ANNALES
POLONICI MATHEMATICI
LVII.3 (1992)

Oscillation criteria for a class of nonlinear differential equations of third order

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Abstract. Oscillation criteria are obtained for nonlinear homogeneous third order differential equations of the form

$$y''' + q(t)y' + p(t)y^{\alpha} = 0$$

and

$$y''' + q(t)y' + p(t)f(y) = 0\,,$$

where p and q are real-valued continuous functions on $[a,\infty)$, f is a real-valued continuous function on $(-\infty,\infty)$ and $\alpha > 0$ is a quotient of odd integers. Sign restrictions are imposed on $p(t)$ and $q(t)$. These results generalize some of the results obtained earlier in this direction.

1. Oscillation, nonoscillation and asymptotic behaviour of solutions of nonlinear third order ordinary differential equations have been the subject of intensive investigation in recent years (see, in particular, Erbe [1], Erbe and Rao [2], Heidel [3], Kura [4], Nelson [6] and Waltman [13] for nonlinear homogeneous equations, and Parhi [7, 8] and Parhi and Parhi [10, 11] for nonlinear nonhomogeneous equations).

In this paper we obtain oscillation criteria for nonlinear homogeneous third order differential equations of the form

$$(\mathrm{E}_1)\qquad y''' + q(t)y' + p(t)y^{\alpha} = 0\,,$$

where $p, q \in C([a,\infty),\mathbb{R})$, $a \in \mathbb{R}$ and $\alpha > 0$ is a quotient of odd integers, and

$$(\mathrm{E}_2)\qquad y''' + q(t)y' + p(t)f(y) = 0\,,$$

where p and q are as in (E_1) and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Our results on (E_1) and (E_2) with $p(t) \ge 0$ and $q(t) \ge 0$ generalize and supplement

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1991 *Mathematics Subject Classification*: Primary 34C10, 34C11.

Key words and phrases: third order equations, oscillation, nonoscillation, asymptotic behaviour.

This work was done under a scheme supported by the University Grants Commission, New Delhi under Grant No. F.8-9/87 (SR-III).

the results of Waltman [13]. In [1], Erbe has obtained oscillation criteria for more general equations

$$y''' + r(t)y'' + q(t)y' + p(t)y^{\alpha} = 0\,,$$

where p, q and α are as in (E_1) and $r \in C([a,\infty),\mathbb{R})$. However, for $r(t) = 0$, $p(t) \geq 0$ and $q(t) \geq 0$, our conditions are simpler and results are more general than Erbe's. It seems that oscillation criteria are not known for (E_1) with $p(t) \leq 0$ and $q(t) \leq 0$ or $p(t) \geq 0$ and $q(t) \leq 0$. We have obtained some results in this direction.

By a *proper solution* of (E_1) or (E_2) we mean a solution $y(t)$ which exists on some half-line $[T_y,\infty) \subseteq [a,\infty)$, where T_y depends on y, and is nontrivial in any neighbourhood of infinity. Here we restrict our attention to real-valued proper solutions of (E_1) or (E_2). A proper solution is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

In Section 2, we consider (E_1) and (E_2) with $p(t) \geq 0$ and $q(t) \geq 0$. Section 3 is concerned with oscillation criteria for (E_1) with $p(t) \geq 0$ and $q(t) \leq 0$ and with $p(t) \leq 0$ and $q(t) \leq 0$.

2. This section deals with oscillation criteria for (E_1) and (E_2) with $p(t) \geq 0$ and $q(t) \geq 0$. Our results generalize and supplement the following theorems of Waltman [13]:

THEOREM 1. *Suppose that $p(t) \geq 0$, $q(t) \geq 0$ and q is continuously differentiable such that $q'(t) < 0$. If*

$$A + Bt - \int_a^t \Big(\int_a^s p(u)\,du \Big)\,ds < 0 \tag{1}$$

for large t and for any real numbers A and B, then a proper solution of (E_1) which has a zero is oscillatory.

THEOREM 2. *Suppose that $p(t) \geq 0$, $q(t) \geq 0$ and q is continuously differentiable. Let $f(y)/y \geq \beta > 0$ for $y \neq 0$ and some β. If $\beta p(t) - q'(t) > 0$ and*

$$\int_a^\infty t(\beta p(t) - q'(t))\,dt = \infty\,,$$

then a proper solution of (E_2) which has a zero is oscillatory.

The following result due to Lazer [5] is required in the sequel.

LEMMA 1. *If $y \in C^3([a,\infty),\mathbb{R})$ such that $y(t) > 0$, $y'(t) > 0$ and $y'''(t) \leq 0$ for $t \geq a$, then*

$$\liminf_{t\to\infty} \frac{y(t)}{ty'(t)} \geq \frac{1}{2}\,.$$

Remark 1. For $f(y) = y^\alpha$, where $\alpha > 0$ ($\neq 1$) is a quotient of odd integers, the condition $f(y)/y \geq \beta \geq 0$ for $y \neq 0$ and some β takes the form $f(y)/y > \beta = 0$, and for $\alpha = 1$, $\beta = 1$.

In view of the above remark, the following theorem holds for both (E_1) and (E_2).

THEOREM 3. *Suppose that $p(t) \geq 0$, $q(t) \geq 0$, q is continuously differentiable, $f(y)/y \geq \beta \geq 0$ for $y \neq 0$ and some β and $2\beta p(t) - q'(t) \geq 0$ but $\not\equiv 0$ on any subinterval of $[0, \infty)$. Let $F(y)$ be given by*

$$F(y) = (y')^2 - 2yy'' - q(t)y^2 .$$

If the second order differential equation

$$z'' + (q(t) + m\beta t p(t))z = 0 , \tag{2}$$

$0 < m < 1/2$, is oscillatory (i.e. all its proper solutions are oscillatory), then a proper solution $y(t)$ of (E_2) for which $F(y(t_0)) \geq 0$ for some $t_0 \geq T_y$, is oscillatory. In particular, any proper solution of (E_2) which has a zero is oscillatory.

Proof. Suppose that $y(t)$ is nonoscillatory. So there exists a $c \geq t_0$ such that $F(y(c)) \geq 0$ and $y(t) \neq 0$ for $t > c$. Let $y(t) > 0$ for $t > c$; the other case is similar.

By (E_2) we have

$$\frac{d}{dt}[F(y(t))] = y^2(t)\left(2p(t)\frac{f(y(t))}{y(t)} - q'(t)\right) . \tag{3}$$

Then the assumption $2\beta p(t) - q'(t) \geq 0$ implies that $F(y(t))$ is increasing and hence $F(y(t)) > F(y(c)) \geq 0$ for $t > c$. If possible, let t_1 and t_2 ($c < t_1 < t_2$) be consecutive zeros of $y'(t)$. So $F(y(t_1)) > 0$ and $F(y(t_2)) > 0$, that is,

$$-2y(t_1)y''(t_1) > q(t_1)y^2(t_1) \geq 0 ,$$
$$-2y(t_2)y''(t_2) > q(t_2)y^2(t_2) \geq 0 .$$

Thus $y''(t_1) < 0$ and $y''(t_2) < 0$. This is impossible because t_1 and t_2 are consecutive zeros of $y'(t)$. So $y'(t)$ has at most one zero in (c, ∞), and hence there exists a $b > c$ such that $y'(t) > 0$ or < 0 for $t \geq b$.

If possible, let $y'(t) > 0$ for $t \geq b$. From (E_2) it is clear that $y'''(t) \leq 0$ for $t \geq b$. Let $d > b$. From Lemma 1 it follows that

$$\frac{y(t)}{y'(t)} > \frac{1}{2}(t - d)$$

for $t > d$. Thus, for $t > d$,

$$\frac{f(y(t))}{y'(t)} = \frac{f(y(t))}{y(t)}\,\frac{y(t)}{y'(t)} > \frac{\beta}{2}(t - d) .$$

This in turn implies that

$$\frac{f(y(t))}{ty'(t)} > \frac{\beta}{2} - \frac{\beta d}{2t}.$$

Let $0 < m < 1/2$. Choose T so large that $\beta/2 - \beta d/(2t) > m\beta$ for $t > T$. Then

$$\frac{f(y(t))}{ty'(t)} > m\beta$$

for $t > T$. Clearly, $y'(t)$ is a nonoscillatory solution of the second order equation

$$z'' + \left[q(t) + p(t)\frac{f(y(t))}{y'(t)}\right]z = 0$$

for $t > T$. From Sturm's comparison theorem it follows that the equation (2) is nonoscillatory, a contradiction.

Hence $y'(t) < 0$ for $t \geq b$. We consider three cases, viz. (i) $y''(t) \leq 0$, (ii) $y''(t) \geq 0$ and (iii) $y''(t)$ changes sign for large t, and derive a contradiction in each case. Clearly, $y'(t) < 0$ and $y''(t) \leq 0$ for large t imply that $y(t) < 0$ eventually, a contradiction. If $y''(t) \geq 0$ for large t, then $\lim_{t\to\infty} y'(t)$ exists and is nonpositive. From (3) we obtain

$$F(y(t)) = F(y(c)) + \int_c^t [2p(s)f(y(s))y(s) - q'(s)y^2(s)]\,ds,$$

that is,

$$(y'(t))^2 \geq \int_c^t [2p(s)f(y(s))y(s) - q'(s)y^2(s)]\,ds. \tag{4}$$

Thus, from (4), $\lim_{t\to\infty}(y'(t))^2$ is positive, that is, $\lim_{t\to\infty} y'(t) = \mu < 0$, and hence $y(t) < 0$ eventually, a contradiction. Suppose that $y''(t)$ changes sign for large t. So $y'(t)$ has maxima for large t. We claim that $\limsup_{t\to\infty} y'(t) = 0$. If not, $\limsup_{t\to\infty} y'(t) = \lambda < 0$. Thus, for $0 < \varepsilon < -\lambda$, there exists a $T > b$ such that $y'(t) < \lambda + \varepsilon$ for $t > T$. Consequently, $y(t) < 0$ for large t, a contradiction. Hence our claim holds. Let $\langle t_n \rangle$ be the sequence of maxima of $y'(t)$. So $\limsup_{n\to\infty} y'(t_n) = 0$. Clearly, $\langle t_n \rangle$ contains a subsequence $\langle s_n \rangle$ such that $\lim_{n\to\infty} y'(s_n) = 0$. However, since $y''(s_n) = 0$, from (3) it follows that

$$\begin{aligned}(y'(s_n))^2 &\geq (y'(s_n))^2 - 2y(s_n)y''(s_n) - q(s_n)y^2(s_n)\\ &= F(y(c)) + \int_c^{s_n} [2p(s)f(y(s))y(s) - q'(s)y^2(s)]\,ds.\end{aligned}$$

Hence $\lim_{n\to\infty}(y'(s_n))^2 > 0$, a contradiction.

Thus the theorem is proved.

Remark 2. Theorem 3 is more general than Theorem 2 due to Waltman. In the following we show that the conditions in Theorem 2 imply those of Theorem 3. Clearly, $\beta p(t)-q'(t)>0$ implies that $2\beta p(t)-q'(t)>0$. Further, if

$$\int_a^\infty t(\beta p(t)-q'(t))\,dt=\infty,$$

then either

$$\int_a^\infty tp(t)\,dt=\infty \quad \text{or} \quad -\int_a^\infty tq'(t)\,dt=\infty.$$

But

$$-\int_a^t sq'(s)\,ds\le aq(a)+\int_a^t q(s)\,ds$$

implies that $\int_a^\infty q(t)\,dt=\infty$ when $-\int_a^\infty tq'(t)\,dt=\infty$. Hence

$$\int_a^\infty t(\beta p(t)-q'(t))\,dt=\infty$$

implies that

$$\int_a^\infty (q(t)+m\beta tp(t))\,dt=\infty.$$

Consequently, equation (2) is oscillatory (see Swanson [12]).

Now we give an example to which Theorem 2 cannot be applied but our Theorem 3 applies.

EXAMPLE 1. Consider

$$y'''+\left(1+\frac{1}{t^2}\right)y'+\frac{1}{t^3}(\beta+e^y)y=0,\qquad t\ge 1,$$

where $\beta>0$. Clearly, $\beta p(t)-q'(t)>0$. But

$$\int_1^\infty t(\beta p(t)-q'(t))\,dt=(\beta+2)\int_1^\infty\frac{dt}{t^2}<\infty,$$

so Waltman's theorem cannot be applied. Now, for $0<m<1/2$,

$$\int_1^\infty (q(t)+m\beta tp(t))\,dt=\int_1^\infty\left(1+\frac{1}{t^2}\right)dt+m\beta\int_1^\infty\frac{dt}{t^2}=\infty.$$

Thus the equation (2) is oscillatory, and Theorem 3 applies.

Remark 3. We show that (1) is equivalent to $\int_a^\infty p(t)\,dt=\infty$. Set

$$G(t)=\int_a^t\Big(\int_a^s p(u)\,du\Big)\,ds.$$

So (1) holds for large t and arbitrary A and B if and only if

$$\lim_{t\to\infty} G(t)/t = \infty. \tag{5}$$

Further, (5) holds if and only if $\lim_{t\to\infty} G'(t) = \infty$, that is, $\int_a^\infty p(t)\,dt = \infty$, which proves our claim.

Remark 4. We may note that, for $\beta \geq 0$,

$$\int_a^\infty q(t)\,dt = \infty \quad \text{implies that} \quad \int_a^\infty (q(t) + m\beta t p(t))\,dt = \infty$$

and hence (2) is oscillatory. If $\beta > 0$, then

$$\int_a^\infty p(t)\,dt = \infty \quad \text{implies that} \quad \int_a^\infty (q(t) + m\beta t p(t))\,dt = \infty$$

and thus (2) is oscillatory. Therefore, Theorem 3 provides alternative conditions, when compared to Theorem 1 of Waltman, under which every proper solution of (E_1) which has a zero is oscillatory. The following examples strengthen our remark.

EXAMPLE 2. Consider

$$y''' + \left(1 + \frac{1}{t^2}\right)y' + \frac{1}{t^3}y^\alpha = 0, \qquad t \geq 1,$$

where $\alpha > 0$ is a quotient of odd integers. Waltman's result cannot be applied because $\int_1^\infty p(t)\,dt < \infty$. However, $\int_1^\infty q(t)\,dt = \infty$, and Theorem 3 applies.

EXAMPLE 3. Consider

$$y''' + \frac{1}{t^2}y' + \frac{1}{t^2}y^\alpha = 0, \qquad t \geq 1,$$

where $\alpha > 0$ is a quotient of odd integers. Clearly, $\int_1^\infty p(t)\,dt < \infty$ and $\int_1^\infty q(t)\,dt < \infty$. But Theorem 3 can be applied because $\lim_{t\to\infty} t^2 q(t) = 1 > 1/4$ and hence the equation $z'' + (1/t^2)z = 0$ is oscillatory (see [12], p. 45).

Remark 5. It is clear that Theorem 3 generalizes the following theorem due to Lazer [5].

THEOREM 4. *If $p(t) \geq 0$, $q(t) \geq 0$, $2p(t) - q'(t) \geq 0$ and not identically zero in any interval and there exists a number $m < 1/2$ such that the second order differential equation*

$$z'' + [q(t) + mtp(t)]z = 0$$

is oscillatory, then the third order differential equation

$$y''' + q(t)y' + p(t)y = 0$$

admits oscillatory solutions.

3. In this section we consider oscillation criteria for (E_1) with $p(t) \geq 0$ and $q(t) \leq 0$ and with $p(t) \leq 0$ and $q(t) \leq 0$.

The following lemma due to Parhi and Nayak [9] is used in the sequel.

LEMMA 2. *If $r, p, f \in C([a,\infty),\mathbb{R})$ such that $r(t) > 0$, $p(t) \geq 0$ and $f(t) \geq 0$, then all solutions of*

$$(r(t)y')' - p(t)y = f(t)$$

are nonoscillatory.

THEOREM 5. *Suppose that $p(t) \geq 0$, $q(t) \leq 0$, $q'(t) \leq 0$ and $q(t)$ is bounded. Let $\alpha \geq 1$. If*

$$\int_a^\infty \left[p(t) - \frac{2}{3\sqrt{3}}(-q(t))^{3/2}\right] dt = \infty\,, \tag{6}$$

then a proper solution of (E_1) which has a zero is oscillatory.

Proof. Let $|q(t)| \leq M$, where M is a positive real number. Let $y(t)$ be a proper solution of (E_1) such that $y(t_0) = 0$ for some $t_0 \geq T_y$. We claim that $y(t)$ is oscillatory. If not, there exists a $c \geq t_0$ such that $y(c) = 0$ and $y(t) \neq 0$ for $t > c$. Without any loss of generality, we may assume that $y(t) > 0$ for $t > c$. Clearly, $-y'(t)$ is a solution of the second order differential equation

$$z'' + q(t)z = p(t)y^\alpha(t)\,.$$

From Lemma 2, it follows that the above equation is nonoscillatory. So there exists a $b > c$ such that $y'(t) \neq 0$ for $t > b$.

Suppose that $y'(t) < 0$ for $t > b$. Since $y(c) = 0$ and $y(t) > 0$ for $t > c$, there exists a t_1, $c < t_1 \leq b$, such that $y'(t_1) = 0$ and $y'(t) < 0$ for $t > t_1$. Now multiplying (E_1) through by $y'(t)$ and integrating the resulting identity from t_1 to t $(t_1 < t)$, we obtain

$$y'(t)y''(t) = \int_{t_1}^{t} [(y''(s))^2 - q(s)(y'(s))^2 - p(s)y^\alpha(s)y'(s)]\, ds > 0\,.$$

Thus $y''(t) < 0$ for $t > t_1$. This in turn implies that $y(t) < 0$ for large t, a contradiction.

So $y'(t) > 0$ for $t > b$. Suppose that $y(t)$ is bounded. Integrating (E_1) from b to t, we obtain

$$y''(t) \leq y''(b) - q(t)y(t) - y^\alpha(b) \int_b^t p(s)\, ds\,.$$

Clearly, (6) implies that $\int_a^\infty p(t)\, dt = \infty$. Hence $\lim_{t\to\infty} y''(t) = -\infty$. This in turn implies that $y'(t) < 0$ for large t, a contradiction. Next, let $y(t)$ be unbounded. So there exists a $t_2 > b$ such that $y(t) > 1$ for $t \geq t_2$. Then

$z(t) = y'(t)/y(t)$, for $t \geq t_2$, satisfies

$$z''(t) + 3z(t)z'(t) \leq -[z^3(t) + q(t)z(t) + p(t)]\,. \tag{7}$$

It is easy to see that the minimum of $z^3(t) + q(t)z(t) + p(t)$ over all positive $z(t)$ is $p(t) - \frac{2}{3\sqrt{3}}(-q(t))^{3/2}$. Thus

$$z''(t) + 3z(t)z'(t) \leq -\left[p(t) - \frac{2}{3\sqrt{3}}(-q(t))^{3/2}\right] \tag{8}$$

for $t \geq t_2$. Integrating (8) from t_2 to t and making use of (6), we obtain $z'(t) \to -\infty$ as $t \to \infty$, which implies that $z(t) < 0$ for large t, a contradiction.

This completes the proof of the theorem.

Remark 6. The above theorem partially generalizes the following theorem due to Lazer [5].

THEOREM 6. *If* $p(t) \geq 0$, $q(t) < 0$ *and*

$$\int_a^\infty \left[p(t) - \frac{2}{3\sqrt{3}}(-q(t))^{3/2}\right] dt = \infty\,,$$

then (E_1) *with* $\alpha = 1$ *admits oscillatory solutions.*

The following example illustrates Theorem 5.

EXAMPLE 4. Consider

$$y''' - \left(2 - \frac{1}{t^2}\right)y' + e^t y^\alpha = 0\,, \qquad t \geq 1\,,$$

where $\alpha \geq 1$ is a quotient of odd integers. Clearly, all the conditions of Theorem 5 are satisfied.

It is interesting to note that Theorem 5 may be put in the following form which may be viewed as a stability theorem for nonoscillatory solutions of (E_1).

THEOREM 7. *Suppose that the conditions of Theorem 5 hold. For any proper nonoscillatory solution* $y(t)$ *of* (E_1), *the following properties hold for large* t:

(i) $\operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t) = \operatorname{sgn} y'''(t)$,
(ii) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0$.

Proof. From Theorem 5, it follows that $y(t) \neq 0$ for $t \geq a$. Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq a$. Proceeding as in Theorem 5 we may show that $y'(t) < 0$ for large t. Consequently, from (E_1) we obtain $y'''(t) < 0$ for large t, and hence $y''(t) > 0$ or < 0 for large t. But $y''(t) < 0$ for large t gives $y(t) < 0$ for large t, a contradiction. Thus

$y''(t) > 0$ eventually. Hence (i) holds. Clearly, (ii) follows from (i) and the observation that (6) implies $\int_a^\infty p(t)\,dt = \infty$.

Thus the theorem is proved.

In the sequel we improve the following result due to Nelson [6]:

THEOREM 8. *Let $q'(t)$ and $p(t)$ be continuous and $p(t) \geq 0$, $q(t) < 0$ with $q'(t) \geq 0$. For any A and B suppose that*

$$A + Bt - \int_a^t \Big(\int_a^s p(u)\,du \Big)\,ds < 0$$

for large t. Then any nonoscillatory solution $y(t)$ of the equation

$$y''' + q(t)y' + p(t)y^{2n+1} = 0\,, \qquad n = 1, 2, 3\ldots,$$

has the following properties:

$$\operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t)\,,$$
$$\lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'(t) = 0\,, \qquad \lim_{t\to\infty} |y(t)| = L \geq 0\,.$$

Further, if

$$p(t) > \varepsilon > 0\,, \tag{9}$$

then $\lim_{t\to\infty} y(t) = 0$.

THEOREM 9. *Suppose that $p(t) \geq 0$, $q(t) \leq 0$ and $q'(t) \geq 0$. Let $\alpha > 1$. If $\int_a^\infty p(t)\,dt = \infty$, then for any proper nonoscillatory solution $y(t)$ of* (E_1), *the following properties hold for large t:*

$$\operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t) = \operatorname{sgn} y'''(t)\,,$$
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0\,.$$

In view of Remark 3, the proof of Theorem 9 is the same as that of Theorem 8 of Nelson, except that we have to show that $\lim_{t\to\infty} y(t) = 0$ without the extra condition $p(t) > \varepsilon > 0$.

Let $y(t) > 0$ for large t. So $y'(t) < 0$ and $y''(t) > 0$ for large t. If possible, let $\lim_{t\to\infty} y(t) = \lambda > 0$. So $y(t) > \lambda$ for large t. Now integrating (E_1) from t_1 to t, t_1 sufficiently large, we get

$$y''(t) \leq y''(t_1) - \lambda^\alpha \int_{t_1}^t p(s)\,ds\,.$$

Hence $y''(t) < 0$ for large t, a contradiction. Thus $\lim_{t\to\infty} y(t) = 0$.

Theorem 9 may be viewed as an oscillation criterion as follows:

THEOREM 10. *Suppose that the conditions of Theorem 9 hold. Then a proper solution of* (E_1) *which has a zero is oscillatory.*

Proof. If possible, suppose that $y(t)$ is a nonoscillatory solution of (E_1) with $y(t_0) = 0$ for some $t_0 \geq a$ and $y(t) \neq 0$ for $t > t_0$. Without any loss of generality, we may assume that $y(t) > 0$ for $t > t_0$. Consequently, from Theorem 9 it follows that $y'(t) < 0$ for $t \geq t_1 > t_0$. Hence there exists a $t_2 \in (t_0, t_1)$ such that $y'(t_2) = 0$ and $y'(t) < 0$ for $t > t_2$. Now multiplying (E_1) through by $y'(t)$ and integrating the resulting identity from t_2 to t $(t_2 < t)$, we get

$$y'(t)y''(t) = \int_{t_2}^{t} [(y''(s))^2 - q(s)(y'(s))^2 - p(s)y^\alpha(s)y'(s)]\, ds > 0\,.$$

Thus $y''(t) < 0$ for $t > t_2$, a contradiction, which completes the proof of the theorem.

THEOREM 11. *Suppose that* $p(t) \leq 0$, $q(t) \leq 0$ *and* $q'(t) \geq 0$. *If* $\int_a^\infty p(t)\, dt = -\infty$, *then every bounded proper solution of* (E_1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded proper solution of (E_1). Suppose that $y(t)$ is nonoscillatory. Without any loss of generality, assume that $y(t) > 0$ for $t \geq t_0 \geq T_y$. We have to show that $\lim_{t\to\infty} y(t) = 0$.

Clearly, $y'(t)$ is a solution of the second order nonhomogeneous equation

$$z'' + q(t)z = -p(t)y^\alpha(t)\,.$$

By Lemma 2, $y'(t)$ is nonoscillatory. So there exists a $t_1 > t_0$ such that $y'(t) > 0$ or < 0 for $t \geq t_1$. Suppose that $y'(t) > 0$ for $t \geq t_1$. Then $y'''(t) \geq 0$ for $t \geq t_1$ from (E_1). Thus $y''(t)$ is nonoscillatory (note that $y''(t) \equiv 0$ implies that $y(t)$ is unbounded). If $y''(t) > 0$ for large t, then $y(t)$ is unbounded. So $y''(t) < 0$ for $t \geq t_2 \geq t_1$. Now multiplying (E_1) through by $y(t)$ and integrating the resulting identity from t_2 to t, we obtain

$$\begin{aligned}(y'(t))^2 &\leq (y'(t))^2 - 2y(t)y''(t) - q(t)y^2(t)\\ &\leq (y'(t_2))^2 - q(t_2)y^2(t_2) - 2y(t_2)y''(t_2) + 2y^{\alpha+1}(t_2) \int_{t_2}^{t} p(s)\, ds\,.\end{aligned}$$

Thus $(y'(t))^2 < 0$ for large t, a contradiction.

Hence $y'(t) < 0$ for $t \geq t_1$. Consequently, $\lim_{t\to\infty} y(t)$ exists. If possible, let $\lim_{t\to\infty} y(t) = \lambda > 0$. Now integrating (E_1) from t_1 to t, we get

$$y''(t) \geq y''(t_1) + q(t_1)y(t_1) - y^\alpha(t) \int_{t_1}^{t} p(s)\, ds\,.$$

Hence $\lim_{t\to\infty} y''(t) = \infty$, and so $y'(t) > 0$ for large t, a contradiction. Thus $\lim_{t\to\infty} y(t) = 0$.

This completes the proof of the theorem.

Acknowledgement. The authors are thankful to the referee for his valuable remarks which helped to improve the paper.

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Reçu par la Rédaction le 1.9.1990
Révisé le 29.7.1991 et 15.5.1992