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On homeomorphic and diffeomorphic solutions of the Abel equation on the plane

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Abstract. We consider the Abel equation

$$\varphi[f(x)] = \varphi(x) + a$$

on the plane $\mathbb{R}^2$, where f is a free mapping (i.e. f is an orientation preserving homeomorphism of the plane onto itself with no fixed points). We find all its homeomorphic and diffeomorphic solutions φ having positive Jacobian. Moreover, we give some conditions which are equivalent to f being conjugate to a translation.

The aim of this paper is to find all homeomorphic and diffeomorphic solutions with positive Jacobian of the Abel equation

$$\varphi[f(x)] = \varphi(x) + a \tag{1}$$

on the plane $\mathbb{R}^2$. We assume that $a \neq (0,0)$ and f is an orientation preserving homeomorphism of the plane onto itself with no fixed points (such a homeomorphism will be called a *free mapping*). By a *curve* is meant a homeomorphic image of a straight line. A curve is said to be a *line* (an *open line* in [4]) if it is a closed set.

1. We note that the existence of homeomorphic solutions φ of (1) is equivalent to f being conjugate to the translation $T(x) = x + a$ (i.e. $f = \varphi^{-1} \circ T \circ \varphi$, where φ is a homeomorphism). S. Andrea [1] has proved that a free mapping f is conjugate to a translation if and only if

(H) for all $x, y \in \mathbb{R}^2$ there exists a curve segment C with endpoints x, y such that $f^n[C] \to \infty$ as $n \to \mp\infty$, where f^n is the nth iterate of f.

In the present paper we give some other conditions equivalent to (H). We introduce the following conditions:

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(A) There exists a homeomorphism φ of the plane *onto* itself satisfying (1).

(A′) There exists a homeomorphism φ of the plane *into* itself satisfying (1).

(B) There exists a line K such that

$$K \cap f[K] = \emptyset, \tag{2}$$

$$U^0 \cap f[U^0] = \emptyset, \tag{3}$$

$$\bigcup_{n\in\mathbb{Z}} f^n[U^0] = \mathbb{R}^2\,, \tag{4}$$

where $U^0 := M^0 \cup f[K]$ and M^0 is the strip bounded by K and $f[K]$. (See Fig. 1.)

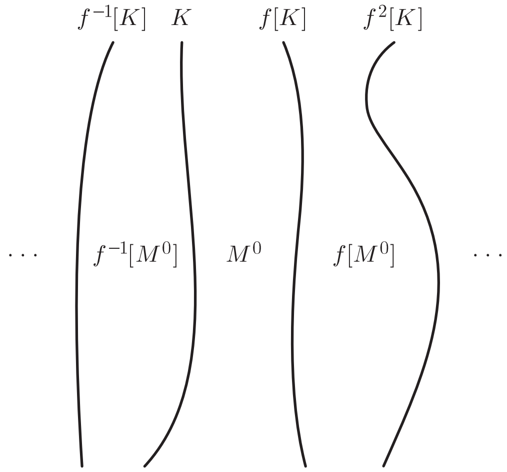

Fig. 1

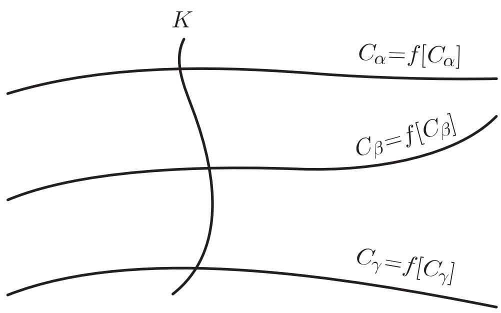

Fig. 2

(C) There exist a family of curves $\{C_\alpha : \alpha \in I\}$ and a line K such that

$$f[C_\alpha] = C_\alpha \quad \text{for } \alpha \in I, \tag{5}$$

(6) $$C_\alpha \cap C_\beta = \emptyset \quad \text{for } \alpha, \beta \in I,\ \alpha \neq \beta\,,$$

(7) $$\operatorname{card}(K \cap C_\alpha) = 1 \quad \text{for } \alpha \in I\,,$$

(8) $$\bigcup_{\alpha\in I} C_\alpha = \mathbb{R}^2. \quad \text{(See Fig. 2.)}$$

We shall show

THEOREM 1. *If f is a free mapping, then conditions* (A), (A′), (B) *and* (C) *are equivalent.*

2. First note the following

LEMMA 1. *Let $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $T(x) := x + a$ for $x \in \mathbb{R}^2$. Then there exists a homeomorphism ψ of the plane onto itself such that*

(9) $$T(x) = \psi^{-1}[\psi(x) + (1,0)]\,.$$

Proof. Set

$$\psi(x_1, x_2) = \left(\frac{1}{a_1}x_1, -\frac{a_2}{a_1}x_1 + x_2\right) \quad \text{if } a_1 \neq 0\,,$$

$$\psi(x_1, x_2) = \left(\frac{1}{a_2}x_2, x_1\right) \quad \text{if } a_1 = 0\,.$$

By Lemma 1, from now on we may assume that $a = (1,0)$.

LEMMA 2. *If f is a free mapping, then* (A′) *implies* (B).

Proof. Since φ is a homeomorphism, $\varphi[\mathbb{R}^2]$ is a region. Moreover, $\varphi[\mathbb{R}^2] = \varphi[\mathbb{R}^2] + (1,0)$. Put $T(x_1, x_2) := (x_1 + 1, x_2)$ and $T^0(x_1, x_2) := (x_1, x_2)$ for $(x_1, x_2) \in \varphi[\mathbb{R}^2]$. Write $L := \{(x_1, x_2) \in \varphi[\mathbb{R}^2] : x_1 = 0\}$. Since $T^n[L] = \{(x_1, x_2) \in \varphi[\mathbb{R}^2] : x_1 = n\}$ for $n \in \mathbb{Z}$, we have

(10) $$T^n[L] \cap T^m[L] = \emptyset \quad \text{for } n, m \in \mathbb{Z},\ n \neq m\,.$$

Put $L^n := T^n[L]$, $N^n := \{(x_1, x_2) \in \varphi[\mathbb{R}^2] : x_1 \in (n, n+1)\}$ and $W^n := N^n \cup L^{n+1}$ for $n \in \mathbb{Z}$. Note that

$$N^n = T^n[N^0] \quad \text{for } n \in \mathbb{Z}\,,$$

$$W^0 \cap W^n = \emptyset \quad \text{for } n \in \mathbb{Z} \setminus \{0\}, \qquad \bigcup_{n\in\mathbb{Z}} W^n = \varphi[\mathbb{R}^2]\,.$$

Put $K := \varphi^{-1}[L]$ and $K^n := f^n[K]$ for $n \in \mathbb{Z}$. By (A′), $\varphi \circ f^n = T^n \circ \varphi$, whence

(11) $$f^n \circ \varphi^{-1} = \varphi^{-1} \circ T^n\,.$$

Hence

(12) $$K^n = f^n[\varphi^{-1}[L]] = \varphi^{-1}[T^n[L]] = \varphi^{-1}[L^n]\,.$$

Therefore, by (10), $K^n \cap K^m = \emptyset$ for $n, m \in \mathbb{Z}, n \neq m$.

Since $K = \varphi^{-1}[L]$ and L is closed in $\varphi[\mathbb{R}^2]$, the curve K is a line, and so is $K^n = f^n[K]$ for every $n \in \mathbb{Z}$.

For each $n \in \mathbb{Z}$, denote by M^n the strip bounded by K^n and K^{n+1}. Since φ is a homeomorphism,

$$M^n = \varphi^{-1}[N^n] \quad \text{for } n \in \mathbb{Z}\,. \tag{13}$$

Hence by (11) we have

$$f^n[M^0] = f^n[\varphi^{-1}[N^0]] = \varphi^{-1}[T^n[N^0]] = \varphi^{-1}[N^n] = M^n \tag{14}$$

for $n \in \mathbb{Z}$.

Put $U^n := M^n \cup K^{n+1}$ for $n \in \mathbb{Z}$. Then by (12) and (13),

$$U^n = \varphi^{-1}[N^n \cup L^{n+1}] = \varphi^{-1}[W^n]$$

and by (14),

$$U^n = f^n[M^0 \cup f[K]] = f^n[U^0]\,.$$

Hence

$$\bigcup_{n\in\mathbb{Z}} f^n[U^0] = \bigcup_{n\in\mathbb{Z}} U^n = \varphi^{-1}\Big[\bigcup_{n\in\mathbb{Z}} W^n\Big] = \mathbb{R}^2$$

and

$$f[U^0] \cap U^0 = U^1 \cap U^0 = \varphi^{-1}[W^1 \cap W^0] = \emptyset\,.$$

This completes the proof.

Theorem 2. *Let f be a free mapping of the plane onto itself and let $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Assume that condition* (B) *is satisfied. Let $\varphi_0 : U^0 \cup K \to \mathbb{R}^2$ be continuous and suppose*

$$\varphi_0[f(x)] = \varphi_0(x) + a \quad \textit{for } x \in K\,. \tag{15}$$

Then:

(a) *There exists a unique solution φ of* (1) *such that*

$$\varphi(x) = \varphi_0(x) \quad \textit{for } x \in U^0 \cup K\,. \tag{16}$$

This solution φ is continuous.

(b) *If φ_0 is one-to-one and $\varphi_0[U^0] \cap (\varphi_0[U^0] + ka) = \emptyset$ for all $k \in \mathbb{Z} \setminus \{0\}$ then φ is a homeomorphism.*

(c) *If φ_0 is one-to-one, $\varphi_0[K]$ is a line and $\varphi_0[K] \cap D_\gamma \neq \emptyset$ for all $\gamma \in \mathbb{R}$, where $D_\gamma = \{(x_1, x_2) \in \mathbb{R}^2 : a_2x_1 - a_1x_2 = \gamma\}$, then φ is a homeomorphism.*

(d) *If φ_0 is one-to-one, $\varphi_0[K]$ is a line, $\varphi_0[K] \cap D_\gamma \neq \emptyset$ for all $\gamma \in \mathbb{R}$, and $\varphi_0[\operatorname{int} U^0] = N^0$, where N^0 is the strip bounded by $\varphi_0[K]$ and $\varphi_0[K]+a$, then φ is a homeomorphism of $\mathbb{R}^2$ onto itself.* (See Fig. 3.)

Proof. Since $K \cap f[K] = \emptyset$,

$$f^n[K] \cap f^{n+1}[K] = \emptyset \quad \text{for } n \in \mathbb{Z}\,.$$

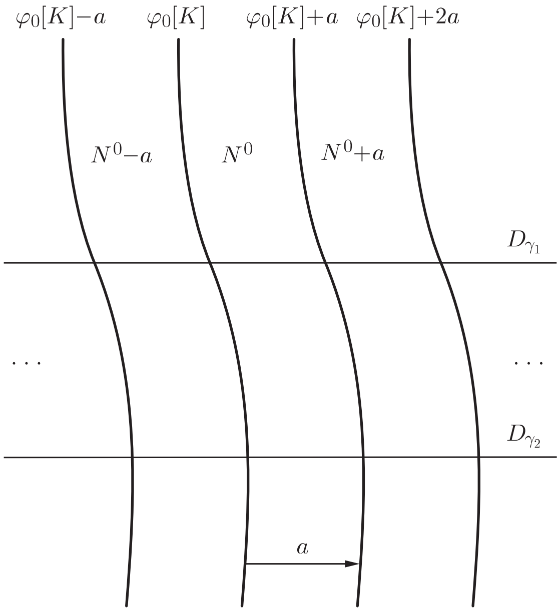

Fig. 3

Moreover, for each $n \in \mathbb{Z}$, the curve $f^n[K]$ is a line, since so is K. Denote by M^n the strip bounded by $f^n[K]$ and $f^{n+1}[K]$. Let $U^n := M^n \cup f^{n+1}[K]$ for $n \in \mathbb{Z}$. Since f is a homeomorphism,

$$U^n = f^n[U^0] \quad \text{for } n \in \mathbb{Z}\,.$$

Furthermore, for every $n \in \mathbb{Z}$, $f^n[K]$ lies in the strip between $f^{n-1}[K]$ and $f^{n+1}[K]$, $f^{n-1}[K] \cap M^n = \emptyset$ and $f^{n+1}[K] \cap M^{n-1} = \emptyset$. Otherwise we would have $f^{n-1}[U^0] \cap f^n[U^0] \neq \emptyset$, which contradicts (3).

Define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$\varphi(x) = \varphi_0[f^{-k}(x)] + ka, \qquad x \in U^k,\ k \in \mathbb{Z}\,. \tag{17}$$

It is clear that φ is a unique solution of (1) satisfying (16) and that φ is continuous in $\bigcup_{n\in\mathbb{Z}} \operatorname{int} U^n$.

Take any $x_0 \in K$. We now show that φ is continuous at x_0. Let P be a closed disc with centre at x_0 such that $P \cap f^{-1}[K] \neq \emptyset$ and $P \cap f[K] \neq \emptyset$. Then $P \cap f^{-1}[K]$ and $P \cap f[K]$ are compact. Let R be an open disc with centre at x_0 and radius smaller than $\min\{\varrho(x_0, P\cap f^{-1}[K]),\ \varrho(x_0, P\cap f[K])\}$, where ϱ is the Euclidean metric on the plane. Then we have

$$R \cap f^{-1}[K] = \emptyset \quad \text{and} \quad R \cap f[K] = \emptyset\,. \tag{18}$$

Put $R_1 := R \cap \operatorname{int} U^0$, $R_2 := R \cap \operatorname{int} U^{-1}$, $R_0 := R \cap K$. Then $\varphi(x) = \varphi_0(x)$ for $x \in R_1$, and $\varphi(x) = \varphi_0[f(x)] - a$ for $x \in R_2 \cup R_0$. As $x_0 \in R_0$ we hence get $\varphi(x_0) = \varphi_0(x_0)$ by (15).

Let $x_k \to x_0$, $x_k \in R$. If $x_k \in R_1 \cup R_0$, then

$$\lim_{k\to\infty} \varphi(x_k) = \lim_{k\to\infty} \varphi_0(x_k) = \varphi_0(x_0) = \varphi(x_0)\,,$$

because φ_0 is continuous in $R_1 \cup R_0 \subset K \cup U^0$. If $x_k \in R_2$, then $f(x_k) \in U^0$ and $f(x_k) \to f(x_0) \in U^0$. Thus

$$\lim_{k\to\infty} \varphi(x_k) = \lim_{k\to\infty} (\varphi_0[f(x_k)] - a) = \varphi_0[f(x_0)] - a = \varphi_0(x_0) = \varphi(x_0)\,,$$

because φ_0 is continuous in U^0 and $x_0 \in K$. Consequently, φ is continuous at $x_0 \in K$.

Let $x_0 \in \mathbb{R}^2 \setminus \bigcup_{k\in\mathbb{Z}} \operatorname{int} U^k$. There exists an $m \in \mathbb{Z}$ such that $x_0 \in U^m \setminus \operatorname{int} U^m = f^{m+1}[K]$. Let V be a neighbourhood of x_0 such that $V \cap f^m[K] = \emptyset$ and $V \cap f^{m+2}[K] = \emptyset$ (proceed as in the proof of the existence of R satisfying (18)). Note that $V \subset U^m \cup \operatorname{int} U^{m+1}$, thus $f^{-m-1}[V] \subset U^{-1} \cup \operatorname{int} U^0$ and $f^{-m-1}[V]$ is a neighbourhood of $f^{-m-1}(x_0) \in K$.

Take any sequence $\{x_k\}$ in V such that $x_k \to x_0$. Then $f^{-m-1}(x_k) \to f^{-m-1}(x_0)$. Since φ is continuous on K and $f^{-m-1}(x_0) \in K$, we have $\varphi[f^{-m-1}(x_k)] \to \varphi[f^{-m-1}(x_0)]$. From (17) we have

$$\varphi(x_0) = \varphi_0[f^{-m-1}(x_0)] + (m+1)a$$

and

$$\varphi(x_k) = \varphi_0[f^{-m-1}(x_k)] + (m+1)a \quad \text{ for } k \in \mathbb{N}\,.$$

Thus $\varphi(x_k) \to \varphi(x_0)$. Consequently, φ is continuous on the plane.

Now assume, in addition, that φ is one-to-one in $U^0 \cup K$ and $\varphi_0(x) + na \notin \varphi_0[U^0]$ for all $x \in U^0$ and $n \in \mathbb{Z} \setminus \{0\}$. We show that φ is one-to-one in the plane. Suppose $x, y \in \mathbb{R}^2$ and $\varphi(x) = \varphi(y)$. By (4), $x \in U^k$ and $y \in U^l$ for some $k, l \in \mathbb{Z}$. From (17) it follows that

$$\varphi(x) = \varphi_0[f^{-k}(x)] + ka, \qquad \varphi(y) = \varphi_0[f^{-l}(y)] + la\,.$$

Therefore

$$\varphi_0[f^{-k}(x)] = \varphi_0[f^{-l}(y)] + (l-k)a\,.$$

Suppose that $l - k \neq 0$. Then $\varphi_0[f^{-l}(y)] + (l-k)a \notin \varphi_0[U^0]$, since $f^{-l}(y) \in U^0$. Hence $\varphi_0[f^{-k}(x)] \notin \varphi_0[U^0]$, which is a contradiction, since $f^{-k}(x) \in U^0$. Thus $l = k$, and consequently $x = y$. Note that φ, being a continuous one-to-one mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$, is a homeomorphism (see [3], p. 186).

Now we show (c). Note that $\varphi[K] + a = \varphi[f[K]]$. Moreover, as $D_\gamma \cap \varphi_0[K] \neq \emptyset$, we have $D_\gamma \cap \varphi_0[f[K]] \neq \emptyset$. Since $\varphi|_{\operatorname{int} U^0}$ is continuous and one-to-one, it is a homeomorphism and $\varphi[\operatorname{int} U^0]$ is a region. Moreover, $\varphi(x) \notin \varphi[\operatorname{int} U^0]$, for every $x \in K \cup f[K]$, since φ is one-to-one in $U^0 \cup K$. Hence each $y \in \varphi[K] \cup \varphi[f[K]]$ is a boundary point of $\varphi[\operatorname{int} U^0]$, since φ_0 is continuous on $U^0 \cup K$. Therefore $\varphi[\operatorname{int} U^0] \subset N^0$, because $\varphi_0[K] \cap D_\gamma \neq \emptyset$ and $\varphi_0[f[K]] \cap D_\gamma \neq \emptyset$ for $\gamma \in \mathbb{R}$ and $\varphi_0[K]$ is a line.

Let $x, y \in \mathbb{R}^2$. Then $x \in U^k$ and $y \in U^l$ for some $k, l \in \mathbb{Z}$. Assume that $\varphi(x) = \varphi(y)$. Then

$$\varphi_0[f^{-k}(x)] = \varphi_0[f^{-l}(y)] + (l-k)a\,.$$

Since $\varphi_0[U^0] \subset N^0 \cup \varphi_0[f[K]]$, we have $l - k = 0$, whence $x = y$. Thus φ is continuous and one-to-one, and hence a homeomorphism.

Assume, in addition, that $\varphi_0[\operatorname{int} U^0] = N^0$. Put $W^0 := N^0 \cup (\varphi[K] + a)$. Then $W^0 = \varphi[U^0]$. Let $y \in \mathbb{R}^2$. Then there exists an $n \in \mathbb{Z}$ such that $y - na \in W^0$. Take an $x \in U^0$ such that $\varphi(x) = y - na$. Then by (1),

$$\varphi[f^n(x)] = \varphi(x) + na = y\,.$$

Thus $\varphi[\mathbb{R}^2] = \mathbb{R}^2$. Consequently, φ is a homeomorphism of $\mathbb{R}^2$ onto itself satisfying (1). This completes the proof.

Obviously, we also have the following

R e m a r k 1. Let f be a free mapping of the plane onto itself and let $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Assume that condition (B) is satisfied. Let φ be any homeomorphic solution of equation (1). Let $\varphi_0 := \varphi|_{U^0 \cup K}$. Then

(a) φ_0 is one-to-one and $\varphi_0[U^0] \cap (\varphi_0[U^0] + ka) = \emptyset$ for $k \in \mathbb{Z} \setminus \{0\}$;

(b) if φ is a homeomorphism of $\mathbb{R}^2$ onto itself, then φ_0 is one-to-one, $\varphi_0[K]$ is a line, $\varphi_0[K] \cap D_\gamma \neq \emptyset$ for $\gamma \in \mathbb{R}$, and $\varphi_0[\operatorname{int} U^0] = N^0$, where D_γ and N^0 are as in the statement of Theorem 2.

From part (d) of Theorem 2 we have

COROLLARY 1. *If f is a free mapping*, *then* (B) *implies* (A).

From Lemma 2, Corollary 1 and the fact that (A) implies (A′) we have

COROLLARY 2. *Let f be a free mapping. Then conditions* (A), (A′) *and* (B) *are equivalent.*

Now we are going to prove

LEMMA 3. *Let f be a free mapping. Then* (A) *implies* (C).

P r o o f. Put $L := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$, and $D_\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \alpha\}$ for $\alpha \in \mathbb{R}$. Let $K := \varphi^{-1}[L]$ and $C_\alpha := \varphi^{-1}[D_\alpha]$ for $\alpha \in \mathbb{R}$, where φ is a homeomorphism satisfying $\varphi \circ f = T \circ \varphi$ with $T(x_1, x_2) = (x_1 + 1, x_2)$ for $x_1, x_2 \in \mathbb{R}$. Let $I := \mathbb{R}$. Since $f \circ \varphi^{-1} = \varphi^{-1} \circ T$ and $T[D_\alpha] = D_\alpha$ for $\alpha \in \mathbb{R}$, we have

$$f[C_\alpha] = \varphi^{-1}[T[D_\alpha]] = \varphi^{-1}[D_\alpha] = C_\alpha \quad \text{for } \alpha \in \mathbb{R}\,.$$

Moreover, note that

$$C_\alpha \cap C_\beta = \varphi^{-1}[D_\alpha \cap D_\beta] = \emptyset \quad \text{for } \alpha, \beta \in \mathbb{R},\ \alpha \neq \beta\,,$$

$$\bigcup_{\alpha \in \mathbb{R}} C_\alpha = \varphi^{-1}\Big[\bigcup_{\alpha \in \mathbb{R}} D_\alpha\Big] = \varphi^{-1}[\mathbb{R}^2] = \mathbb{R}^2$$

and

$$\operatorname{card}(K\cap C_\alpha)=\operatorname{card}\varphi^{-1}[L\cap D_\alpha]=\operatorname{card}(L\cap D_\alpha)=1$$

for $\alpha\in\mathbb{R}$. This completes the proof.

THEOREM 3. *Let f be a free mapping. Then condition* (C) *implies* (B).

Proof. Suppose that (C) holds.

1. First, we show that for the line K which appears in (C), $K\cap f[K]=\emptyset$.

Suppose $x_0\in K\cap f[K]$. On account of (8), $x_0\in C_\alpha$ for some $\alpha\in I$. By (5) we get $f^{-1}(x_0)\in C_\alpha$, and clearly $f^{-1}(x_0)\in K$. Since $\operatorname{card}(K\cap C_\alpha)=1$, $x_0=f^{-1}(x_0)$. Hence x_0 is a fixed point of f, a contradiction.

2. Now we prove that

$$\operatorname{card}(f^n[K]\cap C_\alpha)=1\quad \text{for } \alpha\in I \text{ and } n\in\mathbb{Z}\,.$$

Fix any $\alpha\in I$. Let $n\in\mathbb{Z}\setminus\{0\}$. Take $x_0\in K\cap C_\alpha$. By (5),

$$f^n(x_0)\in f^n[K]\cap C_\alpha\,.$$

Suppose there exist $y_1,y_2\in f^n[K]\cap C_\alpha$ such that $y_1\neq y_2$. Then $f^{-n}(y_1)$, $f^{-n}(y_2)\in K\cap C_\alpha$ and $f^{-n}(y_1)\neq f^{-n}(y_2)$, which contradicts (7).

3. Let $x\in C_\alpha$. We now prove that, for every $n\in\mathbb{Z}$, $f^{n+1}(x)$ lies between $f^n(x)$ and $f^{n+2}(x)$ on the curve C_α. For any $x,y\in C_\alpha$ denote by $\langle x,y\rangle$ the segment of C_α with endpoints x,y. Let $(x,y):=\langle x,y\rangle\setminus\{x,y\}$.

Let $n\in\mathbb{Z}$. If $f^{n+2}(x)\in(f^n(x),f^{n+1}(x))\subset C_\alpha$, then

$$f(\langle f^n(x),f^{n+1}(x)\rangle)=\langle f^{n+2}(x),f^{n+1}(x)\rangle\subset\langle f^n(x),f^{n+1}(x)\rangle\,.$$

Hence by Brouwer's Theorem f has a fixed point, which is impossible. Similarly, if $f^n(x)\in(f^{n+1}(x),f^{n+2}(x))\subset C_\alpha$, then

$$f^{-1}(\langle f^{n+1}(x),f^{n+2}(x)\rangle)=\langle f^{n+1}(x),f^n(x)\rangle\subset\langle f^{n+1}(x),f^{n+2}(x)\rangle\,.$$

Hence f^{-1} has a fixed point, contradiction again. Thus

$$f^{n+1}(x)\in(f^n(x),f^{n+2}(x))\,. \tag{19}$$

4. Now we show that (3) holds. Since, $f^n[K]$ is a line for all $n\in\mathbb{Z}$, $\mathbb{R}^2\setminus f^n[K]$ consists of two unbounded regions, called the *side domains* of $f^n[K]$. Since $K\cap f[K]=\emptyset$, we have $f^n[K]\cap f^{n+1}[K]=\emptyset$ for all $n\in\mathbb{Z}$. For each $n\in\mathbb{Z}$, denote by M^n the strip between the lines $f^n[K]$ and $f^{n+1}[K]$. Let M^n_+ be the side domain of $f^{n+1}[K]$ which does not contain the line $f^n[K]$, and M^n_- the side domain of $f^n[K]$ which does not contain $f^{n+1}[K]$. Then

$$M^n_-\cup f^n[K]\cup M^n\cup f^{n+1}[K]\cup M^n_+=\mathbb{R}^2\quad \text{for } n\in\mathbb{Z}\,.$$

Now we show that $f^{n+2}[K]\subset M^n_+$ for $n\in\mathbb{Z}$. Suppose otherwise. Then for some $n\in\mathbb{Z}$,

$$f^{n+2}[K]\subset M^n_-\cup f^n[K]\cup M^n\,, \tag{20}$$

since $f^{n+2}[K] \cap f^{n+1}[K] = \emptyset$. Take any $x_0 \in K$. By (8), $x_0 \in C_\alpha$ for some $\alpha \in I$. By (5), $f^n(x_0) \in C_\alpha$ for all $n \in \mathbb{Z}$. From (20) we obtain $f^{n+2}(x_0) \in M_-^n \cup f^n[K] \cup M^n$. Thus by (19),

$$C_\alpha \subset M_-^n \cup f^n[K] \cup M^n \cup \{f^{n+1}(x_0)\},$$

since

$$f^n(x_0), f^{n+2}(x_0) \in M_-^n \cup f^n[K] \cup M^n, \qquad f^{n+1}[K] \cap C_\alpha = \{f^{n+1}(x_0)\}$$

and C_α has no self-intersections. Consequently, we have shown that for each $\alpha \in I$,

$$C_\alpha \subset M_-^n \cup f^n[K] \cup M^n \cup f^{n+1}[K] = \mathbb{R}^2 \setminus M_+^n,$$

which contradicts (8). Thus $f^{n+2}[K] \subset M_+^n$ for all $n \in \mathbb{Z}$. Hence M^n, $n \in \mathbb{Z}$, are mutually disjoint and $f^n[K] \cap K = \emptyset$ for $n \in \mathbb{Z} \setminus \{0\}$. Since f is a homeomorphism, we have

$$f^n[M^0] = M^n \quad \text{for } n \in \mathbb{Z}. \tag{21}$$

Thus $f^n[M^0] \cap M^0 = \emptyset$ for $n \in \mathbb{Z} \setminus \{0\}$. Moreover, for every $n \in \mathbb{Z} \setminus \{0\}$,

$$f^n[M^0 \cup f[K]] \cap (M^0 \cup f[K]) = (f^n[M^0] \cup f^{n+1}[K]) \cap (M^0 \cup f[K]) = \emptyset.$$

Thus, for all $n \in \mathbb{Z} \setminus \{0\}$, $f^n[U^0] \cap U^0 = \emptyset$, where $U^0 = M^0 \cup f[K]$.

5. To complete the proof we show that

$$\bigcup_{n \in \mathbb{Z}} f^n[U^0] = \mathbb{R}^2.$$

For each $\alpha \in I$ let $K \cap C_\alpha =: \{x_\alpha\}$ and $C_\alpha^0 = (x_\alpha, f(x_\alpha)) \subset C_\alpha$. First, we show that $\bigcup_{\alpha \in I} C_\alpha^0 = M^0$.

Suppose that $x_0 \in C_\alpha^0$ and $x_0 \notin M^0$. Then C_α^0 has either a common point with K different from x_α or a common point with $f[K]$ different from $f(x_\alpha)$, which is impossible.

For each $\alpha \in I$ denote by C_α^{0+} the set of all $x \in C_\alpha$ such that $f(x_\alpha) \in (x_\alpha, x) \subset C_\alpha$, and by C_α^{0-} the set of all $x \in C_\alpha$ such that $x_\alpha \in (x, f(x_\alpha)) \subset C_\alpha$.

Take any $x_0 \in M^0$. Then $x_0 \in C_\alpha$ for some $\alpha \in I$. Suppose that $x_0 \in C_\alpha^{0+}$. Since $\operatorname{card}(C_\alpha \cap f[K]) = 1$ and $f(x_\alpha) \in C_\alpha \cap f[K]$, we have $C_\alpha^{0+} \cap f[K] = \emptyset$. Hence C_α^{0+} is contained either in M_+^0 or in $M_-^0 \cup K \cup M^0$. Since $f^2(x_0) \in C_\alpha^{0+} \cap M_+^0$, we have $C_\alpha^{0+} \subset M_+^0$, whence $x_0 \in M_+^0$, but this is impossible, since $x_0 \in M^0$.

Now suppose $x_0 \in C_\alpha^{0-}$. Since $\operatorname{card}(C_\alpha \cap K) = 1$ and $x_\alpha \in C_\alpha \cap K$, we have $C_\alpha^{0-} \cap K = \emptyset$. Hence C_α^{0-} is contained either in M_-^0 or in $M^0 \cup f[K] \cup M_+^0$. Since $f^{-1}(x_0) \in C_\alpha^{0-} \cap M_-^0$, we have $C_\alpha^{0-} \subset M_-^0$, whence $x_0 \in M_-^0$, and this is also impossible. Consequently,

$$\bigcup_{\alpha \in I} C_\alpha^0 = M^0. \tag{22}$$

For every $\alpha \in I$ and every $n \in \mathbb{Z}$, let $C_\alpha^n := (f^n(x_\alpha), f^{n+1}(x_\alpha)) \subset C_\alpha$. Since f is a homeomorphism, we have by (5),

$$C_\alpha^n = f^n[C_\alpha^0] \quad \text{for } \alpha \in I \text{ and } n \in \mathbb{Z}\,. \tag{23}$$

Hence, for all $n \in \mathbb{Z}$, we get by (21) and (22),

$$M^n = f^n[M^0] = \bigcup_{\alpha \in I} f^n[C_\alpha^0] = \bigcup_{\alpha \in I} C_\alpha^n\,.$$

Let $x_0 \in \mathbb{R}^2$. If there exists an $n \in \mathbb{Z}$ such that $x_0 \in f^n[K]$, then $x_0 \in f^{n-1}[U^0]$. Now assume that $x_0 \in \mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{Z}} f^n[K]$. Then $x_0 \in C_\alpha$ for some $\alpha \in I$. Since $f^n(x_\alpha) \to \infty$ as $n \to \mp\infty$ (see [1], Prop. 1.2), there is an $n \in \mathbb{Z}$ such that $x_0 \in C_\alpha^n$. Hence by (22) and (23),

$$x_0 \in f^n[C_\alpha^0] \subset f^n[M^0] \subset f^n[U^0]\,.$$

Consequently, $\mathbb{R}^2 = \bigcup_{n\in\mathbb{Z}} f^n[U^0]$. This completes the proof.

Note that Theorem 1 is a consequence of Corollary 2, Lemma 3 and Theorem 3.

Moreover, from the proof of Theorem 3 we have the following

COROLLARY 3. *Let f be a free mapping. Let K be a line on the plane. If K satisfies condition* (C), *then it also satisfies* (B).

3. In this section we study diffeomorphic solutions of equation (1). First we quote the following

LEMMA 4 (see [5]). *If the functions f and φ are of class C^p $(p > 0)$ in a region $U \subset \mathbb{R}^n$ such that $f[U] \subset U$, then for $x \in U$,*

$$\frac{\partial^q}{\partial x_{i_1} \ldots \partial x_{i_q}} \varphi[f(x)] = \sum_{k=1}^{q} \sum_{j_1,\ldots,j_k=1}^{n} b_{i_1\ldots i_q}^{j_1\ldots j_k}(x) \varphi_{j_1\ldots j_k}[f(x)]\,,$$

$q = 1, \ldots, p$, where

$$\varphi_{i_1\ldots i_k}(x) = \frac{\partial^k}{\partial x_{i_1} \ldots \partial_{i_k}} \varphi(x)\,,$$

$b_{i_1\ldots i_q}^{j_1\ldots j_k}(x)$ may be expressed by means of sums and products of $a_i^j(x), \ldots$ $\ldots, a_{i_1,\ldots,i_{q-k+1}}^j(x), a_{i_1,\ldots,i_k}^j(x) = \frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}} f_j(x)$, $k = 1, \ldots, p$, and $f = (f_1, \ldots, f_n)$. Consequently, $b_{i_1\ldots i_q}^{j_1\ldots j_k}$ are of class $C^{p-q+k-1}$. In particular,

$$b_{i_1\ldots i_q}^{j_1\ldots j_q}(x) = a_{i_1}^{j_1}(x) \cdot \ldots \cdot a_{i_q}^{j_q}(x)\,.$$

Now let f be a free mapping. Assume that condition (B) is satisfied.

DEFINITION (see [5]) . Let ψ be a continuous function defined in $U^0 \cup K$, p times continuously differentiable in $\operatorname{int} U^0$. We write

$$\psi_{i_1\ldots i_k}(x_0) = \lim_{\substack{x\to x_0\\ x\in \operatorname{int} U^0}} \frac{\partial^k}{\partial x_{i_1}\ldots \partial x_{i_k}}\psi(x), \qquad k = 1,\ldots,p\,,$$

for $x_0 \in K \cup f[K]$ (provided this limit exists). The function ψ is said to be of class C^p in $U^0 \cup K$ if all the functions $\psi, \psi_i, \ldots, \psi_{i_1\ldots i_p}$ are continuous in $U^0 \cup K$.

All diffeomorphic solutions of equation (1) having positive Jacobian can be obtained from the following

THEOREM 4. *Let f be a free C^p mapping of the plane having positive Jacobian at every $x \in \mathbb{R}^2$ and let $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Assume that condition* (B) *is satisfied. Let ψ be a C^p function from $U^0 \cup K$ into the plane which satisfies*

$$\psi[f(x)] = \psi(x) + a \qquad \text{for } x \in K\,,$$

$$\sum_{k=1}^{q} \sum_{j_1,\ldots,j_k=1}^{2} b^{j_1\ldots j_k}_{i_1\ldots i_q}(x)\psi_{j_1\ldots j_k}[f(x)] = \psi_{i_1\ldots i_q}(x)$$

for $x \in K$, $q = 1,\ldots,p$, $i_1,\ldots,i_q = 1,2$, where the functions $b^{j_1\ldots j_k}_{i_1\ldots i_q}$ are those occurring in Lemma 4. *Then there exists a unique solution φ of equation* (1) *such that*

$$\varphi(x) = \psi(x) \qquad \text{for } x \in U^0 \cup K\,.$$

This solution is of class C^p in the plane. Moreover, if ψ is one-to-one, the Jacobian, $\operatorname{jac}_x \psi$, *is positive for $x \in \operatorname{int} U^0$, and* $\det[\psi_1(x), \psi_2(x)] > 0$ *for $x \in K \cup f[K]$, and either*

$$\psi[U^0] \cap (\psi[U^0] + ka) = \emptyset \qquad \text{for } k \in \mathbb{Z}\setminus\{0\}$$

or

$$\psi[K] \cap D_\gamma \neq \emptyset \quad \text{for } \gamma \in \mathbb{R} \quad \text{and} \quad \psi[K] \text{ is a line},$$

where $D_\gamma = \{(x_1,x_2) \in \mathbb{R}^2 : a_2x_1 - a_1x_2 = \gamma\}$, then φ is an orientation preserving diffeomorphism of class C^p having positive Jacobian.

P r o o f. Define φ by setting

$$\varphi(x) = \psi[f^{-k}(x)] + ka, \qquad x \in U^k,\ k \in \mathbb{Z}, \tag{24}$$

where $U^k = f^k[U^0]$. For $p = 0$ we have Theorem 1. Let $p > 0$. From (24) it follows that φ is of class C^p in $\bigcup_{k\in\mathbb{Z}} \operatorname{int} U^k$.

Let $x_0 \in K$. Then there exists an open disc R with centre at x_0 such that $R \cap f^{-1}[K] = \emptyset$ and $R \cap f[K] = \emptyset$ (see the proof of Theorem 2). The proof of φ being C^p in R runs in the same way as that of Theorem 3.1 in [5], part 2.

Let $x_0 \in \mathbb{R}^2 \setminus \bigcup_{k\in\mathbb{Z}} \operatorname{int} U^k$. There is an $m \in \mathbb{Z}$ such that $f^{-m-1}(x_0) \in K$. We have already proved that φ is C^p in a neighbourhood R of $f^{-m-1}(x_0)$. The function f^{m+1} is a C^p map of R onto a neighbourhood $f^{m+1}[R]$ of x_0. Since φ is a solution of (1), we have

$$\varphi(x_0) = \varphi[f^{-m-1}(x_0)] + (m+1)a\,.$$

Hence φ is C^p in $f^{m+1}[R]$.

Now assume, in addition, that ψ is one-to-one, $\psi(x) + ka \notin \psi[U^0]$ for $x \in U^0$ and $k \in \mathbb{Z} \setminus \{0\}$ [or $\psi[K] \cap D_\gamma \neq \emptyset$ for all $\gamma \in \mathbb{R}$ and $\psi[K]$ is a line], $\operatorname{jac}_x \psi > 0$ for $x \in \operatorname{int} U^0$ and $\det[\psi_1(x), \psi_2(x)] > 0$ for $x \in K \cup f[K]$. On account of Theorem 2, φ is a homeomorphism.

If $x \in U^0$, then $\operatorname{jac}_x \varphi = \operatorname{jac}_x \psi > 0$. If $x \in f[K]$, then $\operatorname{jac}_x \varphi = \det[\psi_1(x), \psi_2(x)] > 0$, since $(\partial\varphi/\partial x_1)(x) = \psi_1(x)$ and $(\partial\varphi/\partial x_2)(x) = \psi_2(x)$ for $x \in f[K]$. Thus $\operatorname{jac}_x \varphi > 0$ for $x \in U^0$.

Let $x \in \mathbb{R}^2$. Then $x \in f^n[U^0]$ for some $n \in \mathbb{Z}$. Since $\varphi(x) = \varphi[f^{-n}(x)] + na$, we have

$$\operatorname{jac}_x \varphi = \operatorname{jac}_{f^{-n}(x)} \varphi \cdot \operatorname{jac}_x f^{-n}\,.$$

Hence $\operatorname{jac}_x \varphi > 0$, since $f^{-n}(x) \in U^0$ and $\operatorname{jac}_x f^{-n} > 0$. Thus φ preserves orientation. Since φ is invertible and of class C^p, and $\operatorname{jac}_x \varphi \neq 0$ for $x \in \mathbb{R}^2$, φ^{-1} is C^p (see e.g. [6], p. 205). This completes the proof.

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