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# A theorem of the Hahn–Banach type and its applications

by Zbigniew Gajda (Katowice), Andrzej Smajdor (Kraków) and Wilhelmina Smajdor (Katowice)

**Abstract.** Let $Y$ be a subgroup of an abelian group $X$ and let $\mathcal{F}$ be a given collection of subsets of a linear space $E$ over the rationals. Moreover, suppose that $F$ is a subadditive set-valued function defined on $X$ with values in $\mathcal{F}$. We establish some conditions under which every additive selection of the restriction of $F$ to $Y$ can be extended to an additive selection of $F$. We also present some applications of results of this type to the stability of functional equations.

**1. Introduction.** Throughout this paper, $\mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}$ stand for the sets of all reals, rationals and integers, respectively. Our main goal is to give a generalization of the following well-known Hahn–Banach theorem:

THEOREM A. *Let $Y$ be a linear subspace of a real linear space $X$. Assume that $p : X \to \mathbb{R}$ is a functional such that*

(i) *$p(x+y) \le p(x) + p(y)$ for all $x, y \in X$;*
(ii) *$p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$ and $x \in X$.*

*If $f : Y \to \mathbb{R}$ is a linear functional satisfying*

$$f(x) \le p(x) \quad \text{for } x \in Y, \tag{1}$$

*then $f$ can be extended to a linear functional $g : X \to \mathbb{R}$ with*

$$g(x) \le p(x) \quad \text{for } x \in X. \tag{2}$$

First we are going to rephrase this theorem in terms of a set-valued function (abbreviated to "s.v. function" in the sequel) with values in the family $\mathrm{cc}(\mathbb{R})$ of all non-empty, compact, convex subsets of $\mathbb{R}$. Clearly, the elements of $\mathrm{cc}(\mathbb{R})$ are just the non-empty compact intervals in $\mathbb{R}$.

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An s.v. function $F$ mapping $X$ into cc($\mathbb{R}$) is said to be *subadditive* iff

$$F(x+y) \subset F(x) + F(y) \quad \text{for all } x, y \in X, \tag{3}$$

and it is called *positively homogeneous* iff

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \geq 0 \text{ and } x \in X. \tag{4}$$

Addition of sets and multiplication of sets by scalars are here understood in the Minkowski sense, i.e.

$$A + B := \{a + b : a \in A, b \in B\}, \qquad \alpha A := \{\alpha a : a \in A\}$$

for any $A, B \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$.

One can easily check that inequalities (1) and (2) of Theorem A are equivalent to

$$-p(-x) \leq f(x) \leq p(x) \quad \text{for } x \in Y, \tag{1'}$$

$$-p(-x) \leq g(x) \leq p(x) \quad \text{for } x \in X, \tag{2'}$$

respectively. Moreover, by (i) and (ii) we have $p(0) = 0$ and

$$-p(-x) \leq p(x) \quad \text{for } x \in X.$$

Therefore, we may correctly define an s.v. function $F : X \to$ cc($\mathbb{R}$) by

$$F(x) := [-p(-x), p(x)] \quad \text{for } x \in X. \tag{5}$$

It is evident that $F$ is subadditive, positively homogeneous and odd, i.e. $F(-x) = -F(x)$ for $x \in X$. Conversely, each s.v. function $F : X \to$ cc($\mathbb{R}$) which is subadditive, positively homogeneous and odd must be of the form (5) with a functional $p : X \to \mathbb{R}$ satisfying (i) and (ii).

Now Theorem A may be interpreted as a result on extending partial additive selections of an s.v. function $F : X \to$ cc($\mathbb{R}$), as follows:

THEOREM B. *Let $Y$ be a linear subspace of a real linear space $X$. Assume that $F : X \to$ cc($\mathbb{R}$) is a subadditive, positively homogeneous and odd s.v. function. If $f : Y \to \mathbb{R}$ is a linear functional such that*

$$f(x) \in F(x) \quad \textit{for all } x \in Y,$$

*then $f$ extends to a linear functional $g$ defined on the whole of $X$ and such that*

$$g(x) \in F(x) \quad \textit{for all } x \in X.$$

In the next section we generalize Theorem B to the following abstract setting. Instead of the linear space $X$ we consider an arbitrary abelian group $(X, +)$, and the family cc($\mathbb{R}$) is replaced by an axiomatically given collection $\mathcal{F}$ of subsets of a linear space $E$ over $\mathbb{Q}$. Among the assumptions imposed on $\mathcal{F}$ the crucial role is played by the so-called *binary intersection property*. It means that every subfamily of $\mathcal{F}$, any two members of which intersect, has a non-empty intersection. This property was first introduced and studied by L. Nachbin in [5]. It is well known that the collection of all non-empty compact intervals in $\mathbb{R}$ has the binary intersection property (see [5]).

**2. Generalizations of the Hahn–Banach theorem.** Since we now assume that $X$ is a group (not a linear space), it is natural to discuss additive (instead of linear) selections of an s.v. function $F : X \to \mathcal{F}$, i.e. functions $g : X \to E$ such that

$$g(x+y) = g(x) + g(y) \quad \text{for } x, y \in X\,,$$
$$g(x) \in F(x) \quad \text{for all } x \in X\,.$$

THEOREM 1. *Let $Y$ be a subgroup of an abelian group $(X,+)$ and let $E$ be a linear space over $\mathbb{Q}$. Furthermore, let $\mathcal{F}$ be a family of non-empty subsets of $E$ having the binary intersection property and satisfying the following conditions*:

$$A \in \mathcal{F},\ v \in E \Rightarrow A + v \in \mathcal{F}; \tag{6}$$

$$A \in \mathcal{F},\ n \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \Rightarrow \frac{1}{n}A \in \mathcal{F}\,. \tag{7}$$

*Assume that $F : X \to \mathcal{F}$ is a subadditive s.v. function such that*

$$F(nx) \subset nF(x) \quad \textit{for all } x \in X \textit{ and } n \in \mathbb{Z}^*\,. \tag{8}$$

*If $f : Y \to E$ is an additive selection of the restriction of $F$ to $Y$ (denoted by $F|_Y$), then $f$ can be extended to an additive selection of $F$.*

Proof. Denote by $\Omega$ the family of all additive maps $\phi : \operatorname{dom}\phi \to E$ such that $Y \subset \operatorname{dom}\phi \subset X$, $\operatorname{dom}\phi$ is a subgroup of $X$, $\phi(x) \in F(x)$ for $x \in \operatorname{dom}\phi$ and $\phi(x) = f(x)$ for $x \in Y$. The family $\Omega$ is partially ordered by the relation $\prec$ defined by

$$\phi \prec \psi \quad \text{iff} \quad \operatorname{dom}\phi \subset \operatorname{dom}\psi \text{ and } \phi = \psi|_{\operatorname{dom}\phi}\,.$$

It is easy to see that every chain $\mathcal{C} \subset \Omega$ has an upper bound in $\Omega$: it is the map $\phi_{\mathcal{C}}$ such that $\operatorname{dom}\phi_{\mathcal{C}} := \bigcup\{\operatorname{dom}\phi : \phi \in \mathcal{C}\}$ and $\phi_{\mathcal{C}}|_{\operatorname{dom}\phi} = \phi$ for each $\phi \in \mathcal{C}$. By the Kuratowski–Zorn lemma, $\Omega$ contains at least one maximal element $g$. To complete the proof it is enough to show that $\operatorname{dom} g = X$.

Suppose that there exists a $z_0 \in X \setminus \operatorname{dom} g$ and put

$$W := \{x + nz_0 : x \in \operatorname{dom} g, n \in \mathbb{Z}\}\,.$$

Obviously $W$ is a subgroup of $X$ properly containing $\operatorname{dom} g$. We distinguish two cases depending on whether the set

$$A := \{k \in \mathbb{Z}^* : kz_0 \in \operatorname{dom} g\}$$

is empty or not.

Case 1: $A \neq \emptyset$. If $k, l \in A$, then $k \cdot l \in A$ and $lg(kz_0) = g(lkz_0) = kg(lz_0)$, whence

$$g(kz_0)/k = g(lz_0)/l\,.$$

Putting

$$u_0 := g(kz_0)/k \qquad \text{for some } k \in A\,,$$

we define an element $u_0 \in E$ which does not depend on the choice of $k \in A$. Next we define $\widetilde{g} : W \to E$ by

$$\widetilde{g}(x + nz_0) := g(x) + nu_0 \qquad \text{for } x \in \operatorname{dom} g \text{ and } n \in \mathbb{Z}\,. \tag{9}$$

If an element of $W$ admits two representations: $x + nz_0 = y + mz_0$ with some $x, y \in \operatorname{dom} g$ and $n, m \in \mathbb{Z}$, then $(m - n)z_0 = x - y \in \operatorname{dom} g$. There are two possibilities: either $m = n$ or $m - n \in A$. In the first case we have $x = y$ and $g(x) + nu_0 = g(y) + mu_0$. If the second possibility holds, then

$$u_0 = \frac{g((m-n)z_0)}{m-n}\,,$$

which implies that

$$g(x) - g(y) = g(x - y) = g((m-n)z_0) = (m-n)u_0$$

and consequently, $g(x) + nu_0 = g(y) + mu_0$. Thus the definition of $\widetilde{g}$ is correct. It is also clear that $\widetilde{g}$ is additive and $\widetilde{g}|_{\operatorname{dom} g} = g$.

Now let $x \in \operatorname{dom} g$, $n \in \mathbb{Z}$ and $k \in A$. Then

$$\begin{aligned}\widetilde{g}(x + nz_0) = g(x) + nu_0 = g(x) + \frac{ng(kz_0)}{k} &= \frac{g(k(x + nz_0))}{k} \\ &\in \frac{1}{k}F(k(x + nz_0)) \subset F(x + nz_0)\,.\end{aligned}$$

Thus $\widetilde{g}$ is an additive selection of $F|_W$, contrary to the maximality of $g$ in $\Omega$.

Case 2: $A = \emptyset$. Then $kz_0 \in X \setminus \operatorname{dom} g$ for every $k \in \mathbb{Z}^*$. Fix $x, y \in \operatorname{dom} g$ and $n, m \in \mathbb{Z}^*$. By the subadditivity of $F$ and by (8) we have

$$\begin{aligned}mg(x) - ng(y) = g(mx - ny) &\in F(mx - ny) \\ &= F(mx + nmz_0 - nmz_0 - ny) \\ &\subset F(m(x + nz_0)) + F(-n(y + mz_0)) \\ &\subset mF(x + nz_0) - nF(y + mz_0)\,.\end{aligned}$$

Consequently,

$$0 \in m[F(x + nz_0) - g(x)] - n[F(y + mz_0) - g(y)]\,,$$

which means that

$$0 \in \frac{1}{n}[F(x + nz_0) - g(x)] - \frac{1}{m}[F(y + mz_0) - g(y)]\,.$$

We conclude that for any $x, y \in \operatorname{dom} g$ and $n, m \in \mathbb{Z}^*$ the intersection

$$\frac{1}{n}[F(x+nz_0) - g(x)] \cap \frac{1}{m}[F(y+mz_0) - g(y)]$$

is non-void. From the hypotheses it now follows that

$$\bigcap \left\{ \frac{1}{n}[F(x+nz_0) - g(x)] : x \in \operatorname{dom} g, n \in \mathbb{Z}^* \right\} \neq \emptyset .$$

Let $u_0$ be in this intersection; then $g(x) + nu_0 \in F(x+nz_0)$ for all $x \in \operatorname{dom} g$ and $n \in \mathbb{Z}$.

Similarly to Case 1 we define $\widetilde{g} : W \to E$ by (9). This definition is unambiguous, since now $x + nz_0 = y + mz_0$ (with $x, y \in \operatorname{dom} g$ and $n, m \in \mathbb{Z}$) only holds if $x = y$ and $n = m$. Moreover, $\widetilde{g}$ is an additive selection of $F|_W$, which again contradicts the maximality of $g$ in $\Omega$. The proof is finished.

If members of $\mathcal{F}$ are $\mathbb{Q}$-convex, i.e. $\alpha A + (1-\alpha)A \subset A$ for all $\alpha \in \mathbb{Q} \cap [0,1]$ and $A \in \mathcal{F}$, then assumption (8) on the s.v. function $F : X \to \mathcal{F}$ can be weakened:

THEOREM 2. *Let $Y$ be a subgroup of an abelian group $(X, +)$ and let $E$ be a linear space over $\mathbb{Q}$. Moreover, let $\mathcal{F}$ be a family of non-empty $\mathbb{Q}$-convex subsets of $E$ having the binary intersection property and satisfying conditions* (6) *and* (7) *of Theorem* 1. *If $F : X \to \mathcal{F}$ is a subadditive s.v. function such that*

$$(8') \qquad F(-x) \subset -F(x) \quad \text{for all } x \in X ,$$

*then every additive selection of $F|_Y$ has an extension to an additive selection of $F$.*

Proof. It is sufficient to observe that in fact $F$ satisfies (8). Indeed, if $x \in X$, $n \in \mathbb{Z}$ and $n > 0$, then by the subadditivity of $F$ and by the $\mathbb{Q}$-convexity of $F(x)$, we derive

$$F(nx) \subset \underbrace{F(x) + \ldots + F(x)}_{n \text{ terms}} \subset nF(x) .$$

If $n \in \mathbb{Z}$ and $n < 0$, then on account of (8′) we have

$$F(nx) \subset -F(-nx) \subset -(-n)F(x) = nF(x) ,$$

which completes the proof.

The next result is an immediate consequence of Theorems 1 and 2 with $Y := \{0\}$ and $f(0) := 0$.

COROLLARY 1. *Let $(X, +)$ be an abelian group and let $E$ be a linear space over $\mathbb{Q}$. Under the hypotheses of either Theorem* 1 *or Theorem* 2 *concerning the family $\mathcal{F}$ and the s.v. function $F : X \to \mathcal{F}$, if $0 \in F(0)$, then $F$ has an additive selection.*

The collection cc($\mathbb{R}$) is a simple example of a family $\mathcal{F}$ satisfying all the conditions in both Theorems 1 and 2. If $X, Y$ and $F : X \to \mathrm{cc}(\mathbb{R})$ satisfy all the assumptions of Theorem B, then by virtue of either Theorem 1 or 2 a given linear selection $f : Y \to \mathbb{R}$ of $F|_Y$ can be extended to an additive (a priori not necessarily linear) selection $g : X \to \mathbb{R}$ of $F$. With each $x \in X$ we associate a function $g_x : \mathbb{R} \to \mathbb{R}$ defined by

$$g_x(\alpha) := g(\alpha x) \qquad \text{for } \alpha \in \mathbb{R},$$

which is additive and

$$g_x(\alpha) \le \sup F(\alpha x) = \alpha \sup F(x) \qquad \text{for } \alpha \ge 0.$$

In particular, $g_x$ is upper bounded on a non-empty open interval and by a classical result (see e.g. [1], Sect. 2.1.1, Theorem 1 and the subsequent remarks) it has the form

$$g_x(\alpha) = \alpha g_x(1) \qquad \text{for all } \alpha \in \mathbb{R}.$$

This assures that $g$ is homogeneous and shows that Theorem B may be easily deduced from both Theorems 1 and 2.

**3. Applications.** Using Theorem 2 we may prove the following result on extending additive maps which approximate a function with Cauchy differences in a given linear space $K$ over $\mathbb{Q}$.

THEOREM 3. *Let $Y$ be a subgroup of an abelian group $(X, +)$ and let $K$ be a linear subspace of a linear space $E$ over $\mathbb{Q}$. If $\phi : X \to E$ is such that*

$$\phi(x+y) - \phi(x) - \phi(y) \in K \qquad \text{for all } x, y \in X \tag{10}$$

*and $f : Y \to E$ is an additive map satisfying*

$$f(x) - \phi(x) \in K \qquad \text{for } x \in Y,$$

*then $f$ can be extended to an additive function $g : X \to E$ such that*

$$g(x) - \phi(x) \in K \qquad \text{for } x \in X.$$

Proof. First we observe that the family

$$\mathcal{F} := \{w + K : w \in E\}$$

has the binary intersection property (in fact, every subfamily of $\mathcal{F}$ any two of whose members intersect consists of a single set). Clearly, all elements of $\mathcal{F}$ are $\mathbb{Q}$-convex and $\mathcal{F}$ satisfies (6) and (7). We consider an s.v. function $F : X \to \mathcal{F}$ given by

$$F(x) := \phi(x) + K \qquad \text{for } x \in X.$$

It is evidently subadditive and $f : Y \to E$ is an additive selection of $F|_Y$. To check that $F$ satisfies (8′) set $x = y = 0$ in (10), whence $\phi(0) \in K$.

Moreover, setting $y = -x$ in (10) we get $\phi(0) - \phi(x) - \phi(-x) \in K$, which combined with the preceding relation implies that

$$F(-x) = \phi(-x) + K \subset -\phi(x) + K = -F(x) \quad \text{for } x \in X .$$

Now the conclusion follows directly from Theorem 2.

The subsequent corollary was first established in a different way by K. Baron (cf. [2]). It results from our Theorem 3 upon setting $Y := \{0\}$.

COROLLARY 2. *Let $(X, +)$ be an abelian group and let $K$ be a linear subspace of a linear space $E$ over $\mathbb{Q}$. If $\phi : X \to E$ satisfies* (10), *then there exists an additive function $g : X \to E$ such that*

$$g(x) - \phi(x) \in K \quad \textit{for all } x \in X .$$

In the sequel we shall say that a normed space $(E, \| \ \|)$ has the binary intersection property iff the collection of all closed balls in $E$ has the binary intersection property in the sense introduced before. We will be concerned with the following inequality:

$$\|\phi(x+y) - \phi(x) - \phi(y)\| \le r(x) + r(y) - r(x+y) \tag{11}$$

for $x, y \in X$, where $(X, +)$ is an abelian group, $\phi$ maps $X$ into $E$ and $r$ is a real-valued subadditive function on $X$. A study of this inequality with the "control function" $r := \| \ \|$ was first proposed by D. Yost (cf. [6] and [7]) and then it was undertaken by R. Ger in connection with some stability questions for functional equations (see [3] and [4]). Here, we prove the following extension theorem:

THEOREM 4. *Let $Y$ be a subgroup of an abelian group $(X, +)$ and let $(E, \| \ \|)$ be a normed space having the binary intersection property. Moreover, suppose that $r : X \to [0, \infty)$ is an even, subadditive function and $\phi : X \to E$ is an odd map satisfying* (11). *If $f : Y \to E$ is an additive function such that*

$$\|f(x) - \phi(x)\| \le r(x) \quad \textit{for } x \in Y , \tag{12}$$

*then $f$ has an extension to an additive function $g : X \to E$ such that*

$$\|g(x) - \phi(x)\| \le r(x) \quad \textit{for } x \in X . \tag{13}$$

Proof. For $v \in E$ and $\varrho \in [0, \infty)$ let $K(v, \varrho)$ denote the closed ball in $E$ with centre $v$ and radius $\varrho$. Then $\mathcal{F} := \{K(v, \varrho) : v \in E, \varrho \in [0, \infty)\}$ is a family of convex sets which, by hypothesis, has the binary intersection property and, evidently, satisfies (6) and (7).

Notice that for any $\varrho_1, \varrho_2 \in [0, \infty)$ we have

$$K(0, \varrho_1) + K(0, \varrho_2) = K(0, \varrho_1 + \varrho_2) .$$

Indeed, the inclusion $\subset$ is clear. Conversely, if $w \in K(0, \varrho_1 + \varrho_2)$, then $w = u + v$, where

$$u := \frac{\varrho_1}{\varrho_1 + \varrho_2} w, \qquad v := \frac{\varrho_2}{\varrho_1 + \varrho_2} w$$

(without loss of generality one may assume that $\varrho_1 > 0$ and $\varrho_2 > 0$).

Inequality (11) may be written as

$$\phi(x+y) - \phi(x) - \phi(y) \in K(0, r(x) + r(y) - r(x+y)),$$

which yields

$$\begin{aligned}\phi(x+y) - \phi(x) - \phi(y) + K(0, r(x+y)) \\ &\subset K(0, r(x) + r(y) - r(x+y)) + K(0, r(x+y)) \\ &= K(0, r(x) + r(y)) = K(0, r(x)) + K(0, r(y)).\end{aligned}$$

Hence

$$\phi(x+y) + K(0, r(x+y)) \subset \phi(x) + K(0, r(x)) + \phi(y) + K(0, r(y))$$

or equivalently,

$$K(\phi(x+y), r(x+y)) \subset K(\phi(x), r(x)) + K(\phi(y), r(y)).$$

Now we define an s.v. function $F : X \to \mathcal{F}$ by

$$F(x) := K(\phi(x), r(x)) \qquad \text{for } x \in X.$$

We have just shown that $F$ is subadditive. It is also odd, because

$$\begin{aligned}F(-x) &= K(\phi(-x), r(-x)) = K(-\phi(x), r(x)) \\ &= -K(\phi(x), r(x)) = -F(x) \qquad \text{for } x \in X.\end{aligned}$$

Moreover, on account of (12), $f$ is a selection of $F|_Y$. Applying Theorem 2 we can extend $f$ to an additive selection $g$ of $F$. In particular, $g$ satisfies (13) and the proof is finished.

COROLLARY 3. *Under the assumptions of Theorem* 4 *on* $X$, $E$, $r$ *and* $\phi$, *there exists an additive function* $g : X \to E$ *such that condition* (13) *holds true.*

Proof. We can use Theorem 4 with $Y := \{0\}$ and $f(0) := 0$, since (11) guarantees that

$$\|f(0) - \phi(0)\| = \|\phi(0)\| \le r(0).$$

COROLLARY 4. *Suppose that the hypotheses of Theorem* 4 *on* $X$, $E$, $r$ *and* $\phi$ *are satisfied except that* $\phi$ *does not have to be odd. Then there exists an additive function* $g : X \to E$ *such that*

$$\|g(x) - \phi(x)\| \le 2r(x) \qquad \textit{for } x \in X.$$

Proof. Let $\phi_e$ and $\phi_o$ stand for the even and odd part of $\phi$, respectively, i.e.

$$\phi_e(x) = \tfrac{1}{2}(\phi(x) + \phi(-x)), \qquad \phi_o(x) = \tfrac{1}{2}(\phi(x) - \phi(-x))$$

for $x \in X$. From (11) we infer that $\|\phi(0)\| \le r(0)$ and

$$\begin{aligned}\|\phi_e(x)\| - \tfrac{1}{2}\|\phi(0)\| &\le \|\phi_e(x) - \tfrac{1}{2}\phi(0)\| \\ &= \tfrac{1}{2}\|\phi(x) + \phi(-x) - \phi(0)\| \\ &\le \tfrac{1}{2}(r(x) + r(-x) - r(0)) \\ &= r(x) - \tfrac{1}{2}r(0) \qquad \text{for } x \in X .\end{aligned}$$

Hence

$$\|\phi_e(x)\| \le r(x) + \tfrac{1}{2}(\|\phi(0)\| - r(0)) \le r(x) \qquad \text{for } x \in X .$$

Moreover, the odd part of $\phi$ also satisfies (11):

$$\begin{aligned}&\|\phi_o(x+y) - \phi_o(x) - \phi_o(y)\| \\ &\quad = \tfrac{1}{2}\|\phi(x+y) - \phi(-x-y) - \phi(x) + \phi(-x) - \phi(y) + \phi(-y)\| \\ &\quad \le \tfrac{1}{2}(\|\phi(x+y) - \phi(x) - \phi(y)\| + \|\phi(-x) + \phi(-y) - \phi(-x-y)\|) \\ &\quad \le \tfrac{1}{2}(r(x) + r(y) - r(x+y) + r(-x) + r(-y) - r(-x-y)) \\ &\quad = r(x) + r(y) - r(x+y) \qquad \text{for } x, y \in X .\end{aligned}$$

Therefore, by Corollary 3, one can find an additive function $g : X \to E$ such that

$$\|g(x) - \phi_o(x)\| \le r(x) \qquad \text{for } x \in X .$$

Finally, we have

$$\begin{aligned}\|g(x) - \phi(x)\| &= \|g(x) - \phi_o(x) - \phi_e(x)\| \\ &\le \|g(x) - \phi_o(x)\| + \|\phi_e(x)\| \le 2r(x) \qquad \text{for } x \in X ,\end{aligned}$$

which was to be shown.

A result similar to our Corollary 4 was proved by R. Ger in [4], where $X$ was assumed to be an amenable (not necessarily abelian) group and the technique of invariant means was used.

We close the paper with the following

COROLLARY 5. *Let $(E, \|\ \|)$ be a normed space which may be equipped with a new norm $\|\ \|_0$ equivalent to $\|\ \|$ and such that $(E, \|\ \|_0)$ has the binary intersection property. Suppose that $X$, $r$ and $\phi$ satisfy the same assumptions as in Corollary 4. Then there exists an additive function $g : X \to E$ such that*

$$\|g(x) - \phi(x)\| \le 2\alpha\beta r(x) \qquad \textit{for } x \in X ,$$

*where $\alpha$ and $\beta$ are positive constants with $\|u\|_0 \le \alpha\|u\|$ and $\|u\| \le \beta\|u\|_0$ for all $u \in E$.*

Proof. If we put $r_0(x) := \alpha r(x)$, then

$$\|\phi(x+y) - \phi(x) - \phi(y)\|_0 \le \alpha\|\phi(x+y) - \phi(x) - \phi(y)\| \\ \le \alpha(r(x) + r(y) - r(x+y)) = r_0(x) + r_0(y) - r_0(x+y)$$

for $x, y \in X$. By virtue of Corollary 4 there exists an additive function $g : X \to E$ such that

$$\|g(x) - \phi(x)\|_0 \le 2r_0(x) = 2\alpha r(x) \qquad \text{for } x \in X .$$

Hence

$$\|g(x) - \phi(x)\| \le \beta\|g(x) - \phi(x)\|_0 \le 2\alpha\beta r(x) \qquad \text{for } x \in X ,$$

which completes the proof.

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Zbigniew Gajda, Wilhelmina Smajdor
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
40-007 KATOWICE, POLAND

Andrzej Smajdor
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
30-084 KRAKÓW, POLAND

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