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A note on the Kobayashi pseudodistance and the tautness of holomorphic fiber bundles

by DO DUC THAI and NGUYEN LE HUONG (Ha noi)

Abstract. We show a relation between the Kobayashi pseudodistance of a holomorphic fiber bundle and the Kobayashi pseudodistance of its base. Moreover, we prove that a holomorphic fiber bundle is taut iff both the fiber and the base are taut.

I. Introduction. In this article we shall show a relation between the Kobayashi pseudodistance of a holomorphic fiber bundle and the Kobayashi pseudodistance of its base. Moreover, we shall give another proof (not using universal coverings as in Nag [5]) that a holomorphic fiber bundle is taut iff both the fiber and the base are taut. An analogous result for hyperbolicity was proved by Nag [5].

In this section we recall some definitions and relevant properties.

We denote the Kobayashi pseudodistance on a complex space M by d_M (see Kobayashi [2], [3] and Lang [4]). M is called (*complete*) *hyperbolic* if d_M is a (complete) distance. M is called *taut* [3] if whenever N is a complex space and $f_i : N \to M$ is a sequence of holomorphic maps, then either there exists a subsequence which is compactly divergent or a subsequence which converges uniformly on compact subsets to a holomorphic map $f : N \to M$. It suffices that this condition should hold when $N = D$ [3, p. 378], where D is the unit disk in $\mathbb{C}$.

Also, a complete hyperbolic space is taut, and a taut complex space is hyperbolic [3, p. 378].

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2. THEOREM A. *Let E be a holomorphic fiber bundle over M with hyperbolic fiber F and projection $\pi : E \to M$, where E, F, M are complex*

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manifolds. Then $d_M(x,y) = d_E(\widetilde{x}, \pi^{-1}(y)) = \inf_{\widetilde{y}\in\pi^{-1}(y)} d_E(\widetilde{x},\widetilde{y})$ *for all* $x, y \in M$ *and all* $\widetilde{x} \in \pi^{-1}(x)$.

Proof. We have $d_M(x,y) \le d_E(\widetilde{x}, \pi^{-1}(y))$ for all $\widetilde{x} \in \pi^{-1}(x)$. We must prove the reverse inequality. Take arbitrary points $x, y \in M$ and $\widetilde{x} \in \pi^{-1}(x)$. Consider $a_1, \ldots, a_k \in D$ and $f_1, \ldots, f_k \in \mathrm{Hol}(D, M)$ such that

$$f_1(0) = x, \quad f_i(a_i) = f_{i+1}(0), \quad f_k(a_k) = y\,.$$

Consider the pull-back diagram

$$\begin{array}{ccc} D \times_M E & \xrightarrow{\sigma_1} & E \\ \Theta_1 \downarrow & & \downarrow \pi \\ D & \xrightarrow{f_1} & M \end{array}$$

By a result of Royden [6], there is an equivalence $\Phi_1 : D \times F \to D \times_M E$ of holomorphic fiber bundles over M. Thus there exists $c_1 \in F$ such that $\sigma_1 \circ \Phi_1(0, c_1) = \widetilde{x}$. We define a holomorphic map $\varphi_1 : D \to E$ by

$$\varphi_1(z) = \sigma_1 \circ \Phi_1(z, c_1) \quad \text{for all } z \in D\,.$$

Consider the pull-back diagram

$$\begin{array}{ccc} D \times_M E & \xrightarrow{\sigma_2} & E \\ \Theta_2 \downarrow & & \downarrow \pi \\ D & \xrightarrow{f_2} & M \end{array}$$

Reasoning as above, there exists an equivalence $\Phi_2 : D \times F \to D \times_M E$ of holomorphic fiber bundles over M and a point $c_2 \in F$ such that $\sigma_2 \circ \Phi_2(0, c_2) = \varphi_1(a_1)$. Define a holomorphic map $\varphi_2 : D \to E$ by

$$\varphi_2(z) = \sigma_2 \circ \Phi_2(z, c_2) \quad \text{for all } z \in D\,.$$

Continuing this process we find maps $\varphi_1, \ldots, \varphi_k \in \mathrm{Hol}(D, E)$ such that $\varphi_1(0)=\widetilde{x}$, $\varphi_i(a_i)=\varphi_{i+1}(0)$, $\varphi_k(a_k)\in\pi^{-1}(y)$. Thus $d_M(x,y)\ge d_E(\widetilde{x}, \pi^{-1}(y))$ for all $\widetilde{x} \in \pi^{-1}(x)$. ∎

3. Lemma. *Let* $\pi : \widetilde{X} \to X$ *be a holomorphic map between two complex spaces satisfying the following*: *For every* $x \in X$ *there exists an open neighbourhood* U_x *of* x *such that* $\pi^{-1}(U_x)$ *is taut. Then* $\widetilde{X}$ *is taut if so is* X.

Proof. Assume that X is taut and a sequence $\{\widetilde{f}_n\} \subset \mathrm{Hol}(D, \widetilde{X})$ is not compactly divergent. Without loss of generality we may suppose that there exists a sequence $\{z_n\} \subset D$ converging to a point $z \in D$ such that $\{\widetilde{f}_n(z_n)\}$ converges to a point $\widetilde{p} \in \widetilde{X}$.

Obviously $\{f_n = \pi \circ \widetilde{f}_n\} \subset \mathrm{Hol}(D, X)$ is not compactly divergent and we can assume that $\{f_n\}$ converges uniformly to a map $F \in \mathrm{Hol}(D, X)$.

Put $p = \pi(\widetilde{p}) = F(z)$. Take an open neighbourhood U_p of p such that $\pi^{-1}(U_p)$ is taut. Since $\{f_n\}$ converges to F, there exists an open subset V of D such that $f_n(V) \subset U_p$ for all $n \geq N$.

On the other hand, since $\{\widetilde{f}_n(z_n)\}$ converges to $\widetilde{p}$ and since $\pi^{-1}(U_p)$ is taut, we can assume that $\{\widetilde{f}_n|V\} \subset \mathrm{Hol}(V, \pi^{-1}(U_p))$ converges uniformly to a map $\widetilde{F} \in \mathrm{Hol}(V, \widetilde{X})$.

Consider the family $\mathcal{V}$ of all pairs (W, Φ), where W is an open subset of D and $\Phi \in \mathrm{Hol}(W, \widetilde{X})$ such that there exists a subsequence $\{\widetilde{f}_{n_k}|W\}$ of $\{\widetilde{f}_n|W\}$ which converges uniformly to Φ in $\mathrm{Hol}(W, \widetilde{X})$.

We define an order relation in $\mathcal{V}$ by setting $(W_1, \Phi_1) \leq (W_2, \Phi_2)$ if $W_1 \subset W_2$ and for every subsequence $\{\widetilde{f}_{n_k}|W_1\}$ of $\{\widetilde{f}_n|W_1\}$ converging to Φ_1 in $\mathrm{Hol}(W_1, \widetilde{X})$ uniformly on compact sets, the sequence $\{\widetilde{f}_{n_k}|W_2\}$ contains a subsequence converging to Φ_2 in $\mathrm{Hol}(W_2, \widetilde{X})$. Assume that $\{(W_\alpha, \Phi_\alpha)\}_{\alpha\in\Lambda}$ is a well-ordered subset of $\mathcal{V}$. Put $W_0 = \bigcup_{\alpha\in\Lambda} W_\alpha$. Define $\Phi_0 \in \mathrm{Hol}(W_0, \widetilde{X})$ by $\Phi_0|W_\alpha = \Phi_\alpha$ for all $\alpha \in \Lambda$. Take a sequence $\{(W_i, \Phi_i)\}_{i=1}^\infty \subset \{(W_\alpha, \Phi_\alpha)\}_{\alpha\in\Lambda}$ such that $(W_1, \Phi_1) \leq (W_2, \Phi_2) \leq \ldots$ and $W_0 = \bigcup_{i=1}^\infty W_i$. Then there exists a subsequence $\{\widetilde{f}_n^1|W_1\}$ of $\{\widetilde{f}_n|W_1\}$ converging to Φ_1 in $\mathrm{Hol}(W_1, \widetilde{X})$.

Consider $\{\widetilde{f}_n^1|W_2\}$. As above $\{\widetilde{f}_n^1\}$ contains a subsequence $\{\widetilde{f}_n^2\}$ such that $\{\widetilde{f}_n^2|W_2\}$ converges to Φ_2 in $\mathrm{Hol}(W_2, \widetilde{X})$.

Continuing this process we get sequences $\{\widetilde{f}_n^k\}$ such that $\{\widetilde{f}_n^k\} \in \{\widetilde{f}_n^{k-1}\}$ for every $k \geq 2$ and $\{\widetilde{f}_n^k|W_k\}$ converges uniformly to Φ_k in $\mathrm{Hol}(W_k, \widetilde{X})$. Therefore $\{\widetilde{f}_k^k\}$ converges to Φ_0 in $\mathrm{Hol}(W_0, \widetilde{X})$.

Thus $(W_0, \Phi_0) \in \mathcal{V}$ and hence the subset $\{(W_\alpha, \Phi_\alpha)\}_{\alpha\in\Lambda}$ has an upper bound. By Zorn's Lemma, the family $\mathcal{V}$ has a maximal element (W, Φ).

Let $\{\widetilde{f}_{n_k}|W\}$ be a subsequence of $\{\widetilde{f}_n|W\}$ converging uniformly to Φ in $\mathrm{Hol}(W, \widetilde{X})$. Take $z \in \overline{W}$ and an open neighbourhood U of $F(z_0)$ in X such that $\pi^{-1}(U)$ is taut. Since $\{f_n\}$ converges uniformly to $F \in \mathrm{Hol}(D, X)$ in $\mathrm{Hol}(D, X)$, there exists an open neighbourhood W_0 of z_0 in D such that $(\pi \circ \widetilde{f}_n)(W_0) \subset U$ for all $n \geq 1$. Hence $\widetilde{f}_n(W_0) \subset \pi^{-1}(U)$ for all $n \geq 1$.

Take $z_1 \in W_0 \cap W$. Then $\{\widetilde{f}_{n_k}(z_1)\}$ is convergent. By the normality of the family $\mathrm{Hol}(W_0, \pi^{-1}(U))$ and by the maximality of (W, Φ) we have $W_0 \subset W$ and $W = D$. ■

Notice that for the hyperbolicity the above lemma was proved by Eastwood [1].

4. Corollary. *Let $\pi : \widetilde{X} \to X$ be a holomorphic covering map between two complex spaces. Then $\widetilde{X}$ is taut if and only if so is X.*

P r o o f. $\Rightarrow$ Assume that $\widetilde{X}$ is taut and a sequence $\{f_n\} \subset \mathrm{Hol}(D, X)$ is not compactly divergent. Then there exists a sequence $\{z_n\} \subset D$ converging to $z_0 \in D$ such that $\{f_n(z_n)\}$ converges to $p \in X$. Put $f_n(z_n) = y_n$.

Since $\widetilde{X}$ is taut, X is hyperbolic. Take $\widetilde{p} \in \pi^{-1}(p)$. Since $d_X(y_n, p) = \inf_{\widetilde{y}_n \in \pi^{-1}(y_n)} d_{\widetilde{X}}(\widetilde{y}_n, \widetilde{p})$ (see [2]), there exists $\widetilde{y}_n \in \pi^{-1}(y_n)$ such that $d_{\widetilde{X}}(\widetilde{y}_n, \widetilde{p}) < d_X(y_n, p) + 1/n$. Lift f_n to a map $\widetilde{f}_n$ such that $\pi \circ \widetilde{f}_n = f_n$ and $\widetilde{f}_n(z_n) = \widetilde{y}_n$. Since $\{\widetilde{f}_n(z_n)\}$ converges to $\widetilde{p}$ and $\widetilde{X}$ is taut, we can assume that $\{\widetilde{f}_n\}$ converges uniformly to a map $\widetilde{f} \in \mathrm{Hol}(D, \widetilde{X})$ in $\mathrm{Hol}(D, \widetilde{X})$. Then $\{f_n\}$ converges to $\pi \circ \widetilde{f}$ in $\mathrm{Hol}(D, X)$.

$\Leftarrow$ The proof is deduced immediately from Lemma 3. ∎

5. Theorem B. *Let E be a holomorphic fiber bundle over M with fiber F and projection $\pi : E \to M$, where E, F, M are complex manifolds. Then E is taut if and only if both F and M are.*

Proof. $\Rightarrow$ Assume that E is taut. Since F embeds as a closed complex subspace in E, F is taut. Assume that a sequence $\{f_n\} \subset \mathrm{Hol}(D, M)$ is not compactly divergent. Without loss of generality we may suppose that there exists a sequence $\{z_n\} \subset D$ converging to $z \in D$ such that $\{y_n = f_n(z_n)\}$ converges to $p \in M$. Take $\widetilde{p} \in \pi^{-1}(p)$. By Theorem A, $d_M(y_n, p) = \inf_{x_n \in \pi^{-1}(y_n)} d_E(\widetilde{p}, x_n)$. Hence there exists $x_n \in E$ such that $d_E(\widetilde{p}, x_n) < d_M(p, y_n) + 1/n$.

Pull back the fiber bundle $\pi : E \to M$ to a fiber bundle $\Theta_n : D \times_M E \to D$ getting a commutative diagram

$$\begin{array}{ccc} D \times_M E & \xrightarrow{\sigma_n} & E \\ \Theta_n \downarrow & & \downarrow \pi \\ D & \xrightarrow{f_n} & M \end{array}$$

As in the proof of Theorem A there exists an equivalence $\Phi_n : D \times F \to D \times_M E$ of holomorphic fiber bundles over D and $c_n \in F$ such that $\sigma_n \circ \Phi_n(z_n, c_n) = x_n$. Put $\varphi_n(z) = \sigma_n \circ \Phi_n(z, c_n)$ for all $z \in D$. It is easy to see that $\{\varphi_n(z_n)\}$ converges to $\widetilde{p} \in E$. Without loss of generality we may assume that $\{\varphi_n\}$ converges to a map $\varphi \in \mathrm{Hol}(D, E)$ in $\mathrm{Hol}(D, E)$. Hence $\{f_n\}$ converges to $\pi \circ \varphi$ in $\mathrm{Hol}(D, M)$.

$\Leftarrow$ The converse is deduced immediately from Lemma 3. ∎

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DEPARTMENT OF MATHEMATICS
PEDAGOGICAL INSTITUTE NO. 1 OF HA NOI
HA NOI, VIET NAM

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