sum_{j=1}^{k}\alpha^{2(k-j)}}{2k+1}=\frac{2\cos\beta+(1-2\cos\beta)\alpha^{2k}-\alpha^{2(k+1)}}{(1-\alpha^2)(2k+1)},$$

for $k\in\mathbb{N}$. This ends the proof of the theorem.

The bound in (2.11) is sharp and achieved by the function $f_{\alpha,\beta}$, $\alpha\in(0,1)$, $\beta\in(-\pi/2,\pi/2)$, which is the solution of the differential equation

$$e^{i\beta}(1-\alpha^2z^2)f'_{\alpha,\beta}(z)=\frac{1+z}{1-z}\cos\beta+i\sin\beta\,,$$

i.e.

$$f_{\alpha,\beta}(z)=e^{-i\beta}\bigg\{\frac{\cos\beta}{1-\alpha^2}\bigg(\log\frac{1-\alpha^2z^2}{(1-z)^2}-\frac{1+\alpha^2}{2\alpha}\log\frac{1+\alpha z}{1-\alpha z}\bigg)$$
$$+\,i\sin\beta\frac{1}{2\alpha}\log\frac{1+\alpha z}{1-\alpha z}\bigg\},\qquad z\in U\,.$$

For the third coefficient a_3 we get the sharp bound

$$|a_3|\le\frac{2\cos\beta+\alpha^2}{3}\,.$$

Equality is attained when $p_2=2e^{i\beta}$ in (2.18). This gives the extremal function $g_{\alpha,\beta}$, $\alpha\in(0,1)$, $\beta\in(-\pi/2,\pi/2)$, which is the solution of the equation

$$e^{i\beta}(1-\alpha^2z^2)g'_{\alpha,\beta}(z)=\frac{1+e^{i\beta/2}z}{1-e^{i\beta/2}z}\cos\beta+i\sin\beta\,,$$

i.e.

$$g_{\alpha,\beta}(z)=\frac{e^{-i\beta}\cos\beta}{2\alpha(\alpha^2-e^{i\beta})}[4\alpha e^{i\beta/2}\log(1-e^{i\beta/2}z)$$
$$+(\alpha-e^{i\beta/2})^2\log(1+\alpha z)-(\alpha+e^{i\beta/2})^2\log(1-\alpha z)]$$
$$+\,ie^{-i\beta}\sin\beta\frac{1}{2\alpha}\log\frac{1+\alpha z}{1-\alpha z},\qquad z\in U\,.$$

It is not known if the bounds for odd-numbered coefficients a_n, $n \ge 5$, of functions $f \in C_\beta(\alpha)$, for $\beta \ne 0$, are sharp. If $\beta = 0$, then the estimates are sharp and are the same as in Corollary 2.4 below.

Corollary 2.4. *If* $f \in C(\alpha)$, $\alpha \in (0,1)$, *and* f *is of the form* (1.1), *then*

$$(2.19)\qquad |a_{2k}| \le \frac{1-\alpha^{2k}}{(1-\alpha^2)k} \quad \text{and} \quad |a_{2k+1}| \le \frac{2-\alpha^{2k}-\alpha^{2(k+1)}}{(1-\alpha^2)(2k+1)}, \quad k \in \mathbb{N}.$$

The above results are sharp. The function

$$f_{\alpha,0}(z) = \frac{1}{1-\alpha^2}\left(\log\frac{1-\alpha^2z^2}{(1-z)^2} - \frac{1+\alpha^2}{2\alpha}\log\frac{1+\alpha z}{1-\alpha z}\right), \quad z \in U,\ \alpha \in (0,1),$$

is extremal for all coefficients.

Observe that the formulas (2.16), (2.17) and (2.18) for the coefficients also hold for $\alpha = 0$ and $\alpha = 1$. Therefore we can also obtain estimates in these two cases. For $\alpha = 0$, from (2.16) we have

$$a_n = \frac{e^{-i\beta}p_{n-1}\cos\beta}{n}, \quad n \in \mathbb{N}.$$

This formula gives the well known result:

Corollary 2.5. *If* $f \in P'(\beta)$, $\beta \in (-\pi/2, \pi/2)$, *and* f *is of the form* (1.1), *then*

$$(2.20)\qquad |a_n| \le \frac{2}{n}\cos\beta, \quad n \in \mathbb{N}.$$

In particular, *for* $\beta = 0$,

$$(2.21)\qquad |a_n| \le \frac{2}{n}, \quad n \in \mathbb{N}$$

(see [7]).

The estimates (2.20) and (2.21) can be obtained from (2.11) and (2.12) by putting $\alpha = 0$. The following functions are extremal for the classes $P'(\beta)$ and $P'(0)$, respectively:

$$f_{0,\beta}(z) = \lim_{\alpha\to 0} f_{\alpha,\beta}(z) = e^{-i\beta}[-e^{-i\beta}z - 2\cos\beta\log(1-z)], \quad z \in U,$$
$$f_{0,0}(z) = \lim_{\alpha\to 0} f_{\alpha,0}(z) = -z - 2\log(1-z), \quad z \in U.$$

Moreover, inequalities (2.21) are satisfied in the class $C(0)$ and equality holds for $f_{0,0}$. The bounds (2.21) can be obtained from (2.19) by putting $\alpha = 0$.

For $\alpha = 1$, from (2.17) and (2.18) we have

$$a_{2k} = \frac{e^{-i\beta}\cos\beta}{2k}\sum_{j=1}^{k} p_{2j-1}, \quad a_{2k+1} = \frac{1 + e^{-i\beta}\cos\beta\sum_{j=1}^{k} p_{2j}}{2k+1}, \quad k \in \mathbb{N}.$$

These two formulas yield the following result due to Hengartner and Schober (see [4], Theorem 3):

Corollary 2.6. *If* $f \in \beta\text{-}CV_2(i)$, $\beta \in (-\pi/2, \pi/2)$, *and* f *is of the form* (1.1), *then*

$$|a_{2k}| \le \cos\beta\,, \tag{2.22}$$

$$|a_{2k+1}| \le \frac{2k\cos\beta + 1}{2k+1}, \qquad k \in \mathbb{N}\,. \tag{2.23}$$

In particular, *for* $\beta = 0$,

$$|a_n| \le 1, \qquad n \in \mathbb{N}\,. \tag{2.24}$$

The function

$$f_{1,\beta}(z) = \lim_{\alpha\to 1} f_{\alpha,\beta}(z) = e^{-i\beta}\left[\frac{z}{1-z}\cos\beta + \frac{i\sin\beta}{2}\log\frac{1+z}{1-z}\right],$$

$\beta \in (-\pi/2, \pi/2)$, $z \in U$, makes (2.22) sharp. On the other hand, if $\beta \ne 0$, then (2.23) is sharp only for $k = 1$ and for the function

$$g_{1,\beta}(z) = \frac{e^{-i\beta}\cos\beta}{2(1-e^{i\beta})}[4e^{i\beta/2}\log(1-e^{i\beta/2}z)$$

$$+(1-e^{i\beta/2})^2\log(1+z) - (1+e^{i\beta/2})^2\log(1-z)] + \frac{ie^{-i\beta}}{2}\sin\beta\log\frac{1+z}{1-z}\,,$$

$z \in U$ (see [4]).

If $\beta = 0$, then (2.24) is sharp and equality is achieved by the function

$$f_{1,0}(z) = \lim_{\alpha\to 1} f_{\alpha,0}(z) = \frac{z}{1-z}, \qquad z \in U\,.$$

Moreover, the estimates (2.24) hold for the class $C(1)$ and $f_{1,0}$ is extremal in this case.

3. Now we give some distortion theorems for the class $C_\beta(\alpha)$.

Theorem 3.1. *If* $f \in C_\beta(\alpha)$, $\alpha \in [0,1]$, $\beta \in (-\pi/2, \pi/2)$, *then*

$$|f'(z)| \le \frac{\sqrt{1+r^4+2r^2\cos 2\beta} + 2r\cos\beta}{(1-\alpha^2r^2)(1-r^2)} = \frac{\exp\left(\operatorname{ar\,sh}\dfrac{2r\cos\beta}{1-r^2}\right)}{1-\alpha^2r^2}, \tag{3.1}$$

$$|f'(z)| \ge \frac{\sqrt{1+r^4+2r^2\cos 2\beta} - 2r\cos\beta}{(1+\alpha^2r^2)(1-r^2)} = \frac{\exp\left(-\operatorname{ar\,sh}\dfrac{2r\cos\beta}{1-r^2}\right)}{1+\alpha^2r^2} \tag{3.2}$$

and

$$|f(z)| \le \int_0^r \frac{\sqrt{1+\varrho^4+2\varrho^2\cos 2\beta} + 2\varrho\cos\beta}{(1-\alpha^2\varrho^2)(1-\varrho^2)}d\varrho\,, \tag{3.3}$$

$$(3.4) \qquad |f(z)| \geq \int_0^r \frac{\sqrt{1+\varrho^4+2\varrho^2\cos 2\beta}-2\varrho\cos\beta}{(1+\alpha^2\varrho^2)(1-\varrho^2)}d\varrho$$

for $z \in U$, $|z| \leq r < 1$.

P r o o f. By Lemma 5 of [4] equation (1.2) may be written as

$$(1-\alpha^2 z^2)f'(z) = \frac{1+e^{-2i\beta}\omega(z)}{1-\omega(z)}, \qquad z \in U,$$

where $\omega \in \Omega$. Thus

$$(3.5) \qquad f'(z) = \frac{1}{1-\alpha^2 z^2}\frac{1+e^{-2i\beta}\omega(z)}{1-\omega(z)}.$$

Moreover, we have

$$(3.6) \qquad \frac{1+e^{-2i\beta}\omega(z)}{1-\omega(z)} \prec \frac{1+e^{-2i\beta}z}{1-z}, \qquad z \in U.$$

In view of (3.6) and by Theorem 2.3 of [11],

$$(3.7) \qquad \frac{|1+e^{-2i\beta}r^2|-|1+e^{-2i\beta}|r}{1-r^2} \leq \left|\frac{1+e^{-2i\beta}\omega(z)}{1-\omega(z)}\right| \leq \frac{|1+e^{-2i\beta}r^2|+|1+e^{-2i\beta}|r}{1-r^2},$$

where $z \in U$, $|z| \leq r < 1$. Now, the upper and lower bounds (3.1) and (3.2) follow from (3.5) and (3.7).

The estimates (3.7) are sharp and in view of (3.6) are realized by the function

$$p_0(z) = \frac{1+e^{-2i\beta}z}{1-z}, \qquad z \in U,$$

at two points z_0 and z_1 of modulus r. Let $z_0 = re^{i\theta_0(\beta)}$ and $z_1 = re^{i\theta_1(\beta)}$, where $0 < r < 1$, $\theta_0(\beta)$, $\theta_1(\beta) \in [0, 2\pi)$, give the lower and upper bound in (3.7) respectively. Now, we denote by $h_{\alpha,\beta}$, $\alpha \in [0,1]$, $\beta \in (-\pi/2, \pi/2)$, the function which is the solution of the equation (3.5) for $\omega = \omega_0$ defined by

$$\omega_0(z) = -ie^{i\theta_0(\beta)}z, \qquad z \in U.$$

The function $h_{\alpha,\beta}$ is extremal for the lower estimate (3.2) and equality is attained at the point $z = ir$.

In the same way we denote by $t_{\alpha,\beta}$, $\alpha \in [0,1]$, $\beta \in (-\pi/2, \pi/2)$, the function which is the solution of the equation (3.5) for $\omega = \omega_1$ given by

$$\omega_1(z) = e^{i\theta_1(\beta)}z, \qquad z \in U.$$

Then $t_{\alpha,\beta}$ gives the maximum modulus in (3.1) at the point $z = r$ and is extremal for the upper estimate.

Now we show the estimates (3.3) and (3.4).

For $z \in U$, $|z| = r$, the upper bound (3.3) follows immediately from (3.1). Let now $\xi \in U$, $|\xi| = r$, be a point such that $|f(\xi)| = \min\{|f(z)| : |z| = r\}$. Moreover, let $I = [0, f(\xi)]$ denote the closed line segment from 0 to $f(\xi)$. Thus for $|z| = r$ we have

$$|f(z)| \ge |f(\xi)| = \int_I |dw| = \int_{f^{-1}(I)} |f'(z)|\,|dz| \ge \int_0^r \frac{\sqrt{1+\varrho^4+2\varrho^2\cos 2\beta} - 2\varrho\cos\beta}{(1+\alpha^2\varrho^2)(1-\varrho^2)} d\varrho\,.$$

The estimates (3.3), (3.4) are sharp and realized by the functions $h_{\alpha,\beta}$ and $t_{\alpha,\beta}$.

Corollary 3.2. *If* $f \in C(\alpha)$, $\alpha \in (0,1)$, *then*

$$\frac{1-r}{(1+r)(1+\alpha^2r^2)} \le |f'(z)| \le \frac{1+r}{(1-r)(1-\alpha^2r^2)}\,, \tag{3.8}$$

$$\begin{aligned} &\frac{1}{1+\alpha^2}\left[\log\frac{(1+r)^2}{1+\alpha^2r^2} - (1-\alpha^2)\frac{1}{\alpha}\arctan(\alpha r)\right] \\ &\qquad \le |f(z)| \le \frac{1}{1-\alpha^2}\left[\log\frac{1-\alpha^2r^2}{(1-r)^2} - \frac{1+\alpha^2}{2\alpha}\log\frac{1+\alpha r}{1-\alpha r}\right], \end{aligned} \tag{3.9}$$

where $z \in U$, $|z| = r < 1$.

The estimates (3.8) and (3.9) are sharp. The upper and lower bounds are achieved when $\beta = 0$. In this case $\theta_1(0) = 0$, $\theta_0(0) = \pi$ and, respectively, $\omega_1(z) = z$, $\omega_0(z) = iz$. The extremal functions $h_{\alpha,0}$ and $t_{\alpha,0}$ have the following form:

$$h_{\alpha,0}(z) = f_{\alpha,0}(z) = \frac{1}{1-\alpha^2}\left(\log\frac{1-\alpha^2z^2}{(1-z)^2} - \frac{1+\alpha^2}{2\alpha}\log\frac{1+\alpha z}{1-\alpha z}\right), \qquad z \in U\,,$$

$$\begin{aligned} t_{\alpha,0}(z) = \frac{i}{1+\alpha^2}\Bigg(&2\log(1-iz) + \frac{1}{2\alpha i}(\alpha-i)^2\log(1+\alpha z) \\ &- \frac{1}{2\alpha i}(\alpha+i)^2\log(1-\alpha z)\Bigg), \qquad z \in U\,. \end{aligned}$$

The function $t_{\alpha,0}$ can be rewritten as

$$\begin{aligned} t_{\alpha,0}(z) &= \frac{i}{1+\alpha^2}\left(2\log(1-iz) - \log(1-\alpha^2z^2) - \frac{1-\alpha^2}{2i\alpha}\log\frac{1+\alpha z}{1-\alpha z}\right) \\ &= \frac{i}{1+\alpha^2}\left(\log\frac{(1-iz)^2}{1-\alpha^2z^2} + (1-\alpha^2)\frac{1}{\alpha}\arctan(\alpha iz)\right). \end{aligned}$$

Putting $\alpha = \beta = 0$ in (3.1)–(3.4) we obtain known results (see [7]):

Corollary 3.3. *If $f \in P'(0)$, then*

$$\frac{1-r}{1+r} \le |f'(z)| \le \frac{1+r}{1-r}\,, \tag{3.10}$$

$$2\log(1+r) - r \le |f(z)| \le -2\log(1-r) - r \tag{3.11}$$

for $z \in U$, $|z| = r < 1$.

The functions

$$h_{0,0}(z) = -z - 2\log(1-z), \qquad z \in U\,,$$

and

$$t_{0,0}(z) = \lim_{\alpha\to 1} t_{\alpha,0}(z) = i\log(1-iz)^2 - z, \qquad z \in U\,,$$

are respective extremal functions for the upper and lower bounds.

The next corollary is obtained from Theorem 3.1 by putting $\alpha = 0$ and $\beta = 1$ (see [4]).

Corollary 3.4. *If $f \in C_0(1)$, then*

$$\frac{1-r}{(1+r)(1+r^2)} \le |f'(z)| \le \frac{1}{(1-r)^2}\,, \tag{3.12}$$

$$\frac{1}{2}\log\frac{(1+r)^2}{1+r^2} \le |f(z)| \le \frac{r}{1-r} \tag{3.13}$$

for $z \in U$, $|z| = r < 1$.

The functions

$$h_{1,0}(z) = \frac{z}{1-z}, \quad z \in U, \quad \text{and} \quad t_{1,0}(z) = \frac{i}{2}\log\frac{(1-iz)^2}{1-z^2}, \quad z \in U\,,$$

are extremal.

In the limit cases as α tends to 0 or to 1, the bounds (3.8) and (3.9) give sharp results for the classes $C(0)$ and $C(1)$ that agree with (3.10), (3.11) and with (3.12), (3.13) respectively.

The lower bound in (3.9) yields

Corollary 3.5. *If $f \in C(\alpha)$, $\alpha \in (0,1]$, then $f(U)$ contains the disk*

$$|w| < \frac{1}{1+\alpha^2}\left[\log\frac{4}{1+\alpha^2} - (1-\alpha^2)\frac{1}{\alpha}\arctan\alpha\right] \tag{3.14}$$

(see [6]).

The constant on the right hand side of (3.14) is best possible and the function $t_{\alpha,0}$ is extremal.

For the class $C(0)$ the following result is known (see [2]):

Corollary 3.6. *If $f \in C(0)$, then $f(U)$ contains the disk*

$$|w| < 2\log 2 - 1\,.$$

This constant can be obtained from (3.14) by letting $\alpha \to 0$.

If $\alpha = 1$, then Corollary 3.5 reduces to the result obtained by Hengartner and Schober [4]:

Corollary 3.7. *If $f \in C(1)$, then $f(U)$ contains the disk*

$$|w| < \frac{1}{2}\log 2\,.$$

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