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A generalization of the Hahn–Banach theorem

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Abstract. If C is a non-empty convex subset of a real linear space E, $p : E \to \mathbb{R}$ is a sublinear function and $f : C \to \mathbb{R}$ is concave and such that $f \le p$ on C, then there exists a linear function $g : E \to \mathbb{R}$ such that $g \le p$ on E and $f \le g$ on C. In this result of Hirano, Komiya and Takahashi we replace the sublinearity of p by convexity.

N. Hirano, H. Komiya and W. Takahashi gave the following generalization of the well-known Hahn–Banach theorem (Theorem 1 of [2]):

If p is a sublinear function on a linear space E, C is a non-empty convex subset of E and f is a concave functional on C such that $f \le p$ on C, then there exists a linear functional g on E such that $f \le g$ on C and $g \le p$ on E.

The main goal of this paper is to give a new version of the above theorem with "sublinear" replaced by "convex". This result can be derived from an abstract Hahn–Banach theorem due to Rodé [6] (cf. also König [4]) or from the Nikodem theorem [5].

Our proof, based on an idea from [2], is an application of a theorem of Fan (Lemma 1).

In the proof of the main theorem we shall use the following two lemmas.

Lemma 1 (Fan). *Let X be a non-empty compact convex subset of a Hausdorff linear topological space and $\{f_\nu : \nu \in I\}$ a family of lower semicontinuous convex functionals on X with values in $(-\infty, +\infty]$. If for any finite set of indices $\nu_1, \ldots, \nu_n$ and for any non-negative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 1$, there is a $y \in X$ such that*

$$\sum_{i=1}^{n} \lambda_i f_{\nu_i}(y) \le 0\,,$$

then there exists an $x \in X$ such that $f_\nu(x) \le 0$, $\nu \in I$.

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Lemma 2. *If p is a convex function on a real linear space E and $x_0 \in E$, than there are a linear functional f and $c \in \mathbb{R}$ such that*

$$c + f(x) \le p(x) \quad \text{for } x \in E\,,$$
$$c + f(x_0) = p(x_0)\,.$$

Lemma 2 is a special case of the Hahn–Banach theorem. In [2] an analogue of Lemma 2 for sublinear functions was proved using the Markov–Kakutani fixed-point theorem. Our lemma can be derived e.g. from Corollary 11.2, p. 91 of [1].

Z. Kominek ([3], Lemma 1) has obtained a more general result (for a midpoint convex functional on a non-empty algebraically open and convex subset).

Using the above lemmas we obtain the following

Theorem. *Let C be a non-empty convex subset of a real linear space E and let $p : E \to \mathbb{R}$ be a convex function. If $f : C \to \mathbb{R}$ is a concave function satisfying*

$$f(x) \le p(x) \quad \text{for } x \in C\,,$$

then there exists a linear function $g : E \to \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that

$$g(x) + a \le p(x) \quad \text{for } x \in E\,,$$
$$f(x) \le g(x) + a \quad \text{for } x \in C\,.$$

Proof. First assume that $0 \in C$. Let F be the linear topological space $\mathbb{R}^E$ with the Tikhonov topology. Then define

$$J(E) := \left\{ g : E \to \mathbb{R} : g\left(\frac{x+y}{2}\right) = \frac{1}{2}[g(x) + g(y)],\ x, y \in E \right\}$$
$$B := \{g \in J(E) : g \le p \text{ on } E\}\,,$$
$$B_n := \{g \in B : g(0) \ge p(0) - n\} \quad \text{for } n \in \mathbb{N}.$$

By Lemma 2, B is non-empty. We also have $B = \bigcup_{n=1}^{\infty} B_n$, $B_n \subset B_{n+1}$ and the B_n are convex and closed in F for all $n \in \mathbb{N}$.

For each $y \in E$ we have

$$p(0) = p\left(\frac{y-y}{2}\right) \le \frac{1}{2}[p(y) + p(-y)]\,,$$

whence

$$2p(0) - p(-y) \le p(y)\,.$$

Consequently, for every $n \in \mathbb{N}$ the set

$$X_n := \prod_{y \in E} [2p(0) - p(-y) - 2n, p(y)]$$

is non-empty, convex and compact in F.

We also have $B_n \subset X_n$ for $n \in \mathbb{N}$. Indeed, each $g \in B_n$ satisfies

$$g(0) \geq p(0) - n\,,$$
$$2g(0) = g(y) + g(-y)\,, \qquad y \in E\,,$$
$$g(y) \leq p(y)\,, \qquad y \in E.$$

Hence for every $y \in E$

$$2p(0) - 2n - p(-y) \leq 2g(0) - g(-y) = g(y) \leq p(y)\,.$$

Now for every $n \in \mathbb{N}$ we define

$$C_n := \{x \in C : 2f(x) - p(2x) \geq p(0) - n\}\,.$$

It is easy to see that $C = \bigcup_{n=1}^{\infty} C_n$, the C_n are convex and $C_n \subset C_{n+1}$ for $n \in \mathbb{N}$.

Take $x \in C_n$. By Lemma 2 there exists $g \in J(E)$ such that $g \leq p$ on E and $g(x) = p(x)$. Then

$$\begin{aligned} f(x) \leq p(x) = g(x) &= g\Big(\frac{2x+0}{2}\Big) \\ &= \frac{1}{2}[g(2x) + g(0)] \leq \frac{1}{2}[p(2x) + g(0)]\,, \end{aligned}$$

whence

$$g(0) \geq 2f(x) - p(2x) \geq p(0) - n\,.$$

This means that $g \in B_n$; that is, for every $x \in C_n$ there exists $g \in B_n$ such that $g(x) = p(x)$. In particular, $g(y) \leq p(y)$ for all $y \in E$.

Fix $n \in \mathbb{N}$ for which $0 \in C_n$. Define $G_x : B_n \to \mathbb{R}$ for $x \in C_n$ by

$$G_x(g) = f(x) - g(x) \qquad \text{for } g \in B_n\,.$$

It is easy to show that each G_x is convex. Moreover, it is lower semicontinuous, for if $c \in \mathbb{R}$ and $\Pi_x(g) = g(x)$ for $g \in F$, then

$$\begin{aligned} \{g \in B_n : G_x(g) > c\} &= \{g \in B_n : g(x) < f(x) - c\} \\ &= \{g \in B_n : \Pi_x(g) < f(x) - c\} \\ &= B_n \cap \Pi_x^{-1}((-\infty, f(x) - c))\,. \end{aligned}$$

The last set is open in B_n in the Tikhonov topology.

Let $x_1, \ldots, x_m \in C_n$ and let $\lambda_1, \ldots, \lambda_m \geq 0$ be such that $\sum_{i=1}^m \lambda_i = 1$. Put $z := \sum_{i=1}^m \lambda_i x_i$. Then there exists $g \in B_n$ for which $g(x) \leq p(x)$, $x \in E$, and $g(z) = p(z)$. Moreover, we have

$$\begin{aligned} \sum_{i=1}^m \lambda_i G_{x_i}(g) &= \sum_{i=1}^m \lambda_i f(x_i) - \sum_{i=1}^m \lambda_i g(x_i) \leq f\Big(\sum_{i=1}^m \lambda_i x_i\Big) - g\Big(\sum_{i=1}^m \lambda_i x_i\Big) \\ &= f(z) - g(z) \leq f(z) - p(z) \leq 0\,. \end{aligned}$$

Hence by virtue of Lemma 1 there exists $g_n \in B_n$ such that $G_x(g_n) \le 0$ for all $x \in C_n$, i.e.,

$$f \le g_n \quad \text{on } C_n\,, \qquad g_n \le p \quad \text{on } E\,.$$

We put

$$X := \prod_{x\in E} [2f(0) - p(-x), p(x)] \cap J(E)\,.$$

This set is compact, convex and non-empty. In fact, we have

$$f(0) \le g_n(0) = \frac{g_n(x) + g_n(-x)}{2} \le \frac{g_n(x) + p(-x)}{2}\,,$$

hence $2f(0) - p(-x) \le g_n(x) \le p(x)$, whence $g_n \in X$.

Now consider the functions $G_x : X \to \mathbb{R}$, $x \in C$, defined by

$$G_x(g) = f(x) - g(x)\,, \qquad g \in X\,.$$

Fix arbitrary $x_1, \ldots, x_m \in C$ and $\lambda_1, \ldots, \lambda_m \ge 0$ with $\sum_{i=1}^m \lambda_i = 1$. For sufficiently large $n \in \mathbb{N}$ we have $0, x_1, \ldots, x_m \in C_n$ and we can find $g_n \in X$ such that $g_n \le p$ on E and $g_n \ge f$ on C_n. Consequently,

$$\begin{aligned}\sum_{i=1}^m \lambda_i G_{x_i}(g_n) &= \sum_{i=1}^m \lambda_i f(x_i) - \sum_{i=1}^m \lambda_i g_n(x_i)\\ &\le f\Big(\sum_{i=1}^m \lambda_i x_i\Big) - g_n\Big(\sum_{i=1}^m \lambda_i x_i\Big) \le 0\,.\end{aligned}$$

By Lemma 1 again there exists $g_0 \in X$ such that

$$\begin{aligned} g_0(x) &\le p(x) \qquad \text{for } x \in E\,,\\ f(x) &\le g_0(x) \qquad \text{for } x \in C\,.\end{aligned}$$

It is not difficult to see that there are a linear functional $g : E \to \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that

$$g_0(x) = g(x) + a \qquad \text{for } x \in E\,.$$

This ends the first part of the proof.

Now suppose that $0 \notin C$ and take an arbitrary $x_0 \in C$. Let $C_1 := C - x_0$ and define $f_1 : C_1 \to \mathbb{R}$ and $p_1 : E \to \mathbb{R}$ by

$$\begin{aligned} f_1(x) &:= f(x + x_0) \qquad \text{for } x \in C_1\,,\\ p_1(x) &:= p(x + x_0) \qquad \text{for } x \in E\,.\end{aligned}$$

It is easily seen that f_1 is concave, p_1 is convex and

$$f_1(x) \le p_1(x)\,, \qquad x \in C_1\,.$$

Then there exists a linear function $g : E \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$f_1(x) \le g(x) + c\,, \qquad x \in C_1\,,$$

$$g(x) + c \leq p_1(x)\,, \qquad x \in E\,.$$

Consequently,

$$f(x) \leq g(x) - g(x_0) + c\,, \qquad x \in C\,,$$
$$g(x) - g(x_0) + c \leq p(x)\,, \qquad x \in E\,.$$

Setting $a := c - g(x_0)$ completes the proof.

Remark. It is easy to check that Theorem 1 of [2] can be obtained as a corollary to ours.

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