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Differential conditions to verify the Jacobian Conjecture

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Abstract. Let F be a polynomial mapping of $\mathbb{R}^2$, $F(0) = 0$. In 1987 Meisters and Olech proved that the solution $y(\cdot) = 0$ of the autonomous system of differential equations $\dot{y} = F(y)$ is globally asymptotically stable provided that the jacobian of F is everywhere positive and the trace of the matrix of the differential of F is everywhere negative. In particular, the mapping F is then injective. We give an n-dimensional generalization of this result.

1. Introduction. The following problem was explicitly stated by Markus and Yamabe (cf. [MY], [O]).

GLOBAL STABILITY PROBLEM IN $\mathbb{R}^2$. *Let $F = (F_1, F_2)$ be a C^1 transformation of $\mathbb{R}^2, F(0,0) = (0,0)$. Assume that the matrix of the differential of F has, at any point x of $\mathbb{R}^2$, all eigenvalues with negative real parts; that is, assume that*

$$\text{(J)} \qquad \operatorname{Jac} F(x_1, x_2) > 0 \quad \textit{for every } (x_1, x_2) \in \mathbb{R}^2$$

and

$$\text{(T)} \qquad \operatorname{Tr} F'(x_1, x_2) := \frac{\partial F_1}{\partial x_1}(x_1, x_2) + \frac{\partial F_2}{\partial x_2}(x_1, x_2) < 0 \quad \textit{for every } (x_1, x_2) \in \mathbb{R}^2 .$$

Does it then follow that the solution $(x_1, x_2) = (0,0)$ of the autonomous system of differential equations

$$(*) \qquad \dot{x}_1(t) = F_1(x_1, x_2), \qquad \dot{x}_2(t) = F_2(x_1, x_2)$$

is globally asymptotically stable? That is, does every solution curve of $(*)$ approach $(0,0)$ as $t \to \infty$?

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It has been showed that the problem has an affirmative solution under some additional conditions (cf. e.g. [H], [HO], [MO], [MY], [O], [P]). In particular, Meisters and Olech proved in [MO] that the answer is positive provided F is a polynomial mapping. As a consequence they showed that a polynomial mapping F is injective, which was a partial affirmative answer to the Global Univalence Problem formulated by Olech (cf. [O]):

GLOBAL UNIVALENCE PROBLEM IN $\mathbb{R}^2$. *Is a C^1 mapping F of $\mathbb{R}^2$ globally univalent (i.e. injective) provided that the assumptions* (J) *and* (T) *are satisfied?*

Note that the Global Univalence Problem is equivalent to the Global Stability Problem (cf. [O]) and it is still not settled. To show how delicate the matter is we recall the following example presented in [P].

EXAMPLE 1.1. Define an analytic map F of $\mathbb{R}^2$ by

$$F(x,y) = (-2e^x + 3y^2 - 1, ye^x - y^3).$$

Then $\operatorname{Jac} F(x,y) = -2e^x < 0$ and $\operatorname{Tr} F'(x,y) = -e^x - 3y^2 < 0$ for every $(x,y) \in \mathbb{R}^2$, but F is not injective because $F(0,1) = (0,0) = F(0,-1)$.

The Global Stability Problem in $\mathbb{R}^2$ has a natural formulation in any dimension:

GLOBAL STABILITY PROBLEM IN $\mathbb{R}^n$. *Let $F = (F_1, \ldots, F_n)$ be a C^1 transformation of $\mathbb{R}^n$, $F(0) = 0$ and assume that for any x in $\mathbb{R}^n$ all eigenvalues of the jacobian matrix $F'(x)$ have negative real parts. Does it then follow that the solution $y(\cdot) = 0$ of the autonomous system of differential equations*

$$(*) \qquad \dot{y} = F(y)$$

is globally asymptotically stable?

The negative answer to the Global Stability Problem in $\mathbb{R}^n$ for $n \geq 4$ was given in 1988 by N. E. Barabanov (cf. [B]); the problem is still open when $n = 2, 3$.

In this paper we give an n-dimensional generalization of Meisters and Olech's just mentioned two-dimensional result (see Section 4 for the precise formulation).

2. Basic facts on stability. Let E be a subset of $\mathbb{R}^n$ and $F : E \to \mathbb{R}^n$ be a C^1 mapping. Consider a real autonomous system of differential equations

$$(*) \qquad \dot{y} = F(y)$$

whose solutions are uniquely determined by initial conditions. Let $y_0(\cdot)$ denote the solution of $(*)$ satisfying the initial condition $y_0(0) = y_0$ and defined for every $t \geq 0$. In the sequel we shall assume that $F(0) = 0$ and $y_0 = 0$, so $y_0(\cdot) = 0$.

The symbol of matrix multiplication is omitted or denoted by "$\circ$", "T" is the matrix tranposition and x is treated as one-column matrix, so x^T is a one-row matrix. The norm $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$, and I denotes the identity mapping or the identity matrix.

We start with a series of definitions.

DEFINITION 2.1. (i) We say that $y_0(\cdot)$ is *locally asymptotically stable* (for short: LAS) if for every $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that if $\|y_0 - y_1\| < \delta$ then the solution $y_1(\cdot)$ of $(*)$ with $y_1(0) = y_1$ exists for every $t \geq 0$, $\|y_0(t) - y_1(t)\| < \varepsilon$ for $t \geq 0$ and $\|y_0(t) - y_1(t)\| \to 0$ as $t \to \infty$.

(ii) The solution $y_0(\cdot)$ is *globally asymptotically stable* (for short: GAS) when it is a LAS solution and the following holds:

If $y_1(\cdot)$ is any solution of $(*)$ defined for small $t \geq 0$, then $y_1(t)$ exists for all $t \geq 0$ and $\|y_0(t) - y_1(t)\| \to 0$ as $t \to \infty$.

(iii) Let $y_0(\cdot) = 0$ be a LAS solution of $(*)$. The *domain of attraction* of $y_0(\cdot)$ (or the domain of attraction of the set $\{0\}$) is the subset A of E consisting of all $a \in E$ such that the solution $y_1(\cdot)$ of $(*)$ starting at a exists for every $t \geq 0$ and $y_1(t) \to 0$ as $t \to \infty$. (Note that if E is open and $y_0(\cdot)$ is LAS, then the domain of attraction is also open.)

(iv) Assume that E is an open set containing 0 and $F(0) = 0$. Let V be a function satisfying the following conditions:

(a) V is defined in a neighbourhood U of 0,
(b) V is of class C^1 in U,
(c) $V \geq 0$ and $V(y) > 0$ if $\|y\| > 0$,
(d) the trajectory derivative of V at a point y (i.e. $\dot{V}(y) := \frac{d}{dt}V[y(t)] = \operatorname{grad} V(y) \circ F(y))$ is negative if $\|y\| > 0$.

We call V a *Lyapunov function* of the equation $(*)$.

Note that if all eigenvalues of $F'(0)$ have negative real parts, then there exists a Lyapunov function of $(*)$.

Now we recall the following classical Lyapunov Theorem.

THEOREM 2.2. *Let E be an open set containing* 0, *let F be a C^1 map of $\mathbb{R}^n$ with $F(0) = 0$ and let V be a Lyapunov function of the equation $(*)$ defined in an open neighbourhood of* 0. *Then $y(\cdot) = 0$ is a LAS solution of $(*)$.*

Let E^* be a connected set. Let $G(y) = [g_{jk}(y) : j, k = 1, \dots, n]$ be a real, symmetric, continuous and positive definite matrix on E^*. We associate with G an element of arc length

$$ds^2 = dy^T G(y)\, dy\,,$$

i.e. if $C : y = y(t)$, $a \le t \le b$, is an arc of class $\mathcal{C}^1$ in E^*, then its length $L(C)$ is given by the formula

$$L(C) = \int_a^b [\dot{y}(t)^T G(y(t))\dot{y}(t)]^{1/2}\, dt .$$

$L(C)$ is independent of a chosen $\mathcal{C}^1$ parametrization of the arc C (cf. [H]).

Take any $y_1, y_2 \in E^*$ and define a metric associated with G by the formula

$$r(y_1, y_2) := \inf\{L(C) \mid C : [a,b] \to E^* ,\ y(a) = y_1,\ y(b) = y_2,\ C \text{ is of class } \mathcal{C}^1\} .$$

Now consider the equation $(*)$ and the "possible" Lyapunov function

$$V(y) := F(y)^T G(y) F(y), \quad y \in E^* .$$

Note that

$$\dot{V}(y) = 2F(y)^T B(y) F(y) ,$$

where

$$B(y) := G(y)F'(y) + \frac{1}{2}\sum_{j=1}^{n} F_j \frac{\partial G}{\partial y_j} .$$

We recall the following correct version of Theorem 14.2 in [H, Chap. 14] (cf. also [HO, Theorem 2.4]).

THEOREM 2.3. *Let $F = (F_1, \ldots, F_n)$ be a mapping of class $\mathcal{C}^1$ defined on an open connected subset E^* of $\mathbb{R}^n$, with $F(y) \neq 0$ for every $y \in E^*$.*

(i) *Let a symmetric matrix $G(y)$ be of class $\mathcal{C}^1$ on E^* and positive definite for $y \in E^*$, and let B and r be defined as above.*

(ii) *Assume that the following "Borg type" condition is satisfied:*

(BC) $$x^T B(y) x \le 0 \quad \textit{if} \quad F(y)^T G(y) x = 0 .$$

(iii) *Further, let $y_0(\cdot)$ be a solution of $(*)$ defined on the right maximal interval of existence $0 \le t < \omega \le \infty$ with the property that there exists $\alpha > 0$ such that*

$$r(y_0(t), \partial E^* \cup \{\infty\}) > \alpha > 0 \quad \textit{for every } t \in [0, \omega) ;$$

i.e. for any $t \in [0,\omega)$ and for any half-open $\mathcal{C}^1$ arc $C : x = \phi(t)$, $\phi(0) = y_0(t)$, $t \in [0,1)$, if $L(C) \le \alpha$, then $\phi(1) := \lim_{t\to 1} \phi(t)$ exists and $\phi(t) \in E^*$ for $t \in [0,1]$.

Then there exist positive constants δ and k such that for any solution $y(\cdot)$ of $(*)$ satisfying $r[y_0(0), y(0)] < \delta$ there exists an increasing, positive function $s(\cdot) : [0,\omega) \to \mathbb{R}$ such that $s(0) = 0$, $[0, s(\omega))$ is the right maximal *interval of existence of* $y(\cdot)$ *and*

$$r[y(s(t)), y_0(t)] \leq kr[y(0), y_0(0)] \quad \text{for } 0 \leq t < \omega\,.$$

3. Remarks on polynomial mappings. Let $F = (F_1, \ldots, F_n)$ be a polynomial map $\mathbb{R}^n \to \mathbb{R}^n$, and let $\operatorname{Jac} F$ denote the jacobian of F, i.e. $\operatorname{Jac} F(x) = \det F'(x)$ for $x \in \mathbb{R}^n$. We begin with the following lemma.

LEMMA 3.1. *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial map such that* $\det F'(y) \neq 0$ *for every* $y \in \mathbb{R}^n$. *Then for every* $b \in \mathbb{R}^n$ *the equation* $F(x) = b$ *has only isolated solutions and*

$$\#\{x \in \mathbb{R}^n : F(x) = b\} \leq \deg F_1 \cdot \ldots \cdot \deg F_n\,.$$

Moreover, $\{y \in \mathbb{R}^n : \#F^{-1}(y) = \max\{\#F^{-1}(b) : b \in \mathbb{R}^n\}\}$ *is a nonempty open subset of* $\mathbb{R}^n$.

Proof. Since $\operatorname{Jac} F \neq 0$ everywhere in $\mathbb{R}^n$ the equation $F(x) = b$ has only isolated roots in $\mathbb{R}^n$. From now on we treat F as a polynomial mapping of $\mathbb{C}^n$.

(i) If $F^{-1}(b) = \{a^1, \ldots, a^p : a^j \in \mathbb{C}^n,\ j = 1, \ldots, p\}$, then the proof is given in [Ł, Chap. 7.13].

(ii) First we recall an important theorem about polynomial mappings (cf. [Md]):

If F has a nontrivial jacobian and $d(F)=[\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)]$, i.e. if $d(F)$ is the so-called geometric (or generic) degree of F, then

$$d(F) = \max\ \{\#F^{-1}(y) : y \in \mathbb{C}^n,\ \#F^{-1}(y) < \infty\}$$

and the *exceptional set* $E := \{y \in \mathbb{C}^n : \#F^{-1}(y) \neq d(F)\}$ is an algebraic proper subset of $\mathbb{C}^n$.

Assume that $\#F^{-1}(b) = \infty$ and let $a^1, \ldots, a^p$ be the isolated points of $F^{-1}(b)$. Choose closed balls $U_j = \overline{B}(a^j, R)$ such that

$$U_j \cap F^{-1}(b) = \{a^j\}, \quad j = 1, \ldots, p\,. \tag{1}$$

Put $r := \inf\{\|F(x) - b\| \ : \ \|x - a_j\| = R,\ j = 1, \ldots, p\} > 0$ and set $B = B(b, r)$. Choose $c \in B \setminus E$, fix j and define holomorphic maps

$$G := b - c, \qquad H := F - b \tag{2}$$

in some neighbourhood of U_j. Evidently

$$\|G(x)\| < r \leq \|H(x)\| \quad \text{when} \quad \|x - a^j\| = R\,. \tag{3}$$

By (1)–(3) we can apply the Rouché Theorem (cf. [Ł, Chap. 5]) to deduce that the holomorphic map $G + H = F - c$ has at least one zero in U_j, i.e.

$$\#\{F^{-1}(c) \cap U_j\} \geq 1, \quad j = 1, \ldots, p\,. \tag{4}$$

From (4) we derive that

$$p \le \# F^{-1}(c) = d(F) \le \deg F_1 \cdot \ldots \cdot \deg F_n .$$

Since $\#\{a \in \mathbb{R}^n : F(a) = b\} \le \#\{c \in \mathbb{C}^n : F(c) = b\}$, we obtain the inequality stated in Lemma 3.1.

(iii) Define $q := \max\{\# F^{-1}(b) : b \in \mathbb{R}^n\}$. Note that if a polynomial map F of $\mathbb{R}^n$ has a nonvanishing jacobian, then by the local inverse function theorem $\{y \in \mathbb{R}^n : \# F^{-1}(y) = q\}$ is a nonempty open subset of $\mathbb{R}^n$. ∎

Remark 3.2. From the Lefschetz Principle we derive that the estimate given in Lemma 3.1 remains true for any field of characteristic 0 (cf. [E], where the existence of some estimate for any field of characteristic 0 is proved, and [BCR, Th. 11.5.2], where an estimate for the number of connected components of a real algebraic set is given).

It is worth remembering that injectivity of a polynomial map implies its bijectivity (cf. [BR, KR]).

4. Global stability of polynomial differential equations. We begin with a proposition which is a consequence of [HO, Theorems 2.2 and 2.4] or [H, Theorem 14.2 and Corollary 14.1].

PROPOSITION 4.1. *Assume that*:

1° $F = (F_1, \ldots, F_n)$ *is a* C^1 *map of* $\mathbb{R}^n$, $F(0) = 0$.

2° *If* $F(b) = 0$, *then* $y(\cdot) = b$ *is a LAS solution of the equation*

$$(*) \qquad \dot{y} = F(y) .$$

3° $\exists p \in C^1(\mathbb{R}^n \setminus F^{-1}(0), (0, \infty))$ *such that* $\dot{p}(y) + p(y) x^T \circ F'(y) \circ x \le 0$ *whenever* $x^T \circ F(y) = 0$, $\|x\| = 1$ *(here* $\dot{p}(y) := p'(y)^T \circ F(y)$*).*

4° $\exists R > 0$, $\exists d > 0$ *such that* $\|F(y)\| > d$ *if* $\|x\| > R$.

Then $y(\cdot) = 0$ *is a GAS solution of* $(*)$. *In particular*, $F^{-1}(0) = \{0\}$.

Proof. (i) Evidently $\# F^{-1}(0) < \infty$, so $F^{-1}(0) := \{b_1 = 0, b_2, \ldots, b_q\}$. By 2° the domain of attraction D_j of the solution $y(\cdot) = b_j$ is nonempty for $j = 1, \ldots, q$. Evidently $D_i \cap D_j = \emptyset$ for $i \ne j$. Put $E^* := \mathbb{R}^n \setminus F^{-1}(0)$.

(ii) Define

$$G(y) := p^2(y) I, \qquad y \in E^* .$$

Evidently $G(y)$ is a symmetric, positive definite matrix of class C^1 on E^* and, by 3°, $x^T \circ B(y) \circ x \le 0$ whenever $F(y)^T \circ x = 0$, $\|x\| = 1$. Thus assumptions (i) and (ii) of Theorem 2.3 are satisfied.

(iii) If $q > 1$, then $E^* \setminus D_1 \ne \emptyset$. This means that there exists $y_0 \in E^* \cap \partial D_1$. Let $y_0(t)$ denote the solution of $(*)$ satisfying $y_0(0) = y_0$ and defined on $[0, \omega)$. Since D_1 is a domain of attraction, therefore, by the classical theorem on continuous dependence on initial values, $y_0(t) \in \partial D_1$ for $t \in [0, \omega)$. Now 4° shows that

$$\exists \alpha > 0 \quad \text{such that} \quad r(y_0(t), \partial E^* \cup \{\infty\}) > \alpha \quad \text{for } 0 \le t < \omega\,,$$

which means that assumption (iii) of Theorem 2.3 is satisfied.

(iv) By Theorem 2.3 there exist positive numbers δ and k such that if $y(0) \in D_1$ and $r[y_0(0), y(0)] < \delta$ and $y(\cdot)$ is the solution of $(*)$ with initial value $y(0)$, then

$$r[y(s(t)), y_0(t)] \le kr[y(0), y_0(0)] \quad \text{for } 0 \le t < \omega$$

for a suitable increasing, positive function $s(\cdot) : [0, \omega) \to (0, \infty)$ such that $s(0) = 0$ and $[0, s(\omega))$ is the right maximal interval of existence of $y(\cdot)$. Since $y(t) \to 0$ as $t \to \infty$ we get a contradiction, hence $q = 1$ and $y_0(\cdot) = 0$ is a GAS solution in the whole $\mathbb{R}^n$. ∎

Note that Proposition 4.1 remains true if instead of 4° we assume that

$$\forall r > 0 \quad \int_r^\infty \varrho(s)\, ds = \infty, \qquad \text{where} \quad \varrho(s) := \min\{p(y) : \|y\| = s\}\,,$$

but the proof is a little more complicated.

A typical candidate for p is $p(y) = \|F(y)\|^{2c}$, where c is a fixed nonnegative number. Then $p(y)$ is positive of class C^1, and p satisfies 4° if

$$\begin{aligned}&\dot p(y) + p(y) x^T \circ F'(y) \circ x \\ &\quad = c\|F(y)\|^{2c-2} F(y)^T \circ F'(y) \circ F(y) + \|F(y)\|^{2c} x^T \circ F'(y) \circ x \le 0\,.\end{aligned}$$

This yields

COROLLARY 4.2. *Proposition* 4.1 *remains true if* 3° *is replaced by*

$$\text{(C)} \quad \exists c \ge 0 \quad cF(y)^T \circ F'(y) \circ F(y) + \|F(y)\|^2 x^T \circ F'(y) \circ x \le 0 \quad \textit{whenever } x^T \circ F(y) = 0,\ \|x\| = 1\,.$$

From Proposition 4.1, Theorem 2.2 and Corollary 4.2 we obtain at once the following.

Remark 4.3. Let $F = (F_1, \dots, F_n)$ be a polynomial map of $\mathbb{R}^n$ with $F(0) = 0$ such that

$$\text{(1)} \qquad \operatorname{Jac} F(y) \ne 0 \quad \text{for every } y \in \mathbb{R}^n\,,$$

$$\text{(2)} \qquad x^T \circ F'(b) \circ x < 0 \quad \text{if } \|x\| = 1,\ F(b) = 0\,,$$

$$\text{(3)} \qquad \exists c \ge 0 \quad c\,[F(y)]^T \circ F'(y) \circ F(y) + \|F(y)\|^2 x^T \circ F'(y) \circ x \le 0 \quad \text{whenever } x^T \circ F(y) = 0,\ \|x\| = 1\,.$$

Then $y(\cdot) = 0$ is a GAS solution of $(*)$. In particular, $F^{-1}(0) = \{0\}$.

At this moment we want to show a connection between Borg type condition (C) and the eigenvalues of the matrix $H(y)$ which is the symmetric part of $F'(y)$, i.e. $H(y) = \frac{1}{2}[F'(y) + F'(y)^T]$. Let $\lambda_1(y), \ldots, \lambda_n(y)$ be those eigenvalues. Define

$$\alpha(y) := \max\{(\lambda_j(y) + \lambda_k(y)) : j \neq k,\ j,k = 1,\ldots,n\}, \qquad y \in \mathbb{R}^n .$$

Remark 4.4. Inequality (2.5) in [HO] implies

$$F(y)^T \circ F'(y) \circ F(y) + \|F(y)\|^2 x^T \circ F'(y) \circ x \leq \alpha(y)\ \|F(y)\|^2$$
$$\text{whenever } x^T \circ F(y) = 0,\ \|x\| = 1 .$$

Now we formulate and prove the main theorem of the paper.

THEOREM 4.5 (Main Theorem). *Let $F = (F_1, \ldots, F_n)$ be a polynomial map of $\mathbb{R}^n$, $F(0) = 0$. Let $H(y)$, $\lambda_1(y), \ldots, \lambda_n(y)$ and $\alpha(y)$ be as defined above. Assume that*

(a) $\operatorname{Jac} F(y) \neq 0$ *for every* $y \in \mathbb{R}^n$,
(b) *all eigenvalues of* $F'(b)$ *have negative real parts if* $F(b) = 0$,
(c) $\alpha(y) \leq 0$ *for every* $y \in \mathbb{R}^n$.

Then $y(\cdot) = 0$ is a GAS solution of $(*)$, $y \in \mathbb{R}^n$, and the mapping F is bijective.

Proof. Case I. We assume additionally $\#F^{-1}(0) = \max\{\#F^{-1}(w) : w \in \mathbb{R}^n\}$. Now it is sufficient to prove that $y_0(\cdot) = 0$ is a GAS solution of $(*)$. By (b) and Theorem 2.2, $y(\cdot) = b$ is a LAS solution of $(*)$ whenever $F(b) = 0$.

We check that the function

$$p(y) := \|F(y)\|^2, \qquad y \in \mathbb{R}^n \setminus F^{-1}(0),$$

satisfies the assumptions of Corollary 4.2. By Remark 4.4,

$$\dot{p}(y) + p(y)x^T \circ F'(y) \circ x = F(y)^T \circ F'(y) \circ F(y) + \|F(y)\|^2 x^T \circ F'(y) \circ x$$
$$\leq \alpha(y)\|F(y)\|^2 \leq 0$$

whenever $x^T \circ F(y) = 0$, $\|x\| = 1$. Therefore, assumption (C) of Corollary 4.2 is satisfied.

Since $\#F^{-1}(0) = \max\{\#F^{-1}(w) : w \in \mathbb{R}^n\}$, there exists $d > 0$ such that $A := \{y \in \mathbb{R}^n : \|F(y)\| \leq d\}$ is compact. Hence,

$$\exists R > 0 \quad \text{such that} \quad \|F(y)\| > d \text{ if } \|x\| > R,$$

i.e. 4° is also satisfied. Thus, by Corollary 4.2, $y_0(\cdot) = 0$ is a GAS solution of $(*)$ and F is bijective.

Case II. If $\#F^{-1}(0) < \max\{\#F^{-1}(w) : w \in \mathbb{R}^n\}$, then, by Lemma 3.1, we can choose $b \in \mathbb{R}^n$ such that $\#F^{-1}(b) = \max\{\#F^{-1}(w) : w \in \mathbb{R}^n\}$. Put

$$G(y) := F(b + y) - F(b), \qquad y \in \mathbb{R}^n .$$

Obviously G satisfies all assumptions which F satisfied in Case I. Hence G is bijective, so is F, and F has to satisfy the assumptions of Case I. ∎

Now we prove that Theorem 4.5 generalizes Meisters and Olech's result (cf. [MO]).

THEOREM 4.6. *Let $F = (F_1, F_2)$ be a polynomial mapping of $\mathbb{R}^2$. If $F(0) = 0$, $\operatorname{Jac} F(y) > 0$ and*

$$\operatorname{Tr} F'(y) := \frac{\partial F_1}{\partial y_1}(y) + \frac{\partial F_2}{\partial y_2}(y) < 0 \quad \textit{for every } y \in \mathbb{R}^2 ,$$

then $y(\cdot) = 0$ is a GAS solution of the autonomous system

$$(*) \qquad \dot{y}_1 = F_1(y_1, y_2), \quad \dot{y}_2 = F_2(y_1, y_2), \quad (y_1, y_2) \in \mathbb{R}^2 .$$

In particular, $F^{-1}(0) = \{0\}$.

Proof. We show that the assumptions of Theorem 4.6 imply those of Theorem 4.5 for $n = 2$.

Since $\operatorname{Jac} F(y) > 0$ and $\operatorname{Tr} F'(y) < 0$, all eigenvalues of $F'(y)$ have negative real parts. If $H(y) := \frac{1}{2}[F'(y) + F'(y)^T]$, $y \in \mathbb{R}^n$, and $\lambda_1(y), \lambda_2(y)$ are the eigenvalues of $H(y)$, then

$$\alpha(y) = \lambda_1(y) + \lambda_2(y) = \operatorname{Tr} H'(y) = \operatorname{Tr} F'(y) < 0 .$$

Thus all assumptions of Theorem 4.5 are satisfied. ∎

It is known that if a complex polynomial map of $\mathbb{C}^2$ has nontrivial constant jacobian and symmetric jacobian matrix, then the map is bijective (cf. [D]). For the sake of comparison we present a real counterpart of this fact.

Remark 4.7. If F is a polynomial mapping of $\mathbb{R}^2$, $\operatorname{Jac} F(y) > 0$ and $F'(y)$ is symmetric for any $y \in \mathbb{R}^2$, then F is bijective.

One can check this by applying Theorem 4.6. The result is also true for any C^1 map and it is a consequence of Corollary 1 in [MO1].

Note that assumption (C) of Corollary 4.2 in fact concerns the symmetric part $H(y)$ of $F'(y)$ because

$$\begin{aligned} B(x,y) &:= c\, F(y)^T \circ F'(y) \circ F(y) + \|F(y)\|^2 x^T \circ F'(y) \circ x \\ &= c\, F(y)^T \circ H'(y) \circ F(y) + \|F(y)\|^2 x^T \circ H'(y) \circ x ; \end{aligned}$$

thus, it can be very restrictive even in the case of a linear system of differential equations. The following simple example, presented to us by A. van den Essen, shows this fact.

EXAMPLE 4.8. Define the linear mapping F of $\mathbb{R}^3$ by

$$F(y_1, y_2, y_3) = (-y_1 + 21y_3, -y_2 + 12y_3, -y_3).$$

Then the zero solution of the equation $\dot{y} = F(y)$ is globally asymptotically stable, but assumption (C) of Corollary 4.2 is not satisfied because if

$$x = \frac{1}{\sqrt{2}}(1,1,0) \quad \text{and} \quad y = \frac{-1}{\sqrt{3}}(22,11,1),$$

then

$$F(y) = \frac{-1}{\sqrt{3}}(-1,1,-1), \quad x^T \circ F(y) = 0, \quad B(x,y) = 2c-1,$$

while if

$$x = \frac{1}{\sqrt{3}}(1,-1,1) \quad \text{and} \quad y = \frac{-1}{\sqrt{2}}(1,1,0),$$

then

$$F(y) = \frac{-1}{\sqrt{2}}(-1,-1,0), \quad x^T \circ F(y) = 0, \quad B(x,y) = 2-c.$$

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