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Most random walks on nilpotent groups are mixing

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Abstract. Let G be a second countable locally compact nilpotent group. It is shown that for every norm completely mixing (n.c.m.) random walk μ, $\alpha\mu + (1-\alpha)\nu$ is n.c.m. for $0 < \alpha \le 1$, $\nu \in P(G)$. In particular, a generic stochastic convolution operator on G is n.c.m.

1. Introduction. Let G be a locally compact group with a left Haar measure m. We denote by $M(G)$ the convolution Banach algebra of finite Radon measures on G. The convex closed subset of (Radon) probabilities on G will be denoted by $P(G)$. For a Banach space X, $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators on X. An operator $T \in \mathcal{L}(M(G))$ is called a *convolution operator* if for some $\mu \in M(G)$ it is equal to T_μ or ${}_\mu T$, where $T_\mu \nu = \nu * \mu$ and ${}_\mu T\nu = \mu * \nu$, $\nu \in M(G)$. Thus the mapping $\mu \to T_\mu$ (${}_\mu T$) is a representation of $M(G)$ by a semigroup of right (left) convolution operators on $M(G)$. We let $L^1(m)$ be the Banach space of real-valued m-integrable functions on G. Then for each $\mu \in P(G)$, T_μ and ${}_\mu T$ are stochastic operators on $L^1(m)$, i.e. they take $P(G) \cap L^1(m)$ into itself.

It is well known that there is a 1-1 correspondence between the stochastic convolution operators on $L^1(m)$ and the random walks on G. This means that for a given random walk with law $\mu \in P(G)$, the right transition probability $p_\mu(g,\cdot) = \delta_g * \mu$ defines a stochastic convolution operator $T_\mu(\nu) = \int p_\mu(g,\cdot)\,d\nu(g)$, $\nu \in M(G)$. We say that T_μ is *induced* by the random walk μ. Analogously ${}_\mu T$ is induced by the left transition probability ${}_\mu p$.

Consider a right random walk with law μ. A bounded Borel function f on G is called *μ-harmonic* if it is p_μ-invariant, or equivalently, if $\int f(gh)\,d\mu(h) = f(g)$ for every $g \in G$ (for the left random walk ${}_\mu p$ the definition is similar). If all the μ-harmonic functions are constant, we say that the ran-

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dom walk is *ergodic*. It is easy to check that the random walk is ergodic iff the induced stochastic convolution operator is ergodic, i.e. constant functions are the only fixed points for the adjoint to T_μ (resp. to ${}_\mu T$) acting on $L^\infty(m)$.

We say that a random walk is *norm completely mixing* (n.c.m.) if the corresponding stochastic convolution operator is norm completely mixing, i.e. if for every pair $\nu_1, \nu_2 \in P(G) \cap L^1(m)$ we have $\lim \|T_\mu^n(\nu_1 - \nu_2)\| = 0$ (resp. for ${}_\mu T$).

Rosenblatt proved that there exists at least one n.c.m. random walk iff G is σ-compact and amenable ([9], Thm. 1.10 and Prop.). As was observed in [6], if G is second countable and abelian, the set of n.c.m. random walks forms a dense G_δ subset of the set of all stochastic convolution operators on $L^1(m)$ for both strong operator topology (s.o.t.) and norm operator topology (n.o.t.) ([6], Thm. 5).

We shall extend the above result to second countable nilpotent groups.

2. Generic stochastic convolution operator on a nilpotent group is n.c.m. Let G be a second countable l.c. group with a left Haar measure m. The proof of Thm. 5 in [6] was based on the classical Choquet–Deny theorem, which says that in abelian groups, a random walk μ is ergodic iff the closed subgroup generated by the support of μ is the whole group G (see e.g. [8], Ch. 5). Unfortunately, this is not true in general l.c. groups. As follows from the theory of Poisson spaces, for some random walks on nilpotent groups the Choquet–Deny theorem does hold (see [1], Prop. IV.10). For the reader's convenience we present some of the relevant facts of this theory. For the proofs see [1].

Take a random walk with law $\mu \in P(G)$. Then the space of all uniformly continuous μ-harmonic functions is isometrically isomorphic to a certain C^* commutative algebra with unit. Its spectrum Π_μ, which is a compact G-space, is called the *Poisson space* of G corresponding to the random walk μ. The Gelfand transform of this algebra is called the *Poisson formula.* The Poisson formula determines all μ-harmonic functions if μ is *spread-out*, i.e. μ^{*n} is not singular with respect to m for some positive integer n ([1], Thm. I.3).

Now assume G is amenable. By combining Thm. I.3, Prop. IV.7 and Prop. IV.8 of [1], the following conditions are seen to be equivalent for a (right) random walk with a spread-out law μ:

(i) *The space of μ-harmonic functions is finite-dimensional*;

(ii) *Π_μ is finite*;

(iii) *Π_μ is isomorphic to G/H as a G-space, where H is the closed subgroup generated by the support of μ*;

(iv) Π_μ *is a homogeneous G-space.*

Now, we see that the stochastic convolution operator T_μ is ergodic iff $G = H$ and Π_μ is homogeneous. Since in the case of nilpotent groups and μ spread-out, Π_μ is homogeneous iff H has a finite index in G ([1], Prop. IV.10), the above remark shows that for nilpotent groups with spread-out measures the classical Choquet–Deny theorem holds.

THEOREM. *Let G be a second countable nilpotent l.c. group. For every n.c.m. random walk with spread-out law μ the random walk $\mu_\alpha = \alpha\mu + (1-\alpha)\nu$ is n.c.m. for every $\nu \in P(G)$ and $\alpha \in (0, 1]$.*

Proof. From §5, Ch. 2 of [5], the assumption that μ is spread-out and n.c.m. is equivalent to $\|(\nu_1 - \nu_2) * \mu^{*n}\| \to 0$ $(\nu_1, \nu_2 \in P(G))$. This means that the random walk induces an ergodic "space-time" random walk on $G \times \mathbb{Z}$, where $\mathbb{Z}$ is the group of integers (see Lemma 3 of [7]). Now it is clear that the support of μ is not contained in a coset of a proper closed subgroup of G. Therefore, the same holds for the support of μ_α. To complete the proof apply the Choquet–Deny theorem and Thm. 2 of [3]. ∎

It is worth pointing out that the above theorem also follows from Proposition 2.5 of [9] and from the fact that for nilpotent groups the Choquet–Deny theorem holds.

COROLLARY 1. *The set of n.c.m. random walks on a nilpotent group G is dense in the norm topology of $P(G)$.*

Proof. We only need to show that on a nilpotent group there is at least one n.c.m. random walk with a spread-out law. Since nilpotent groups are amenable, this follows from the Rosenblatt theorem [9] (alternatively we can use the Choquet–Deny theorem). ∎

COROLLARY 2. *If G is nilpotent, then the set of n.c.m. stochastic convolution operators is a dense G_δ set in the set of all stochastic convolution operators for both s.o.t. and n.o.t.*

Proof. First note that the representation $\mu \to T_\mu$ $({}_\mu T)$ is norm continuous. Now the set of n.c.m. stochastic convolution operators, being the intersection of the sets of n.c.m. stochastic operators and the stochastic convolution operators, is a G_δ in s.o.t. (see Thm. 3 of [6] and the Wendel Theorem of [4]). Therefore, Corollary 2 follows from Corollary 1. ∎

Recently, the author was informed by W. Bartoszek that Corollary 1 holds for arbitrary amenable σ-compact l.c. groups [2]. It is not known whether our Theorem is also true in that case.

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