Some division theorems for vector fields

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Abstract. This paper is concerned with the problem of divisibility of vector fields with respect to the Lie bracket \([X,Y]\). We deal with the local divisibility. The methods used are based on various estimates, in particular those concerning prolongations of dynamical systems. A generalization to polynomials of the adjoint operator \(\text{ad}(X)\) is given.

0. Introduction. The Lie bracket of differentiable vector fields on a smooth manifold is one of the fundamental operations not only in differential geometry. We deal with the following problem of division:

Given vector fields \(X, Z\), does there exist a vector field \(Y\) such that \([X,Y] = Z\)?

The problem has been considered only for local vector fields and the full and positive answer is known whenever \(X\) has a nonvanishing germ. In this case \(X\) has local representation \(\partial/\partial x_1\) and the “quotient” \(Y\) can be taken to be

\[ Y(x_1, \ldots, x_n) = \int_{-\delta}^{x_1} Z(t, x_2, \ldots, x_n) \, dt \]

for \(\|x\| = \sup |x_i| \leq \delta\). This fact has been broadly exploited in papers concerning the well-known Pursell–Shanks theorem and its generalizations.

Since our problem will also be of local character it can be assumed that \(X\) and \(Z\) are vector fields defined in a neighbourhood of the origin 0 of \(\mathbb{R}^n\) and the equality \([X,Y] = Z\) is meant in the sense of germs, that is, there exists a neighbourhood \(U\) of 0 in which it holds.

Thus \(X, Y, Z\) will be elements of the Lie algebra \(\mathfrak{X}(\mathbb{R}^n)\) of local \(\mathcal{C}^\infty\) vector fields defined near the origin of \(\mathbb{R}^n\). In view of the above, the question remains open only for homogeneous vector fields \(X, Z\), that is, with \(X(0) = Z(0) = 0\). From now on the notation \(\mathfrak{X}(\mathbb{R}^n)\) will be used for the subalgebra of homogeneous elements.

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In order to justify what we deal with in the section that follows, let us see how the flow \( \Psi_t \) of a given vector field \( X \) can be involved in the problem of divisibility by \( X \).

For any field \( Z \) the transfer of \( Z \) along the trajectories of \( X \) is defined by

\[
(\Psi_t)_* Z = (D\Psi_{-t} \circ \Psi_t) Z \circ \Psi_t,
\]

i.e.

\[
(\Psi_t)_* Z(x) = D\Psi_{-t}(\Psi_t(x)) Z(\Psi_t(x)) .
\]

The Lie bracket \([Z, X]\) is just the infinitesimal version of that and we have

\[
[Z, X] = \left. \frac{d}{dt} \right|_{t=0} (\Psi_t)_* Z .
\]

More generally,

\[
\frac{d}{dt} (\Psi_t)_* Z = [(\Psi_t)_* Z, X] .
\]

Setting \( Y_t = (\Psi_t)_* Z \) we can write

\[
Y_t' = [Y_t, X] \quad \left( Y' = \frac{d}{dt} Y \right) .
\]

This gives

\[(0.1) \quad Z = - \left[ \int_0^t Y_s ds, X \right] + Y_t \]

since \( Y_0 = Z \). Without loss of generality we can assume that \( X \) is complete so the range of \( t \) is \((-\infty, \infty)\). If necessary we can replace \( X \) by \( fX \) where \( f \) is a \( C^\infty \) function which is 1 in a neighbourhood of 0 and has a compact support in the set where \( X \) is defined. Suppose that

1° \( Y_t \to 0 \) as \( t \to \infty \),

2° the integral

\[(0.2) \quad Y(x) = \int_0^\infty Y_t(x) dt \]

is convergent and \( Y \) is \( C^\infty \) in a neighbourhood of 0. Then \( Z = [X, Y] \) so \( Y \) is a solution to the problem.

Since \( X(0) = 0 \) we have \( \Psi_t(0) = 0 \) for all \( t \). If \( x = 0 \) is asymptotically stable then \( \Psi_t(x) \to 0 \) as \( t \to \infty \) for small \( \|x\| \). As also \( Z(0) = 0 \), it follows that \( Z(\Psi_t(x)) \to 0 \), but what we need is that \( D\Psi_{-t}(\Psi_t(x)) Z(\Psi_t(x)) \) and all its \( x \)-derivatives converge to 0 as \( t \to \infty \), uniformly in \( x \). The study of this question will be the subject of the next section.
1. **Some bounds to flows.** Consider the system of differential equations
\begin{equation}
(1.1) \quad x' = X(x) \quad (x' = dx/dt)
\end{equation}
where \( x, X(x) \) are \( n \)-vectors, \( X \) is \( C^\infty \) in a neighbourhood of \( x = 0 \) and \( X(0) = 0 \). Thus \( X \) can be written
\begin{equation}
(1.1)'
X(x) = Ax + h(x)
\end{equation}
with \( A = DX(0) \) and \( \|h(x)\| \leq L\|x\|^2 \). We may assume that the Lipschitz constant \( L \) is global, so that solutions to (1.1) are defined globally.

Assume that all the eigenvalues \( \lambda_i \) of \( A \) satisfy \( \text{Re} \lambda_i < 0 \) for \( i = 1, \ldots, n \) (for short: \( \text{Re} \lambda < 0 \)). It is known that under this condition there exist positive constants \( K \) and \( c \) such that
\[ \|e^{tA}\| \leq Ke^{-ct} \quad \text{for } t > 0, \]
and \( \delta > 0 \) such that
\begin{equation}
(1.2)
\|\Psi_t(x)\| \leq Ke^{-ct/2}\|x\| \quad \text{for } \|x\| \leq \delta.
\end{equation}
Here \( \Psi_t(x) \) is the solution of (1.1) passing through \( x \) at \( t = 0 \) (the flow of \( X \)). For the constant \( c \) we may take any number \( < \min(\text{Re} \lambda) \) (this is easily seen by writing \( A \) in Jordan canonical form).

**Lemma 1.1.** If there is a bound
\[ \|\Psi_t(x)\| \leq Ke^{c(t)}\|x\| \quad \text{for } \|x\| \leq \delta, \quad t \geq 0, \]
with \( K \) a constant and \( c(t) \) depending only on the eigenvalues of \( A \) and not on their multiplicities (as in the above case), then the derivatives \( D^k\Psi_t(x) \), \( k = 1, 2, \ldots \) also have similar bounds with the same \( \delta \) and \( c(t) \) and different constants \( K_k \).

**Proof.** Consider the following variational equation (\( k \)th prolongation of (1.1) with respect to \( x \)):
\begin{equation}
(1.3)
\begin{cases}
x' = X(x), \\
\xi_1' = DX(x)\xi_1, \\
\xi_2' = D^2X\xi_1\xi_1 + DX\xi_2, \\
\cdots \\
\xi_k' = \sum_{s=1}^{k} D^sX \sum_{\alpha_1+\cdots+\alpha_s=k, \alpha_i>0} \xi_{\alpha_1} \cdots \xi_{\alpha_s}.
\end{cases}
\end{equation}
with \( \xi_{\alpha} \in \mathbb{R}^n \) for \( \alpha = 1, \ldots, k \). With brief notation \( (x, \xi'_1, \ldots, \xi'_k) = F(x, \xi_1, \ldots, \xi_k) \) the Hessian of this equation, i.e. \( DF(0) \), is of the form
\begin{equation}
\begin{pmatrix}
A & A & \cdots \\
A & A & \cdots \\
& & & & A
\end{pmatrix}
\end{equation}
(of dimension \( (k+1)n \)).
Thus $DF(0)$ has the same eigenvalues as $A$.

For any constant vector $v \in \mathbb{R}^n$ the system
\[
(\Psi_t(x), D\Psi_t(x)v, \ldots, D^k\Psi_t(x)v^k)
\]
is a solution to (1.3) passing through $(x, v, 0, \ldots, 0) \in \mathbb{R}^{(k+1)n}$. In fact,
\[
(D^k\Psi_t v^k)' = (D^k\Psi_t)' v^k = D^k\Psi_t v^k = D^k(X \circ \Psi_t)v^k
\]
and $(D^{\alpha_1}\Psi_t \ldots D^{\alpha_k}\Psi_t)v^k = (D^{\alpha_1}\Psi_t v^{\alpha_1}) \ldots (D^{\alpha_k}\Psi_t v^{\alpha_k})$. Therefore, if a bound $\|\Psi_t(x)\| \leq k e^{\xi(t)} \|x\|$ holds for $\|x\| \leq \delta$ and $t \geq 0$, then
\[
\|D^l\Psi_t(x)v^l\| \leq K_1 e^{\xi(t)} \|(x, v, 0, \ldots, 0)\|,
\]
for all $\|x\| \leq \delta$ and any $\|v\| \leq 1$. This gives $\|D^l\Psi_t\| \leq K_1 e^{\xi(t)}$.

**Lemma 1.2.** We have
\[ |\det D\Psi_t(x)| \geq M e^{(\operatorname{tr} A)t} \quad \text{for } \|x\| \leq \delta, \]
with a positive constant $M$.

**Proof.** Set $\Delta_t(x) = \det D\Psi_t(x)$. Then $\Delta_{t+s}(x) = \Delta_t(\Psi_s(x))$. Hence
\[
\Delta_t(x) = \Delta_0(\Psi_0(x)) \Delta_s(x).
\]
A routine computation leads to $\Delta_0'(x) = \operatorname{tr} DX(x)$ and finally
\[ \Delta_s(x) = \exp \int_0^t \operatorname{tr} DX(\Psi_s(x)) \, ds, \]
since $\Delta_0(x) = 1$. By applying (1.1)' this can be written as
\[
\Delta_s(x) = e^{(\operatorname{tr} A)t} \exp \int_0^t \operatorname{tr} Dk(\Psi_s(x)) \, ds.
\]
Since
\[
|\operatorname{tr} Dk(\Psi_s(x))| \leq C \|\Psi_s(x)\|^2 \leq C\delta K e^{-ct},
\]
the integral $\int_0^t \operatorname{tr} Dk(\Psi_s(x)) \, ds$ is bounded from below by $-C\delta K/c$. Thus we can take $M = \exp(-C\delta K/c)$.

**Lemma 1.3.** There are constants $K_k$ and $L_k$ such that
\[ |D^k\Psi_t(x)| \leq K_k e^{-ct/2} \]
and
\[
|D^k[X(\Psi_t(x))]| \leq L_k e^{(k+1)at},
\]
where $a = -\operatorname{tr} A - (n - 1)c$ and $\|x\| \leq \delta$.

**Proof.** The bounds (1.5) follow immediately from Lemma 1.1 with reference to (1.2).
For (1.6), from the identity $\Psi_t(\Psi_t(x)) = x$ it follows that $D\Psi_t(\Psi_t(x))$ is equal to the inverse matrix to $D\Psi_t(x)$. In view of (1.5) and Lemma 1.2 the elements of $(D\Psi_t(x))^{-1}$ are majorized in absolute value by $e^{(-\text{tr} A - (n-1)c)t} (= e^{at})$

up to a constant multiplicative factor.

Now from $(D\Psi_t)^{-1} \circ D\Psi_t = I$ we get

$$D(D\Psi_t)^{-1}D\Psi_t - (D\Psi_t)^{-1}D^2\Psi_t = 0,$$

which gives

$$\|D(D\Psi_t)^{-1}\| \leq L_1 e^{2at},$$

and (1.6) follows by induction.

We denote by $X_m(\mathbb{R}^n)$ the space of local vector fields, $m$-flat at $0$.

**Theorem 1.4.** Suppose $X$ is as above and $Z \in X_m(\mathbb{R}^n)$. If $(k+1)a - mc/2 < 0$ then there exists a $C^k$ vector field $Y$ in a neighbourhood of $0$ such that $[X, Y] = Z$.

**Proof.** $Z$ being $m$-flat satisfies $\|D^k Z(x)\| \leq M_k \|x\|^{m-k}$ for $0 \leq k \leq m - 1$ and it is bounded for $k \geq m$ when $\|x\| \leq \delta$. We have

$$\|D^k(D\Psi_t^{-1}(\Psi_t(x))Z(\Psi_t(x)))\| \leq \sum_{r+s=k} \|D^r(D\Psi_t^{-1} \circ \Psi_t)\| \|D^s(Z \circ \Psi_t)\|.$$

By (1.2) and (1.5)

$$\|(D^u Z) \circ \Psi_t\| \leq e^{-(m-u)ct/2} \quad \text{for } u \leq m - 1,$$

and the left hand side is bounded for $u \geq m$. Here and below, $\text{\leq}$ indicates that the bound holds up to a multiplicative constant.

As the other term in (1.7) is bounded by $e^{-uct/2}$ we get

$$\|D^s(Z \circ \Psi_t)\| \leq e^{-mct/2}$$

for all $s$ since $u \geq m$. In view of (1.6)–(1.8) we have

$$\|D^k(D\Psi_t^{-1} \circ \Psi_t)Z \circ \Psi_t\| \leq e^{((k+1)a - mc/2)t}.$$

Suppose $(k+1)a - mc/2 < 0$ for a positive integer $k$. Then the integral

$$F(X, Z) = \int_0^\infty (\Psi_t)_* Z \, dt \quad \left( = \int_0^\infty (\exp tX)_* Z \, dt \right)$$

is a vector field of class $C^k$ in the ball $\|x\| \leq \delta$.

Since clearly $\|\Psi_t\| \to 0$ as $t \to \infty$, we get by (0.1)

$$[X, F(X, Z)] = Z,$$

which was to be proved.
2. Divisibility by linear vector fields. Suppose \( X = Ax \). Then \( \Psi_t(x) = e^{tA}x \). As previously we assume that \( A \) satisfies \( \text{Re} \lambda < 0 \).

Let \( c \) be any constant \( < \min(|\text{Re} \lambda|) \) and \( b \) any constant \( > \max(|\text{Re} \lambda|) \). Then

\[
\|e^{ta}\| \leq Ke^{-ct}, \quad \|e^{-tA}\| \leq Le^{bt}, \quad t \geq 0.
\]

We call

\[
d(X) = \frac{\max(|\text{Re} \lambda|)}{\min(|\text{Re} \lambda|)}
\]

the dispersion of \( X \). Obviously \( b/c > d(X) \).

**Theorem 2.1.** Suppose that \( Z \) is \( m \)-flat at \( x = 0 \) and \( m \geq d(X) + 1 \). Then \( Z \) is divisible by \( X \) with a quotient \( F(X,Z) \) defined by (1.9).

**Proof.** Now

\[
(\Psi_t)_*Z(x) = e^{-tA}Z(e^{tA})
\]

and

\[
D^k((\Psi_t)_*Z(x)) = e^{-tA}D^kZ(e^{tA})e^{ktA}, \quad k \geq 0.
\]

Exploiting the \( m \)-flatness of \( Z \) as in the proof of Theorem 1.4 we come to the following estimate:

\[
\|D^k((\Psi_t)_*Z(x))\| \leq e^{(b-mc)t}, \quad k = 0, 1, \ldots
\]

The constants \( b \) and \( c \) may be taken such that \( b/c < d(X) + 1 \). It follows that \( b - mc < 0 \) for \( m \geq d(X) + 1 \). Consequently, the integral (1.9) converges uniformly together with all its derivatives. Thus \( F(X, Z) \) is a \( C^\infty \) vector field in any ball contained in the domain of \( Z \).

In particular, if \( X = x \) then \( d(X) = 1 \) and taking \( m = 2 \) we conclude from Theorem 2.1 that each \( Z \) from \( X_2(\mathbb{R}^n) \) is divisible by \( X \).

3. Divisibility by means of linearization. Consider again the general case \( X = Ax + h(x) \) as in (1.1). The field \( X_0 = Ax \) is called the linearization of \( X \) at 0. From now on the vector field \( X \) will be thought of locally as the germ at 0 of a smooth map \( X : \mathbb{R}^n \to \mathbb{R}^n \).

Suppose that \( X \) is \( C^\infty \)-equivalent to its linearization \( X_0 \), that is, there exists a \( C^\infty \)-diffeomorphism \( f \) of \( \mathbb{R}^n \), with \( f(0) = 0 \), such that \( f_*X = X_0 \) in a neighbourhood of 0.

For a given \( Z \) set \( Z_0 = f_*Z \) and assume that there is a \( Y_0 \) such that \( Z_0 = [Y_0, X_0] \). This means

\[
f_*Z = [Y_0, f_*X] = f_*[(f^{-1})_*Y_0, X].
\]

Hence \( Z = [Y, X] \) with \( Y = (f^{-1})_*Y_0 \), and we obtain

**Lemma 3.1.** If \( f_*Z \) is divisible by the linearization of \( X \) then \( Z \) is divisible by \( X \).
Note that the transformation $f^*$ does not change the order of flatness of $Z$.

Which (germs of) vector fields are linearizable? The answer is: almost all. This can be concluded from the following theorems of Sternberg:

Either of the conditions below implies that a vector field $X$ with $X(0) = 0$ is $C^\infty$-equivalent to its linearization $DX(0)x$.

(i) Each eigenvalue $\lambda$ of $DX(0)$ satisfies $\operatorname{Re} \lambda < 0$ and

\begin{equation}
X(x) = DX(0)x + o(x^\infty).
\end{equation}

(ii) Each eigenvalue $\lambda_i$ ($i = 1, \ldots, n$) satisfies

\begin{equation}
\lambda_i \neq m_1 \lambda_1 + \ldots + m_n \lambda_n
\end{equation}

whenever the $m_j$ are non-negative integers with $m_1 + \ldots + m_n \geq 2$ ([1], [2]).

Combining these facts with our results of previous sections, via Lemma 3.1, we come to the following conclusion.

**Theorem 3.2.** Suppose that $X$ is a $C^\infty$ vector field and $DX(0)$ has all eigenvalues with negative real parts. If $X$ satisfies either (3.1) or (3.2) then every vector field $Z$, $m$-flat with $m \geq d(X) + 1$, is $C^\infty$-divisible by $X$.

Sternberg’s algebraic condition (3.2) is also directly involved in the problem of divisibility of vector fields. Namely, let

$$
\sum a^i_\alpha x^\alpha, \quad \sum b^j_\alpha x^\alpha, \quad \sum c^j_\alpha x^\alpha
$$

be the Taylor series at $x = 0$ for $X, Y, Z$ respectively. The equality $[X, Y] = Z$ gives

\begin{equation}
\sum_{\alpha+\beta=\gamma} b^j_{\alpha+1} a^i_{\beta} - a^i_{\alpha+1} b^j_{\beta} = c^j_{\gamma}
\end{equation}

with $1_j$ standing for the multiindex $(0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the $j$th place. For given coefficients $a$ and $c$ there is a purely algebraic problem of solvability of this equation with respect to the unknown coefficients $b$.

Let us take $X = \sum_{i=1}^n \lambda_i x_i \partial/\partial x_i$. Then the $\lambda_i$ are the eigenvalues of $DX(0)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1 + \ldots + \alpha_n = |\alpha|$. In this case all $a^i_\alpha$ in formula (3.3) vanish for $|\alpha| \geq 2$. Hence (3.3) is now

$$
\sum_{\alpha} \left( \sum_{k} b^j_{\alpha+1-k} a^i_k - a^i_{\alpha+1} b^j_k \right) = c^j_\alpha.
$$

Since $a^i_j = \lambda^i \delta^i_j$ and the number of the indices $j$ is $\alpha_j$ we get

$$
\left( \sum_{j} \alpha_j \lambda_j - \lambda_i \right) b^i_\alpha = c^i_\alpha.
$$
Suppose that $Z$ is $m$-flat and $c^i_\alpha \neq 0$ for $|\alpha| \geq m$; then for the existence of $Y$ such that $[X,Y] = Z$ it is necessary to have
\begin{equation}
\lambda_i \neq \sum_j \alpha_j \lambda_j
\end{equation}
for any non-negative integers $\alpha_1, \ldots, \alpha_n$ satisfying $|\alpha| \geq m$. This is exactly Sternberg’s condition for $m = 2$ (in the regularity class $k = \infty$).

If $\lambda_i = \sum \alpha_j \lambda_j$ and $\text{Re} \lambda < 0$ (or $\text{Re} \lambda > 0$) then
\[|\text{Re} \lambda_i| = \sum \alpha_j |\text{Re} \lambda_j| \geq |\alpha| \min(|\text{Re} \lambda_j|).
\]
This implies $|\alpha| \leq \max(|\text{Re} \lambda_j|)/\min(|\text{Re} \lambda_j|) = d(X)$. Thus for $m \geq d(X) + 1$ we have $|\alpha| \leq m - 1$ and the condition (3.4) is satisfied (as it should be in view of Theorem 3.2). This also shows that the lower bound $d(X) + 1$ for $m$ in Theorem 3.2 is sharp.

On the other hand, if there are both negative and positive numbers in $\text{Re} \lambda$ then the equality $\lambda_i = \sum \alpha_j \lambda_j$ may occur for all $|\alpha|$.

**4. Generalization to polynomials.** For some applications to actions of infinite Lie groups it is useful to know when polynomials of the adjoint mapping $\text{ad}(X)$ act surjectively in the space of infinitely flat vector fields.

An answer to this question is given in the following:

**Theorem 4.1.** Let $P(\xi) = a_0 + a_1 + \ldots + a_r \xi^r$ be a polynomial of degree $r > 0$. Suppose that $X$ satisfies $\text{Re} \lambda < 0$. For any vector field $Z$ vanishing up to infinite order at $x = 0$ there exists a vector field $Y$ such that $Z = P(\text{ad}(X))Y$. The $Y$ can be defined by
\begin{equation}
Y(x) = -\int_0^\infty f(t)(\Psi_t)_* Z(x) \, dt
\end{equation}
where $f(t)$ is the solution of the differential equation
\begin{equation}
a_0 \xi - a_1 \xi' + \sum_{k=2}^r (k-1)a_k \xi^{(k)} = 0,
\end{equation}
with initial conditions $\xi(0) = \ldots = \xi^{(r-2)}(0) = 0$, $\xi^{(r-1)}(0) = 1/(r-1)a_r$ for $r \geq 2$ and $\xi(0) = -1/a_1$ for $r = 1$.

**Proof.** Equation (4.2) being with constant coefficients, there are positive constants $\alpha, \beta$ such that
\begin{equation}
|f(t)| \leq \alpha e^{\beta t} \quad \text{for } t \geq 0.
\end{equation}
As in Section 1, we have the following bounds:
\begin{equation}
\|f(t)D^k(\Psi_t)_* Z(x)\| \leq \alpha M^k_m e^{(\beta + \gamma \lambda - mc)t}, \quad c > 0,
\end{equation}
for $t \geq 0$ and $\|x\| \leq \delta$. With $k$ fixed we can choose $m$ great enough so that
$\beta + \gamma_k - mc < 0$. This makes the integral (4.1) uniformly convergent in $B(\delta)$
together with all derivatives. Thus $Y$ is $C^\infty$ in $B(\delta)$.

Set $Y_t = (\Phi_t)_* Z$. In the introduction we saw that $Y_t' = \text{ad}(X)Y_t$. Hence
$$Y_t^{(k)} = [\text{ad}(X)]^k Y_t, \quad k \geq 1.$$ Therefore
\begin{equation}
(4.5) \quad P(\text{ad}(X))f Y_t = a_0 f Y_t + a_1 f Y_t' + \ldots + a_r f Y_t^{(r)}.
\end{equation}
From
$$(f Y_t)^{(k)} = f Y_t^{(k)} + k f'(Y_t)^{(k-1)} + (1 - k) f^{(k)} Y_t$$
for $k \geq 1$, we get
$$f Y_t^{(k)} = (f Y_t)^{(k)} - k f'(Y_t)^{(k-1)} + (k - 1) f^{(k)} Y_t.$$ On inserting this into (4.5) one gets for $r \geq 2$
$$P(\text{ad}(X))f Y_t = \left( a_0 f - a_1 f' + \sum_{k=2}^{r} a_k (k - 1) f^{(k)} \right) Y_t + a_1 (f Y_t)'$$
$$+ \sum_{k=2}^{r} a_k [(f Y_t)^{(k)} - k f'(Y_t)^{(k-1)}]$$
$$= a_1 (f Y_t)' + \sum_{k=2}^{r} a_k [(f Y_t)^{(k)} - k f'(Y_t)^{(k-1)}]$$
according to our assumption on $f$. Now, by integrating either side with respect to $t$ over
the interval $[0, \infty)$ and using notation (4.1) we obtain
\begin{equation}
(4.6) \quad P(\text{ad}(X))Y = a_1 f Y_t^{(\infty)} + \left\{ \sum_{k=1}^{r} a_k [(f Y_t)^{(k-1)} - k f'(Y_t)^{(k-2)}] \right\}_0^{\infty},
\end{equation}
and $f$ satisfies $f(0) = f'(0) = \ldots = f^{(r-2)}(0) = 0$, $f^{(r-1)}(0) = 1/(r - 1) a_r$.
So, in view of (4.4) for $k = 0$, we have
$$f Y_t^{(\infty)} = - f(0) Y_0 = - f(0) Z = 0.$$ As the bound (4.3) can be extended to all derivatives of $f$ and the operator $\text{ad}(X)$
is bounded in $B(\delta)$, there is a constant $M$ such that
$$\|f^{(p)} Y_t^{(q)}\| \leq M e^{\beta t} \|\text{ad}(X)\| t^p \|Y_t\| \leq M e^{(\beta + \gamma_0 - mc) t}, \quad p, q \geq 0,$$
with $\beta + \gamma_0 - mc < 0$. Therefore
$$I_k = [(f Y_t)^{(k-1)} - k f'(Y_t)^{(k-2)}]_0^\infty = 0$$
for $2 \leq k \leq r - 1$. For $k = r$
$$I_r = f^{(r-1)}(0) Y_0 + rf^{(r-1)}(0) Y_0 = (r - 1) f^{(r-1)}(0) Z.$$
Coming back to (4.6) we finally get

\[ P(\text{ad}(X))Y = a(r - 1)f^{(r-1)}(0)Z = Z, \]

as required.

For \( r = 1 \), we take \( f(0) = -1/a_1 \). Then

\[ P(\text{ad}(X))Y = a_1fY|_0^\infty = -a_1f(0)Z = Z. \]

In particular:

(i) If \( P(u) = a + u \), then \( f(t) = -e^{at} \) and

\[ Y = \int_0^\infty e^{at}(\Psi_t)_* Z \, dt. \]

(ii) If \( P(u) = u^r \), \( r \geq 2 \), then

\[ Y = \frac{1}{(r-1)!} \int_0^\infty \frac{t^{r-1}}{(r-1)!} (\Psi_t)_* Z \, dt. \]

This \( Y \) satisfies

\[ Z = [X, \ldots [X, [X, Y]]] \quad (r \text{ commutators}). \]

As we see from the proof one can expect existence of a solution to the equation \( P(\text{ad}(X))Y = Z \) also in the case where \( Z \) vanishes at \( x = 0 \) up to a finite order \( m \). This would depend on the polynomial \( P \) and the required regularity class of \( Y \) which is to be defined by formula (4.1).

References