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A generalization of the saddle point method with applications

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Abstract. We show that one can drop an important hypothesis of the saddle point theorem without affecting the result. We then show how this leads to stronger results in applications.

1. Introduction. The saddle point theorem is a useful extension of the mountain pass theorem (cf., e.g., [Ra2]). In its simplest form, it can be described as follows. Let G be a continuously differentiable functional on a Hilbert space H, and let $H = M \oplus N$ be an orthogonal decomposition of H into closed subspaces M, N with $\dim N < \infty$. Assume that

$$\inf_M G = m_0 > -\infty \tag{1.1}$$

and

$$\limsup_{\|v\|\to\infty,\, v\in N} G(v) = m < \infty\,. \tag{1.2}$$

The theorem states that if $m < m_0$, then there is a constant $c \geq m_0$ and a sequence $\{u_k\} \subset H$ such that

$$G(u_k) \to c, \qquad G'(u_k) \to 0\,. \tag{1.3}$$

This in itself does not provide a solution of

$$G'(u) = 0 \tag{1.4}$$

but an additional hypothesis such as the Palais–Smale (PS) condition does indeed provide such a solution.

The purpose of the present paper is to show that the assumption $m < m_0$ is unnecessary. Indeed, the conclusion can be reached under hypotheses (1.1)

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and (1.2) even if $m_0 \leq m$ and even if

$$\sup_{\|v\|=R,\, v\in N} G(v) > m_0 \tag{1.5}$$

for every $R > 0$. In fact we have

THEOREM 1.1. *Assume* (1.1), (1.2). *Let* ψ *be a nonincreasing function in* $(0,\infty)$ *such that*

$$\limsup_{R\to\infty} \int_R^{2R} \psi(r)\,dr = \infty\,. \tag{1.6}$$

Let

$$m_1 = \sup_N G\,. \tag{1.7}$$

Then there is a constant c *satisfying*

$$m_0 \leq c \leq m_1 \tag{1.8}$$

and a sequence $\{u_k\} \subset H$ *such that*

$$G(u_k) \to c, \qquad G'(u_k)/\psi(\|u_k\|) \to 0\,. \tag{1.9}$$

It should be noted (as we shall show) that if there is an $R > 0$ such that the left hand side of (1.5) is less than m_0, then the same conclusion holds even if ψ satisfies the weaker condition

$$\int_1^{\infty} \psi(r)\,dr = \infty\,. \tag{1.10}$$

However, in many applications the existence of such an R is practically impossible to verify when $m > m_0$.

As an application, consider the Dirichlet problem

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.11}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|f(x,t)| \leq C(|t|+1), \qquad x\in\Omega,\ t\in\mathbb{R}\,, \tag{1.12}$$

$$f(x,t)/t \to b_\pm(x) \qquad \text{a.e. as } t \to \pm\infty\,. \tag{1.13}$$

If

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_l < \lambda_{l+1} < \ldots \tag{1.14}$$

are the eigenvalues of the linear problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.15}$$

we assume that for some $l \geq 0$

$$\lambda_l t^2 - W_1(x) \leq 2F(x,t) \leq \lambda_{l+1} t^2 + W_2(x) \tag{1.16}$$

where

$$F(x,t) := \int_0^t f(x,s)\,ds \tag{1.17}$$

and the $W_j(x)$ are functions in $L^1(\Omega)$. We shall prove

THEOREM 1.2. *Under the above hypotheses, problem* (1.11) *has a solution provided there does not exist an eigenfunction v of* (1.15) *corresponding to λ_l such that*

$$\begin{aligned} b_+(x) &= \lambda_l \quad \text{a.e. when } v(x) > 0\,, \\ b_-(x) &= \lambda_l \quad \text{a.e. when } v(x) < 0\,, \end{aligned}$$

and there does not exist an eigenfunction w of (1.15) *corresponding to λ_{l+1} such that*

$$\begin{aligned} b_+(x) &= \lambda_{l+1} \quad \text{a.e. when } w(x) > 0\,, \\ b_-(x) &= \lambda_{l+1} \quad \text{a.e. when } w(x) < 0\,. \end{aligned}$$

Theorem 1.2 is essentially known (cf. [BF]). The novelty is that it is a simple consequence of Theorem 1.1. An example of a new application is

THEOREM 1.3. *Let* $u^+ = \max(u,0)$, $u^- = (-u)^+$,

$$b(u) = \int_\Omega \{b_+(u^+)^2 + b_-(u^-)^2\}\,dx \tag{1.18}$$

$$N = \bigoplus_{\lambda \le \lambda_l} N(A-\lambda), \qquad D = H_0^{1,2}(\Omega), \qquad M = N^\perp \cap D\,.$$

Assume (1.12), (1.13) *and*

$$|2F(x,t) - b_+(t^+)^2 - b_-(t^-)^2| \le W(x) \in L^1(\Omega) \tag{1.19}$$

$$\|v\|_D^2 \le b(v), \qquad v \in N\,, \tag{1.20}$$

$$\|w\|_D^2 \ge b(w), \qquad w \in M\,. \tag{1.21}$$

Assume also that the only functions $v \in N$, $w \in M$ satisfying

$$-\Delta(v+w) = b_+(v^+ + w^+) - b_-(v^- + w^-)\,, \tag{1.22}$$

$$(b_+v^+ - b_-v^-, w) = (v, b_+w^+ - b_-w^-)\,, \tag{1.23}$$

$$b_+(x) = b_-(x) \quad \text{when } v(x)w(x) < 0 \tag{1.24}$$

are $v = w = 0$. Then (1.11) *has at least one solution.*

The problem (1.11) is called *resonant at infinity* if

$$\liminf_{|t|\to\infty} f(x,t)/t \le \lambda_k \le \limsup_{|t|\to\infty} f(x,t)/t$$

for some eigenvalue λ_k of (1.15). In the case of Theorem 1.2 we have

$$\lambda_l \le \liminf_{|t|\to\infty} f(x,t)/t \le \limsup_{|t|\to\infty} f(x,t)/t \le \lambda_{l+1}$$

and we allow

$$\lambda_l = \inf_x \liminf_{|t|\to\infty} f(x,t)/t, \qquad \lambda_{l+1} = \sup_x \limsup_{|t|\to\infty} f(x,t)/t.$$

Thus our situation can be called *double resonance*. There is quite an extensive literature concerning resonance problems beginning with the work of Landesman and Lazer [LL]. Many are quoted in our bibliography and in the references quoted in them. Of particular relevance to the problem we consider are Berestycki–de Figueiredo [BF], Cac [Cac1], de Figueiredo–Gossez [DFG] and Lazer–McKenna [LM2,3]. Our proof of Theorem 1.1 is based upon ideas found in Brezis–Nirenberg [BN].

2. The basic theorem. In this section we present the theorem from which our results are obtained. Let

$$H = M \oplus N, \qquad M \ne \{0\},\ M \ne H, \tag{2.1}$$

be an orthogonal decomposition of a Hilbert space H into subspaces M, N with $\dim N < \infty$. Let $G(u)$ be a continuously differentiable functional on H, and let

$$B_R := \{u \in H : \|u\| \le R\}, \tag{2.2}$$

$$\partial B_R := \{u \in H : \|u\| = R\}. \tag{2.3}$$

We assume

$$\inf_M G = m_0 > -\infty. \tag{2.4}$$

Let $R > 0$ be fixed, and let

$$m := \max_{\partial B_R \cap N} G, \qquad m_1 := \max_{B_R \cap N} G. \tag{2.5}$$

(Clearly $m \le m_1$.) Let Ψ denote the set of nonincreasing functions $\psi(r)$ on $(0,\infty)$ such that

$$\int_1^\infty \psi(r)\,dr = \infty. \tag{2.6}$$

Our basic theorem is

THEOREM 2.1. *For each $\psi \in \Psi$ satisfying*

$$\int_R^{2R} \psi(r)\,dr > m - m_0 \tag{2.7}$$

there are a constant c and a sequence $\{u_k\} \subset H$ such that

$$m_0 \le c \le m_1\,, \tag{2.8}$$

$$G(u_k) \to c, \quad \|G'(u_k)\| \le \psi(\|u_k\|)\,. \tag{2.9}$$

Proof. Let ψ be a function in Ψ satisfying (2.7). If the conclusion of the theorem did not hold, there would be an $\varepsilon > 0$ such that

$$\psi(\|u\|) < \|G'(u)\| \tag{2.10}$$

holds for all u in the set $Q_0 := \{u \in H : m_0 - 3\varepsilon \le G(u) \le m_1 + 3\varepsilon\}$. If necessary, reduce ε so that

$$m - m_0 + \varepsilon < \alpha \int_R^{R+T} \psi(r)\,dr \tag{2.11}$$

holds for some $T < R$ and $\alpha < 1$. Let

$$Q = \{u \in H : m_0 - 2\varepsilon \le G(u) \le m_1 + 2\varepsilon\}\,,$$

$$Q_1 = \{u \in H : m_0 - \varepsilon \le G(u) \le m_1 + \varepsilon\}\,,$$

$$Q_2 = H \setminus Q, \quad \eta(u) := d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]\,.$$

There is a locally Lipschitz continuous map $Y(u)$ of $\widehat{H} := \{u \in H : G'(u) \neq 0\}$ into H such that

$$\|Y(u)\| \le 1 \quad \text{and} \quad (G'(u), Y(u)) \ge \alpha\|G'(u)\|, \quad u \in \widehat{H} \tag{2.12}$$

(cf., e.g., [Sc1, 3, 6]). Thus for each $u \in H$ there is a unique solution $\sigma(t, u)$ of

$$\sigma'(t) = -\eta(\sigma)Y(\sigma), \quad t \ge 0, \quad \sigma(0) = u\,. \tag{2.13}$$

Consequently,

$$\|\sigma(t, u) - u\| \le t\,, \tag{2.14}$$

$$\begin{aligned} dG(\sigma(t, u))/dt &= (G'(\sigma), \sigma') = -\eta(\sigma)(G'(\sigma), Y(\sigma)) \\ &\le -\alpha\eta(\sigma)\|G'(\sigma)\| \le -\alpha\eta(\sigma)\psi(\|\sigma\|) \le 0\,. \end{aligned} \tag{2.15}$$

Let P denote the (orthogonal) projection of H onto N. For $v \in N$ we have

$$\|v\| - \|P\sigma(t, v)\| \le \|v - P\sigma(t, v)\| \le \|v - \sigma(t, v)\| \le t\,.$$

Hence for $v \in N \cap \partial B_R$ and $t \in [0, T]$ we have

$$\|P\sigma(t, v)\| \ge R - T > 0\,. \tag{2.16}$$

Let $\varphi_t(v)$ be any continuous map of $[0, T] \times (B_R \cap N)$ to H such that

$$\varphi_t(v) = \sigma(t, v), \quad v \in \partial B_R \cap N\,. \tag{2.17}$$

By (2.16), $P\varphi_t(v) \neq 0$ for $t \in [0, T]$ and $v \in \partial B_R \cap N$. Hence the Brouwer index $i(P\varphi_t, B_R \cap N, 0)$ is defined and satisfies

$$i(P\varphi_t, B_R \cap N, 0) = i(P, B_R \cap N, 0) = 1\,. \tag{2.18}$$

This means that

$$\varphi_t(B_R \cap N) \cap M \neq \emptyset, \quad 0 \le t \le T. \tag{2.19}$$

Let v be any element in $\partial B_R \cap N$. If there is a $t_1 \le T$ such that $\sigma(t_1, v) \notin Q_1$, then $G(\sigma(T,v)) \le G(\sigma(t_1,v)) < m_0 - \varepsilon$ since $G(\sigma(t,v))$ cannot go above m. On the other hand, if $\sigma(t,v) \in Q_1$ for $0 \le t \le T$, then

$$\begin{aligned} G(\sigma(T,v)) - G(v) &\le -\alpha \int_0^T \psi(\|\sigma(t,v)\|)\,dt \\ &\le -\alpha \int_0^T \psi(R+t)\,dt \le -(m - m_0 + \varepsilon) \end{aligned}$$

by (2.11), (2.14) and (2.15). Hence

$$G(\sigma(T,v)) < m_0 - \varepsilon, \quad v \in \partial B_R \cap N. \tag{2.20}$$

Let S denote the set of mappings φ from $B_R \cap N$ to H such that

$$\varphi(v) = \sigma(T,v), \quad v \in \partial B_R \cap N. \tag{2.21}$$

Since $\sigma(T,v) \in S$, $S \neq \emptyset$. Define

$$c := \inf_{\varphi \in S} \max_{v \in B_R \cap N} G(\varphi(v)). \tag{2.22}$$

By (2.19), $\varphi(B_R \cap N)$ intersects M, and consequently $m_0 \le c$ in view of (2.4). Also

$$c \le \max_{v \in B_R \cap N} G(\sigma(T,v)) \le \max_{v \in B_R \cap N} G(v) \le m_1$$

by (2.15) and (2.21). Let $\delta = \varepsilon/3$ and define

$$Q' = \{u \in Q_0 : |G(u) - c| \le 2\delta\}, \quad Q_1 = \{u \in Q' : |G(u) - c| \le \delta\},$$
$$Q_2' = H \setminus Q', \quad \eta_1(u) = d(u, Q_2')/[d(u, Q_1') + d(u, Q_2')].$$

For each $u \in H$ there is a unique solution $\sigma_1(t,u)$ of

$$\sigma'(t) = -\eta_1(\sigma) Y(\sigma), \quad t \ge 0, \quad \sigma(0) = u. \tag{2.23}$$

As before we have

$$\|\sigma_1(t,u) - u\| \le t, \tag{2.24}$$

$$\begin{aligned} dG(\sigma_1(t,u))/dt &= (G'(\sigma_1), \sigma_1') = -\eta_1(\sigma_1)(G'(\sigma_1), Y(\sigma_1)) \\ &\le -\alpha\eta_1(\sigma_1)\|G'(\sigma_1)\| \le -\alpha\eta_1(\sigma_1)\psi(\|\sigma_1\|) \le 0. \end{aligned} \tag{2.25}$$

From the fact that c satisfies (2.8) we see that (2.10) holds for all $u \in Q'$. From the definition (2.22) of c we see that there must be a $\varphi \in S$ such that

$$G(\varphi(v)) < c + \delta, \quad v \in B_R \cap N. \tag{2.26}$$

Let

$$M = \max_{B_R \cap N} \|\varphi(v)\| \tag{2.27}$$

and pick T_1 so that

$$\alpha \int_M^{M+T_1} \psi(r)\, dr > 2\delta . \tag{2.28}$$

Let v be any element in $B_R \cap N$. If there is a $t_1 \le T_1$ such that $\sigma_1(t_1, \varphi(v)) \notin Q_1'$, then (2.25) implies

$$G(\sigma_1(T_1, \varphi(v))) \le G(\sigma_1(t_1, \varphi(v))) < c - \delta$$

since $G(\sigma_1(t, \varphi(v)))$ cannot go above $c+\delta$ by (2.25) and (2.26). On the other hand, if $\sigma_1(t, \varphi(v)) \in Q_1'$ for $0 \le t \le T_1$, then (2.25) implies

$$\begin{aligned} G(\sigma_1(T_1, \varphi(v))) - G(\varphi(v)) &\le -\alpha \int_0^{T_1} \psi(\|\sigma_1(t, \varphi(u))\|)\, dt \\ &\le -\alpha \int_0^{T_1} \psi(M+t)\, dt \\ &= -\alpha \int_M^{M+T_1} \psi(r)\, dr < -2\delta . \end{aligned} \tag{2.29}$$

Thus by (2.26)

$$G(\sigma_1(T_1, \varphi(v))) < c - \delta, \qquad v \in B_R \cap N . \tag{2.30}$$

Let

$$\varphi_1(v) := \sigma_1(T_1, \varphi(v)), \qquad v \in B_R \cap N .$$

Since $\varphi \in S$, it satisfies (2.21). Consequently,

$$G(\varphi(v)) < m_0 - \varepsilon, \qquad v \in \partial B_R \cap N , \tag{2.31}$$

by (2.20). Since $\varepsilon = 3\delta$, this means that $\varphi(v) \in Q_2'$ for $v \in \partial B_R \cap N$. Thus $\eta_1(\varphi(v)) = 0$ for such v. This implies $\sigma_1(t, \varphi(v)) \equiv \varphi(v) = \sigma(T, v)$ for such v. Hence $\varphi_1 \in S$. But this causes (2.30) to contradict (2.22). Since the entire argument is based solely upon assumption (2.10), the conclusion of the theorem must hold. ∎

COROLLARY 2.2. *Under the same hypotheses, if* $m < m_0$, *then there is a constant* c *satisfying* (2.8) *such that for each* $\psi \in \Psi$ *there is a sequence* $\{u_k\} \subset H$ *for which*

$$G(u_k) \to c, \qquad G'(u_k)/\psi(\|u_k\|) \to 0 . \tag{2.32}$$

Proof. In this case we do not have to use $\sigma(T, v)$ in (2.20). It can be replaced by

$$G(v) < m_0 - \varepsilon, \qquad v \in \partial B_R \cap N , \tag{2.33}$$

for ε sufficiently small. We can define c immediately not depending on ψ. We let S denote the set of all continuous maps φ of $B_R \cap N$ into H such that $\varphi(v) = v$ on $\partial B_R \cap N$. We then take

$$c := \inf_{\varphi \in S} \max_{v \in B_R \cap N} G(v) . \tag{2.34}$$

We then proceed with the proof of Theorem 2.1. If $\psi \in \Psi$, then for each $k > 0$, $\psi/k \in \Psi$. By what has been proved, there is a $u_k \in H$ such that $|G(u_k) - c| < 1/k$ and $\|G'(u_k)\| < \psi(\|u_k\|)/k$. This gives (2.32). ∎

Proof of Theorem 1.1. For each $k > 0$ there is an R_k such that

$$\int_{R_k}^{2R_k} \psi(r)\, dr > k(m_1 - m_0) .$$

Then $\psi_k(r) = \psi(r)/k \in \Psi$ and satisfies

$$\int_{R_k}^{2R_k} \psi_k(r)\, dr > m_1 - m_0 .$$

Since

$$\max_{\partial B_{R_k} \cap N} G \le \max_{B_{R_k} \cap N} G \le m_1, \qquad k = 1, 2, \ldots ,$$

we can apply Theorem 2.1 for each k to find a $u_k \in H$ and a constant c_k such that

$$m_0 \le c_k \le m_1, \quad |G(u_k) - c_k| < 1/k, \quad \|G'(u_k)\| < \psi_k(\|u_k\|) .$$

A subsequence will satisfy (2.8) and (2.32). ∎

3. An application. We now consider a semilinear boundary value problem which can be solved by means of Theorem 1.1. Let Ω be a bounded domain in $\mathbb{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$. We assume that $A \ge \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega) \tag{3.1}$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions on Ω and $H^{m,2}(\Omega)$ is the Sobolev space with norm $\|u\|_{m,2}$. For m an integer, this norm is equivalent to the sum of the $L^2(\Omega)$ norms of u and all its derivatives up to order m. For m not an integer, the norm can be defined by interpolation (cf., e.g., [LM]). We assume that the spectrum of A consists only of eigenvalues λ_k of finite multiplicity satisfying

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l < \lambda_{l+1} < \ldots \tag{3.2}$$

We consider the semilinear problem

$$Au = f(x, u), \qquad u \in D , \tag{3.3}$$

where $f(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ (continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$). We assume

(A) The function $f(x,t)$ satisfies

$$|f(x,t)| \le C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{3.4}$$

$$f(x,t)/t \to b_{\pm}(x) \quad \text{a.e. as } t \to \pm\infty. \tag{3.5}$$

(B) If

$$F(x,t) := \int_0^t f(x,s)\,ds, \tag{3.6}$$

then

$$\lambda_l t^2 - W_1(x) \le 2F(x,t) \le \lambda_{l+1}t^2 + W_2(x) \tag{3.7}$$

for some $l \ge 0$, where the $W_j(x)$ are functions in $L^1(\Omega)$.

(C) No eigenfunction of A corresponding to either λ_l or λ_{l+1} is a solution of

$$Au = b_+(x)u^+(x) - b_-(x)u^-(x) \tag{3.8}$$

where $u^+(x) = \max[u(x), 0]$ and $u^-(x) = u^+(x) - u(x)$.

THEOREM 3.1. *Under hypotheses* (A)–(C), *equation* (3.3) *has a solution.*

Proof. We define

$$G(u) = a(u) - 2\int_\Omega F(x,u)\,dx \tag{3.9}$$

where $a(u,v) = (Au, v)$, $a(u) = a(u,u)$, $u \in D$. Let $N := \bigoplus_{\lambda \le \lambda_l} N(A-\lambda)$, $M = N^\perp \cap D$. By (3.5) and (3.7) we have

$$\lambda_l \le b_{\pm}(x) \le \lambda_{l+1}. \tag{3.10}$$

If we put

$$B_j = \int_\Omega W_j(x)\,dx, \quad j = 1,2,$$

we have

$$G(v) \le a(v) - \lambda_l\|v\|^2 + B_1 \le B_1, \qquad v \in N,$$
$$G(w) \ge a(w) - \lambda_{l+1}\|w\|^2 - B_2 \ge -B_2, \quad w \in M,$$

by (3.7). Thus (1.1) and (1.2) are verified. Moreover, it is easily checked that (3.4) implies that $G(u)$ is continuously differentiable on D. Hence the hypotheses of Theorem 1.1 are satisfied. Thus there is a sequence $\{u_k\} \subset D$ and a number c such that $-B_2 \le c \le B_1$ and

$$G(u_k) \to c, \quad G'(u_k) \to 0. \tag{3.11}$$

Since

$$(G'(u), h) = 2a(u, h) - 2(f(x, u), h), \qquad u, h \in D, \tag{3.12}$$

a solution of

$$G'(u) = 0, \qquad u \in D, \tag{3.13}$$

is a solution of (3.3). I claim that the sequence satisfying (3.11) satisfies

$$t_k^2 := a(u_k) \le C_1. \tag{3.14}$$

To see this, assume that there is a renamed subsequence such that $t_k \to \infty$. Let $\widetilde{u}_k = u_k/t_k$. Then $a(\widetilde{u}_k) = 1$. Hence there is a renamed subsequence such that $\widetilde{u}_k \to \widetilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. By (3.11) and (3.12)

$$a(u_k, h) - (f(x, u_k), h) \to 0, \qquad h \in D. \tag{3.15}$$

Consequently, $\widetilde{u}$ is a solution of (3.8). We shall show that this implies that $\widetilde{u} \equiv 0$. But (3.11) also gives

$$t_k^2 - 2 \int_\Omega F(x, u_k)\, dx \to c.$$

This implies

$$\int_\Omega \{b_+(x)(\widetilde{u}^+)^2 + b_-(x)(\widetilde{u}^-)^2\}\, dx = 1, \tag{3.16}$$

which cannot hold if $\widetilde{u} \equiv 0$. Thus the t_k are bounded. Hence there is a renamed subsequence such that $u_k \to u$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. In this case (3.15) implies that u is a solution of (2.13) and consequently of (3.3).

It remains to show that every solution of (3.8) vanishes identically. This is done in

LEMMA 3.2. *If $u \in D$ is a solution of* (3.8) *and* (A)–(C) *hold, then* $u \equiv 0$.

Proof. We can write $u = v + w$, where $v \in N$ and $w \in M$. Let $\widehat{u} = w - v$ and

$$q(x) = \begin{cases} b_+(x) & \text{when } u(x) \ge 0, \\ b_-(x) & \text{when } u(x) < 0. \end{cases}$$

Then u is a solution of

$$Au = qu \tag{3.17}$$

and we have

$$\begin{aligned} a(w) - a(v) &= a(w + v, w - v) = (Au, \widehat{u}) = (qu, \widehat{u}) \\ &= (q[w + v], w - v) = (qw, w) - (qv, v). \end{aligned}$$

Hence $0 \le a(w) - (qw, w) = a(v) - (qv, v) \le 0$ since

$$\lambda_l \le q(x) \le \lambda_{l+1}\,, \tag{3.18}$$

$$a(v) \le \lambda_l \|v\|^2, \quad a(w) \ge \lambda_{l+1}\|w\|^2\,. \tag{3.19}$$

Consequently,

$$a(v) = (qv, u), \quad a(w) = (qw, w)\,. \tag{3.20}$$

This implies

$$q(x) = \begin{cases} \lambda_l & \text{when } v(x) \ne 0\,, \\ \lambda_{l+1} & \text{when } w(x) \ne 0\,, \end{cases}$$

and $v \in N(A - \lambda_l)$, $w \in N(A - \lambda_{l+1})$. In particular, $v(x)w(x) \equiv 0$ and

$$q(x) = \begin{cases} b_+ = \lambda_l & \text{when } v(x) > 0 \text{ since } w(x) = 0 \text{ and } u(x) > 0\,, \\ b_- = \lambda_l & \text{when } v(x) < 0 \text{ since } w(x) = 0 \text{ and } u(x) < 0\,, \\ b_+ = \lambda_{l+1} & \text{when } w(x) > 0 \text{ since } v(x) = 0 \text{ and } u(x) > 0\,, \\ b_- = \lambda_{l+1} & \text{when } w(x) < 0 \text{ since } v(x) = 0 \text{ and } u(x) < 0\,. \end{cases}$$

Thus

$$Av = \lambda_l v = b_+ v^+ - b_- v^-, \quad Aw = \lambda_{l+1} w = b_+ w^+ - b_- w^-\,.$$

By hypothesis (C), $v = w = 0$. Hence $u = 0$, and the lemma and theorem are proved. ■

Proof of Theorem 1.2. If we take $A = -\Delta$, $D = H_0^{1,2}(\Omega)$, then hypotheses (A) and (B) of Theorem 3.1 are satisfied. Hypothesis (C) is also satisfied because no eigenfunction corresponding to λ_l or λ_{l+1} satisfies (3.8). Thus by Theorem 3.1 there is a $u \in D$ which satisfies (3.3). By (1.12), $f(x, u) \in L^2(\Omega)$. Elliptic regularity theory now shows that $u \in H^{2,2}(\Omega)$. Thus u is a strong solution of (1.11). ■

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