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Strangely sweeping one-dimensional diffusion

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Abstract. Let $X(t)$ be a diffusion process satisfying the stochastic differential equation $dX(t) = a(X(t))\,dW(t) + b(X(t))\,dt$. We analyse the asymptotic behaviour of $p(t) = \operatorname{Prob}\{X(t) \ge 0\}$ as $t \to \infty$ and construct an equation such that $\limsup_{t\to\infty} t^{-1} \int_0^t p(s)\,ds = 1$ and $\liminf_{t\to\infty} t^{-1} \int_0^t p(s)\,ds = 0$.

1. Introduction. In the present paper we investigate the stochastic differential equation

$$dX(t) = a(X(t))\,dW(t) + b(X(t))\,dt\,, \tag{1.1}$$

where $W(t)$ is a Wiener process on $\mathbb{R}$. Assuming that a and b are differentiable bounded functions and $a(x) > 0$ for $x \in \mathbb{R}$, the asymptotic behaviour of the trajectories of $X(t)$ is described by the integrals

$$I_1(x) = \int_{-\infty}^{x} \exp\left(- \int_0^z \frac{2b(u)}{a(u)^2}\,du \right) dz\,,$$

$$I_2(x) = \int_{x}^{\infty} \exp\left(- \int_0^z \frac{2b(u)}{a(u)^2}\,du \right) dz\,.$$

Namely,

(1.2) if $I_1(x) = \infty$ and $I_2(x) = \infty$, then

$$\operatorname{Prob}\{\sup_{t>0} X(t) = \infty\} = \operatorname{Prob}\{\inf_{t>0} X(t) = -\infty\} = 1\,,$$

(1.3) if $I_1(x) < \infty$ and $I_2(x) = \infty$, then

$$\operatorname{Prob}\{\lim_{t\to\infty} X(t) = -\infty\} = 1\,,$$

(1.4) if $I_1(x) = \infty$ and $I_2(x) < \infty$, then

$$\operatorname{Prob}\{\lim_{t\to\infty} X(t) = \infty\} = 1\,,$$

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(1.5) if $I_1(x) < \infty$ and $I_2(x) < \infty$, then

$$\operatorname{Prob}\{\lim_{t\to\infty} X(t) = \infty\} = 1 - \operatorname{Prob}\{\lim_{t\to\infty} X(t) = -\infty\} = \mathbf{M}\frac{I_1(X(0))}{I_1(X(0)) + I_2(X(0))},$$

where $\mathbf{M}X$ denotes the mean value of the random variable X (see [1] for the proof).

Although the trajectories of the process $X(t)$ admit a rather simple asymptotic description, the behaviour of the distribution of $X(t)$ can be complicated. It is well known that under some regularity conditions on a and b the distribution of $X(t)$ has a density for every $t > 0$. Let f_t and g_t be the densities of two solutions of Eq. (1.1). In the next section we check that if $I_1(0) = \infty$ or $I_2(0) = \infty$, then

$$\|f_t - g_t\| \to 0 \quad \text{as } t \to \infty, \tag{1.6}$$

where $\|\cdot\|$ is the norm in $L^1(\mathbb{R})$. This condition means that the asymptotic behaviour of the distribution of $X(t)$ does not depend on the distribution of $X(0)$. From this it follows that if there exists a *stationary* solution of (1.1), i.e., a solution whose distribution does not depend on t, then $f_t \to g$ in $L^1(\mathbb{R})$ as $t \to \infty$, where g is the density of the stationary solution of (1.1) and f_t is the density of a solution $X(t)$ of (1.1). Moreover, in the next section we check that if there is no stationary solution of (1.1), then for every $c > 0$ we have

$$\int_{-c}^{c} f_t\,dx \to 0 \quad \text{as } t \to \infty, \tag{1.7}$$

where f_t is the density of a solution $X(t)$ of (1.1).

From the above results it follows that if $I_1(0) = \infty$ or $I_2(0) = \infty$ and if Eq. (1.1) has no stationary solution, then the asymptotic behaviour of the function

$$p(t) = \operatorname{Prob}\{X(t) \ge c\}$$

does not depend on c and on the initial distribution of $X(0)$. This leads to the following basic question: does the function $p(t)$ have a limit as $t \to \infty$?

Our paper is devoted to answering this question. Section 2 contains basic notations and results used in the paper. In Section 3, using some results of Gushchin and Mikhailov [2] we give a sufficient condition for the existence of this limit. Section 4 contains the main result of the paper. We show that the behaviour of $p(t)$ can be surprisingly chaotic. Namely, we construct an equation such that (1.6) and (1.7) hold and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t p(s)\,ds = 1, \tag{1.8}$$

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t p(s)\,ds = 0\,. \tag{1.9}$$

In this example $a(x) = 1$ and $b(x) \to 0$ as $|x| \to \infty$. It is interesting that even a small drift coefficient $b(x)$ can cause the synchronous oscillation of molecules between $+\infty$ and $-\infty$.

2. Preliminaries. In this section we assume that $a \in C^3(\mathbb{R})$, $b \in C^2(\mathbb{R})$ and $a(x) > \alpha$, where α is a positive constant and $C^n(\mathbb{R})$ is the space of n times differentiable bounded functions whose derivatives of order $\le n$ are continuous and bounded. It is well known that under these assumptions for every $t > 0$ each solution $X(t)$ of Eq. (1.1) has a density $u(t,x)$ and the function u satisfies the Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}a(x)^2 u\right) - \frac{\partial}{\partial x}(b(x)u), \qquad (t,x) \in (0,\infty)\times\mathbb{R}\,. \tag{2.1}$$

Let the distribution of $X(0)$ have a density f. Then the solution $u(t,x)$ of Eq. (2.1) can be written in the form

$$u(t,x) = \int_{\mathbb{R}} K(t,x,y) f(y)\,dy\,, \tag{2.2}$$

where the kernel K is independent of the initial density f and $\|u(t,\cdot) - f\|_{L^1} \to 0$ as $t \to 0$.

Eq. (2.1) generates a semigroup $\{P^t\}_{t\ge0}$ of Markov operators on $L^1(\mathbb{R})$ defined by

$$P^0 f = f, \qquad (P^t f)(x) = \int_{\mathbb{R}} K(t,x,y) f(y)\,dy\,, \qquad t > 0\,. \tag{2.3}$$

We recall that a linear operator P on $L^1(\mathbb{R})$ is called a *Markov operator* if $P(D) \subset D$, where D is the set of all densities, i.e., $D = \{\, f \in L^1(\mathbb{R}) : f \ge 0,\ \int f\,dx = 1\}$. In [4] it is proved that if

$$\int_{-\infty}^{\infty} \exp\left(-\int_0^x \frac{2b(y)}{a(y)^2}\,dy\right) dx = \infty\,, \tag{2.4}$$

then for any two densities f and g we have

$$\|P^t f - P^t g\|_{L^1} \to 0 \qquad \text{as } t \to \infty\,. \tag{2.5}$$

A semigroup $\{P^t\}_{t\ge0}$ of Markov operators on $L^1(\mathbb{R})$ is called *sweeping* if for every $c > 0$ and for every $f \in L^1(\mathbb{R})$ we have

$$\int_{-c}^{c} P^t f\,dx \to 0 \qquad \text{as } t \to \infty\,. \tag{2.6}$$

The notion of sweeping was introduced by Komorowski and Tyrcha [3]. They proved that if $\{P^t\}_{t\geq 0}$ is a semigroup of integral Markov operators, if $\{P^t\}_{t\geq 0}$ has no invariant density and if there exists a positive locally integrable function f_* invariant with respect to $\{P^t\}_{t\geq 0}$, then this semigroup is sweeping (see [3] for details). Using this criterion we can prove the following.

LEMMA 1. *The semigroup* $\{P^t\}_{t\geq 0}$ *generated by Eq.* (2.1) *is sweeping iff*

$$\int_{-\infty}^{\infty} \exp\left(\int_0^x \frac{2b(y)}{a(y)^2}\,dy\right) dx = \infty\,. \tag{2.7}$$

Proof. Let

$$f_*(x) = \frac{1}{a(x)^2}\exp\left(\int_0^x \frac{2b(y)}{a(y)^2}\,dy\right).$$

Then f_* is a positive locally integrable function such that $P^t f_* = f_*$ for every $t \geq 0$. Since a is a bounded function and $a(x) > \alpha > 0$, (2.7) holds iff $\int f_*\,dx = \infty$. If $\int f_*\,dx < \infty$, then $f = f_*/\|f_*\|_{L^1}$ is an invariant density which does not satisfy (2.6). If $\int f_*\,dx = \infty$ we check that there is no invariant density. Suppose, on the contrary, that g is one. Then g satisfies the differential equation

$$\frac{d^2}{dx^2}\left(\frac{1}{2}a(x)^2 g(x)\right) - \frac{d}{dx}(b(x)g(x)) = 0\,. \tag{2.8}$$

A solution of (2.8) is given by

$$g(x) = f_*\Big(c_1 + c_2 \int_0^x \psi(y)\,dy\Big)\,, \tag{2.9}$$

where

$$\psi(x) = \exp\left(-\int_0^x \frac{2b(y)}{a(y)^2}\,dy\right)$$

and c_1, c_2 are constants. Since $\int f_*\,dx = \infty$, the function g can be nonnegative and integrable only if

$$g(x) = cf_* \int_{-\infty}^x \psi(y)\,dy \quad \text{or} \quad g(x) = cf_* \int_x^{\infty} \psi(y)\,dy\,,$$

where c is a positive constant. We consider the first case, the second one is analogous. Since a and b are bounded and $a(x) \geq \alpha > 0$, there exists $\gamma > 0$ such that if $|x-y| \leq 1$, then $\psi(y)/\psi(x) \geq \gamma$. This implies that

$$g(x) \geq c\gamma a(x)^{-2} \geq c\gamma(\sup a(x))^{-2} > 0$$

for $x \in \mathbb{R}$. Consequently, g is not a density. This completes the proof that (2.7) implies sweeping. ■

Conditions (1.6) and (1.7) mentioned in the introduction follow from the analogous conditions for the semigroup $\{P^t\}_{t\ge 0}$, because for every $t > 0$ each solution of (1.1) has a density. Let

$$p(t) = \int_c^\infty u(t,x)\,dx = \mathrm{Prob}\{X(t) \ge c\}\,.$$

If (2.4) and (2.7) hold, then $\lim_{t\to\infty} p(t)$ does not depend on c and on the distribution of $X(0)$. Now (1.3) and (1.4) immediately yield.

Corollary 1. *If $I_1(0) < \infty$ and $I_2(0) = \infty$, then $p(t) \to 0$ as $t \to \infty$. If $I_1(0) = \infty$ and $I_2(0) < \infty$, then $p(t) \to 1$ as $t \to \infty$.*

Another consequence of condition (2.5) and the sweeping property of the semigroup $\{P^t\}_{t\ge 0}$ is the following

Corollary 2. *Assume that for some constant c we have $a(x) = a(c-x)$ and $b(x) = -b(c-x)$ for all x. Suppose that $I_1(0) = \infty$ and (2.7) holds. Then $\lim_{t\to\infty} p(t) = 1/2$.*

We will also need the following time-homogeneous version of the Kolmogorov equation (see [5]). Let $\varphi \in C^2(\mathbb{R})$ and let $u(t,x)$ be the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} a(x)^2 \frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} \tag{2.10}$$

with the initial condition $u(0,x) = \varphi(x)$. Then $u(t,x) = \mathbf{M}\varphi(X(t))$, where $X(t)$ is the solution of (1.1) with the initial condition $X(0) = x$.

3. Convergence of $p(x)$**.** The main result of this section is the following

Theorem 1. *Let $a \in C^3(\mathbb{R})$, $b \in C^2(\mathbb{R})$, $a(x) > \alpha > 0$ for $x \in \mathbb{R}$ and let*

$$B(x) = \int_0^x \frac{b(y)}{a(y)^2}\,dy$$

be a bounded function and

$$g(x) = \int_0^x e^{-2B(y)}\,dy\,.$$

Assume that the limits

$$\lim_{T\to\infty} \frac{1}{g(T)} \int_0^T 2(g'(x)a(x)^2)^{-1}\,dx = \beta^2\,,$$

$$\lim_{T\to-\infty} \frac{1}{g(T)} \int_0^T 2(g'(x)a(x)^2)^{-1}\,dx = \gamma^2$$

exist, where $\beta > 0$ *and* $\gamma > 0$. *Then for every solution* $X(t)$ *of* (1.1) *and* $c \in \mathbb{R}$ *the function* $p(t) = \operatorname{Prob}\{X(t) \ge c\}$ *satisfies*

$$\lim_{t\to\infty} p(t) = \frac{\beta}{\beta+\gamma}\,. \tag{3.1}$$

The proof of Theorem 1 is based on the following theorem.

THEOREM 2 (Gushchin, Mikhailov [2]). *Let* $q \in C^1(\mathbb{R})$ *and* $q(x) \ge \alpha > 0$ *for* $x \in \mathbb{R}$. *Let* $u(t,x)$ *be the solution of the equation*

$$q(x)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{3.2}$$

with the initial condition $u(0,x) = \varphi(x)$, *where* φ *is a continuous bounded function. Assume that the limits*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T q(s)\,ds = \beta^2\,,$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 q(s)\,ds = \gamma^2\,,$$

$$\lim_{T\to\infty} \frac{1}{T(\beta+\gamma)}\int_{-T/\gamma}^{T/\beta} \varphi(s)q(s)\,ds = A$$

exist, where $\beta > 0$ *and* $\gamma > 0$. *Then* $u(t,x) \to A$ *as* $t \to \infty$ *for each* $x \in \mathbb{R}$.

Proof of Theorem 1. Since B is a bounded function, conditions (2.4) and (2.7) hold. This implies that the limit (3.1) does not depend on the initial condition $X(0)$ and on c. Let $X(t)$ be the solution of Eq. (1.1) with the initial condition $X(0) = 0$, and $Y(t) = g(X(t))$. Since the function g satisfies the equation

$$\tfrac{1}{2}a(x)^2 g''(x) + b(x)g'(x) = 0\,,$$

Itô's formula implies

$$\begin{aligned} dY(t) &= [b(X(t))g'(X(t)) + \tfrac{1}{2}a(X(t))^2 g''(X(t))]\,dt \\ &\quad + a(X(t))g'(X(t))\,dW(t) \\ &= a(X(t))g'(X(t))\,dW(t)\,. \end{aligned}$$

Let $\overline{a}(x) = g'(g(x)^{-1})a(g(x)^{-1})$. Then $\overline{a} \in C^3(\mathbb{R})$ and the process $Y(t)$ satisfies the stochastic equation $dY(t) = \overline{a}(Y(t))\,dW(t)$, $Y(0) = 0$. Let $\varphi \in C^2(\mathbb{R})$ be such that $\varphi(x) = 1$ for $x > 0$ and $\varphi(x) = 0$ for $x \le -1$. Then $\mathbf{M}\varphi(Y(t)) = u(t,0)$, where $u(t,x)$ is the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\overline{a}(x)^2\frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

with the initial condition $u(0,x)=\varphi(x)$ (see (2.10)). Let $q(x)=2\overline{a}(x)^{-2}$. From (3.3) it follows that u is the solution of Eq. (3.2) with the initial condition $u(0,x)=\varphi(x)$. It is easy to check that q and φ satisfy the assumptions of Theorem 2 and $A=\beta/(\beta+\gamma)$. Consequently,

$$\lim_{t\to\infty}\mathbf{M}\varphi(g(X(t)))=\lim_{t\to\infty}\mathbf{M}\varphi(Y(t))=\frac{\beta}{\beta+\gamma}\,. \tag{3.4}$$

Since the semigroup (2.3) is sweeping, we have

$$\lim_{t\to\infty}\mathrm{Prob}\{|X(t)|\le c\}=0 \tag{3.5}$$

for every $c>0$. From (3.4) and (3.5) we obtain

$$\lim_{t\to\infty}p(t)=\frac{\beta}{\beta+\gamma}\,,$$

because $\varphi(g(x))\to1$ as $x\to\infty$ and $\varphi(g(x))\to0$ as $x\to-\infty$. ■

One of the implications of Theorem 2 is the following proposition.

PROPOSITION 1. *Let a and b be functions satisfying the assumptions of Theorem* 1 *and let* $\overline{B}(x)=B(x)-\frac12\log a(x)$. *Assume that* $\lim_{x\to\infty}\overline{B}(x)=r$ *and* $\lim_{x\to-\infty}\overline{B}(x)=s$. *Then*

$$\lim_{t\to\infty}p(t)=\frac{e^{2r}}{e^{2r}+e^{2s}}\,. \tag{3.6}$$

Proof. Since $B(x)=\overline{B}(x)+\frac12\log a(x)$, the function g is given by the formula

$$g(x)=\int_0^x\frac{1}{a(y)}e^{-2\overline{B}(y)}\,dy\,.$$

This implies that

$$\beta^2=\lim_{T\to\infty}\int_0^T\frac{2}{a(x)}e^{2\overline{B}(x)}\,dx\Big/\int_0^T\frac{1}{a(x)}e^{-2\overline{B}(x)}\,dx\,.$$

Since $\int_0^\infty\frac{1}{a(x)}\,dx=\infty$ and $\lim_{x\to\infty}\overline{B}(x)=r$, we have $\beta^2=2e^{4r}$. Analogously $\gamma^2=2e^{4s}$. Finally, (3.6) follows from (3.1). ■

4. Example. In this section we construct a function $b\in C^2(\mathbb{R})$ such that every solution $X(t)$ of the stochastic equation

$$dX(t)=dW(t)+b(X(t))\,dt \tag{4.1}$$

satisfies conditions (1.8) and (1.9). We check these conditions only for the solution which satisfies the initial condition $X(0)=0$ and for $c=0$, because (1.8) and (1.9) imply that the semigroup (2.3) is sweeping and satisfies (2.5).

The function $b(x)$ will be the limit of some sequence of functions $b_n \in C^2(\mathbb{R})$, $n = 2, 3, \ldots$ Set

$$I_1^n = \int_{-\infty}^{0} \exp\Big(-\int_0^z 2b_n(u)\,du\Big)\,dz\,,$$

$$I_2^n = \int_{0}^{\infty} \exp\Big(-\int_0^z 2b_n(u)\,du\Big)\,dz\,.$$

Let $X^n(t)$, $n = 2, 3, \ldots$, be the solution of the stochastic equation

$$dX^n(t) = dW(t) + b_n(X^n(t))\,dt \tag{4.2}$$

with the initial condition $X^n(0) = 0$.

We now define inductively a sequence of functions $\{b_n\}$. Let $b_2 \in C^2(\mathbb{R})$ be a function such that $b_2(x) = 1$ for $x \ge 0$, $b_2(x) = -\alpha_2 = -1/8$ for $x \le -1$ and b_2 is increasing in $[-1, 0]$. Then $I_2^2 = 1/2$ and $I_1^2 \ge 1/(2\alpha_2)$. From (1.5) it follows that

$$\mathrm{Prob}\{\lim_{t\to\infty} X^2(t) = \infty\} \ge 1 - \alpha_2\,.$$

This implies that there exists $t_2 \ge 0$ such that

$$\mathrm{Prob}\{\inf_{t\ge t_2} X^2(t) \ge 0\} \ge 1 - \frac{1}{4}\,. \tag{4.3}$$

Denote the set in braces in (4.3) by F_2 and let

$$F_{2,j} = \{\omega \in F_2 : \max_{0\le t\le 2t_2} |X^2(t)| \le j\}\,.$$

From (4.3) it follows that there exists a positive integer j_2 such that $\mathrm{Prob}\{F_{2,j_2}\} > 1/2$. Assume that b_{n-1}, j_{n-1} and t_{n-1} have already been defined. If n is odd we set $b_n(x) = b_{n-1}(x)$ for $x \le j_{n-1}$ and $b_n(x) = \alpha_n$ for $x \ge 1 + j_{n-1}$, where

$$\alpha_n = (8nI_1^{n-1})^{-1} e^{-2(1+j_{n-1})}\,. \tag{4.4}$$

We assume that $b_n \in C^2(\mathbb{R})$ and b_n is decreasing in $[j_{n-1}, 1 + j_{n-1}]$. Since $I_1^n = I_1^{n-1}$, from (1.5) it follows that

$$\mathrm{Prob}\{\lim_{t\to\infty} X^n(t) = \infty\} \le I_1^{n-1}/I_2^n \le 1/(4n)\,.$$

This implies that there exists $t_n > (n-1)t_{n-1}$ such that

$$\mathrm{Prob}\{\sup_{t\ge t_n} X^n(t) \le 0\} \ge 1 - 1/(2n)\,. \tag{4.5}$$

Denote the set in braces in (4.5) by F_n. Then there exists an integer j_n such that $j_n > j_{n-1}$ and the probability of the event

$$F_{n,j_n} = \{\omega \in F_n : \max_{0\le t\le nt_n} |X^n(t)| \le j_n\}$$

is greater than $1-1/n$. Analogously, if n is even, then $b_n \in C^2(\mathbb{R})$ is decreasing in $[-1-j_{n-1}, -j_{n-1}]$, $b_n(x) = b_{n-1}(x)$ for $x \geq -j_{n-1}$ and $b_n(x) = -\alpha_n$ for $x \leq -1-j_{n-1}$, where

$$\alpha_n = (8nI_2^{n-1})^{-1} e^{-2(1+j_{n-1})} .$$

The constants t_n and j_n are chosen in such a way that $t_n > (n-1)t_{n-1}$, $j_n > j_{n-1}$ and the probability of the event

$$F_{n,j_n} = \{ \inf_{t \geq t_n} X^n(t) \geq 0 \text{ and } \max_{0 \leq t \leq nt_n} |X^n(t)| \leq j_n \}$$

is greater than $1-1/n$. The functions b_n can be chosen in such a way that the sequences $\{b_n'\}$ and $\{b_n''\}$ are uniformly bounded. Let $b(x) = \lim_{n\to\infty} b_n(x)$. Then $b \in C^2(\mathbb{R})$. Since $\alpha_n \to 0$ as $n \to \infty$, $b(x) \to 0$ as $|x| \to \infty$. Let $X(t)$ be the solution of Eq. (4.1) with the initial condition $X(0) = 0$. Since $b(x) = b_n(x)$ for $|x| \leq j_n$, we have $X(t)(\omega) = X^n(t)(\omega)$ for $t \in [0, nt_n]$ and $\omega \in F_{n,j_n}$ (see [1]). This gives

$$\mathrm{Prob}\{(-1)^n X(t) \geq 0 \text{ for } t \in [t_n, nt_n]\} \geq \mathrm{Prob}\{F_{n,j_n}\} \geq 1 - 1/n .$$

Thus $p(t) \geq 1 - 1/n$ for even n and $t \in [t_n, nt_n]$, and $p(t) \leq 1/n$ for odd n and $t \in [t_n, nt_n]$. The last inequalities imply (1.8) and (1.9).

References

[1] I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*, Springer, Berlin 1972.

[2] A. K. Gushchin and V. P. Mikhailov, *The stabilization of the solution of the Cauchy problem for a parabolic equation with one space variable*, Trudy Mat. Inst. Steklov. 112 (1971), 181–202 (in Russian).

[3] T. Komorowski and J. Tyrcha, *Asymptotic properties of some Markov operators*, Bull. Polish Acad. Sci. Math. 37 (1989), 221–228.

[4] R. Rudnicki, *Asymptotical stability in L^1 of parabolic equations*, J. Differential Equations, in press.

[5] Z. Schuss, *Theory and Applications of Stochastic Differential Equations*, Wiley, New York 1980.

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