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On the Łojasiewicz exponent at infinity for polynomial mappings of $\mathbb{C}^2$ into $\mathbb{C}^2$ and components of polynomial automorphisms of $\mathbb{C}^2$

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Abstract. A complete characterization of the Łojasiewicz exponent at infinity for polynomial mappings of $\mathbb{C}^2$ into $\mathbb{C}^2$ is given. Moreover, a characterization of a component of a polynomial automorphism of $\mathbb{C}^2$ (in terms of the Łojasiewicz exponent at infinity) is given.

1. Introduction. Let $H = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping and $N(H) = \{\nu \in \mathbb{R} : \exists A > 0,\ \exists B > 0,\ \forall |z| > B,\ A|z|^\nu \le |H(z)|\}$. By the *Łojasiewicz exponent at infinity* of H we shall mean $\sup N(H)$ when $N(H) \neq \emptyset$, and $-\infty$ when $N(H) = \emptyset$. We shall denote it by $\mathcal{L}_\infty(H)$. In [CK] the exponent $\mathcal{L}_\infty(H)$, called there the exponent of growth of H, was defined only for $N(H) \neq \emptyset$.

In the case $\mathcal{L}_\infty(H) > 0$, an exact formula for $\mathcal{L}_\infty(H)$ in the n-dimensional case was given by Płoski [P_2]. In the case $\mathcal{L}_\infty(H) < 0$ where H is the gradient of a polynomial function $h : \mathbb{C}^2 \to \mathbb{C}$, an exact formula for $\mathcal{L}_\infty(H)$ was given by Ha [H].

The main results of our paper are: a characterization of $\mathcal{L}_\infty(H)$ in the general case (Theorems 3.1–3.3) and a characterization of a component of a polynomial automorphism of $\mathbb{C}^2$ (Theorem 3.4), which we obtain as a corollary from the first result.

Moreover, some properties of $\mathcal{L}_\infty(H)$ in the case $H = (h'_x, h'_y)$ where $h : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial function (Sec. 9) and other characterizations of a component of a polynomial automorphism of $\mathbb{C}^2$ (Theorems 10.1, 10.2) are given.

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In the remarks at the end of the paper we indicate some possible weakenings of the assumptions in the main results and show all possible values of $\mathcal{L}_\infty(H)$ for polynomial mappings and the gradients of polynomial functions.

2. Notations and definitions. We shall use notations and definitions as in [CK], except the ones mentioned in the introduction (concerning the name and the notation of $\mathcal{L}_\infty(H)$).

3. The main results. Let $\mathbb{C}^2 \ni z = (x, y) \mapsto H(z) = (f(z), g(z)) \in \mathbb{C}^2$ be a polynomial mapping. In the sequel, we shall assume that H satisfies the condition

$$(*) \qquad 0 < \deg f = \deg_y f, \quad 0 < \deg g = \deg_y g\,.$$

The above assumptions do not restrict our considerations. This follows, on the one hand, from the fact that, for $f = $ const. or $g =$ const., we evaluate $\mathcal{L}_\infty(H)$ directly and, on the other hand, that $\mathcal{L}_\infty(H)$ is invariant with respect to linear automorphisms of the domain of H.

Let $w = (u, v) \in \mathbb{C}^2$ be arbitrary and let $Q(w, x) = \mathrm{Res}_y(f - u, g - v)$ be the resultant of $f - u$ and $g - v$ with respect to y. From the properties of the resultant it follows that Q does not vanish identically. Put

$$(1) \qquad Q(w, x) = Q_0(w)x^N + \ldots + Q_N(w), \quad Q_0 \neq 0.$$

3.1. THEOREM. *If a polynomial mapping* $H : \mathbb{C}^2 \to \mathbb{C}^2$ *satisfies* $(*)$, *then*

(i) $Q_0 =$ const. *if and only if* $\mathcal{L}_\infty(H) > 0$,

(ii) $Q_0 \neq$ const. *and* $Q_0(0) \neq 0$ *if and only if* $\mathcal{L}_\infty(H) = 0$,

(iii) *there exists* r *such that* $Q_0(0) = \ldots = Q_r(0) = 0$ *and* $Q_{r+1}(0) \neq 0$ *if and only if* $-\infty < \mathcal{L}_\infty(H) < 0$,

(iv) $Q_0(0) = \ldots = Q_N(0) = 0$ *if and only if* $\mathcal{L}_\infty(H) = -\infty$.

The above theorem gives an effective formula for $\mathcal{L}_\infty(H)$ only in cases (ii), (iv). In the theorems below we shall also give effective formulae for $\mathcal{L}_\infty(H)$ in the remaining cases.

3.2. THEOREM. *For* $\mathcal{L}_\infty(H) > 0$, *we have*

$$\mathcal{L}_\infty(H) = \left[\max_{1 \le i \le N} \frac{\deg Q_i}{i}\right]^{-1}$$

3.3. THEOREM. *For* $-\infty < \mathcal{L}_\infty(H) < 0$ *when* $Q_0(0) = \ldots = Q_r(0) = 0$ *and* $Q_{r+1}(0) \neq 0$, *we have*

$$\mathcal{L}_\infty(H) = \left[-\min_{0 \le i \le r} \frac{\mathrm{ord}_0 Q_i}{r + 1 - i}\right]^{-1}$$

Now, let $h : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function satisfying the condition

$$(**) \qquad 0 < \deg h - 1 = \deg_y h'_x = \deg_y h'_y .$$

Using a linear automorphism of the domain of h, we easily note that $(**)$ does not restrict the considerations.

Put $Q(w, x) = \mathrm{Res}_y(h'_x - u,\, h'_y - v)$. Let Q have the form (1), as above.

3.4. THEOREM. *A necessary and sufficient condition for a polynomial function h to be a component of a polynomial automorphism of $\mathbb{C}^2$ is that* $\mathrm{ord}_0 Q_N = 0$ *and, provided $N > 0$, that* $\mathrm{ord}_0 Q_i > N - i$ *for each* $i \in \{0, \ldots, N-1\}$.

4. Properties of the resultant Q. In this section we use the same notations and assumptions as in Section 3.

First, we give a proposition (without proof) following from the elementary properties of the resultant.

4.1. PROPOSITION. *Let $w_0 = (u_0, v_0) \in \mathbb{C}^2$. The polynomials $f - u_0$, $g - v_0$ have a common divisor in $\mathbb{C}[x, y]$ of positive degree if and only if* $Q_0(w_0) = \ldots = Q_N(w_0) = 0$.

We now prove a simple criterion for H to be proper.

4.2. PROPOSITION. *The mapping H is proper if and only if* $Q_0 = \mathrm{const.}$

Proof. $\Rightarrow$ Assume to the contrary that there exists w_0 such that $Q_0(w_0) = 0$. Then either $Q_0(w_0) = \ldots = Q_N(w_0) = 0$ or there exists r such that $Q_{r+1}(w_0) \neq 0$. In the first case, by Proposition 4.1, the fibre $H^{-1}(w_0)$ is not compact, which contradicts H being proper. In the second case, from the properties of the resultant it follows that there exists a sequence $\{z_n\}$ such that $|z_n| \to \infty$ and $H(z_n) \to w_0$, again contrary to H being proper.

$\Leftarrow$ If $Q_0 = \mathrm{const.}$ and $K \subset \mathbb{C}^2$ is bounded, then so is $\{x \in \mathbb{C} : Q(w, x) = 0,\ w \in K\}$. Hence $\{z \in \mathbb{C}^2 : H(z) = w,\ w \in K\}$ is also bounded, which easily implies that H is proper.

5. Proof of Theorem 3.1. Before giving the proof we quote an easy corollary from Main Theorem of [CK] (taking into account that, for $\mathcal{L}_\infty(H) \neq -\infty$, the fibre $H^{-1}(0)$ is finite).

5.1. PROPOSITION. *If a polynomial mapping $H = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies the conditions* $\deg f > 0$, $\deg g > 0$ *and* $\mathcal{L}_\infty(H) \neq -\infty$, *then*

(a) *there exist positive constants A, B such that*

$$A|z|^{\mathcal{L}_\infty(H)} \le |H(z)| \quad \text{for } |z| > B\,,$$

(b) *there exists a branch* Γ *of the curve* $f = 0$ *or* $g = 0$ *in a neighbourhood of infinity such that*

$$|x| \sim |z|, \quad |z|^{\mathcal{L}_\infty(H)} \sim |H(z)| \quad as\ |z| \to \infty,\ z \in \Gamma\,.$$

We now pass to the proof of Theorem 3.1.

(i) $\Leftrightarrow$ By Proposition 4.2, the condition $Q_0 =$ const. is equivalent to H being proper. On the other hand, by Corollary 3.3 of [CK], the latter is equivalent to the condition $\mathcal{L}_\infty(H) > 0$.

(iv) $\Rightarrow$ By Proposition 4.1, f and g have a common factor of positive degree. Hence there exists a sequence $\{z_n\}$ such that $|z_n| \to \infty$ and $H(z_n) = 0$. Then $N(H) = \emptyset$, which gives $\mathcal{L}_\infty(H) = -\infty$.

$\Leftarrow$ From $\mathcal{L}_\infty(H) = -\infty$ it follows that $N(H) = \emptyset$. Then, by Main Theorem (ii) of [CK], the fibre $H^{-1}(0)$ is infinite. This easily gives that $Q(0, x) = 0$ for an infinite number of x.

(iii) $\Rightarrow$ Analogously to the proof of Proposition 4.2, there exists a sequence $\{z_n\}$ such that $|z_n| \to \infty$ and $H(z_n) \to 0$. Hence, from (iv) and Proposition 5.1(a) we get $-\infty < \mathcal{L}_\infty(H) < 0$.

$\Leftarrow$ By Proposition 5.1(b), there exists Γ such that

$$(2) \qquad |x| \to \infty \quad \text{and} \quad |H(z)| \to 0 \quad \text{as } |z| \to \infty,\ z \in \Gamma\,.$$

On the other hand, from an elementary property of the resultant we have, for $x \neq 0$,

$$Q_0(H(z)) + Q_1(H(z))\, x^{-1} + \ldots + Q_N(H(z))\, x^{-N} = 0\,.$$

Hence and from (2) we get $Q_0(0) = 0$. The existence of r follows from (iv).

(ii) $\Leftrightarrow$ It is a direct consequence of (i), (iii), (iv). This ends the proof.

6. The exponent $\mathcal{L}_\infty(H, x)$. As above, we assume that $H = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies condition $(*)$. Let us introduce one more notion. Let $N(H, x) = \{\nu \in \mathbb{R} : \exists A > 0,\ \exists B > 0,\ \forall |x| > B,\ A|x|^\nu \le |H(z)|\}$. Put $\mathcal{L}_\infty(H, x) = \sup N(H, x)$ when $N(H, x) \neq \emptyset$, and $\mathcal{L}_\infty(H, x) = -\infty$ when $N(H, x) = \emptyset$.

6.1. PROPOSITION. *If a polynomial mapping* $H = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ *satisfies* $(*)$, *then*

$$(3) \qquad \mathcal{L}_\infty(H, x) = \mathcal{L}_\infty(H)\,.$$

Proof. First, we show that $\mathcal{L}_\infty(H) \le \mathcal{L}_\infty(H, x)$ for $\mathcal{L}_\infty(H) \ge 0$. In fact, from the inequality $|x| \le |z|$ we then have

$$(4) \qquad |x|^{\mathcal{L}_\infty(H)} \le |z|^{\mathcal{L}_\infty(H)}\,.$$

Hence and from Proposition 5.1(a) we get $A|x|^{\mathcal{L}_\infty(H)} \le |H(z)|$ for $|x| > B$. Then $\mathcal{L}_\infty(H) \in N(H, x)$ and, in consequence, $\mathcal{L}_\infty(H) \le \mathcal{L}_\infty(H, x)$ in this case.

We now show that $\mathcal{L}_\infty(H) \le \mathcal{L}_\infty(H,x)$ for $-\infty < \mathcal{L}_\infty(H) < 0$. Let

$$f(x,y) = a_0(x)\,y^m + a_1(x)\,y^{m-1} + \ldots + a_m(x), \qquad a_0 \neq 0.$$

From $(*)$ it follows that $\deg a_i \le i$, $i = 0,1,\ldots,m$. Hence, for each $i \in \{1,\ldots,m\}$, there exists a constant c_i such that, for any k,x,y with $k > 1$, $|x| \ge 1$, $|y| \ge k|x|$, we have $|a_i(x)/y^i| \le c_i/k^i$. Fix a sufficiently large k such that, for any x,y with $|x| \ge 1$ and $|y| \ge k|x|$, the inequality

$$|a_0| - |(a_1(x)/y) + \ldots + (a_m(x)/y^m)| \ge A_1 > 0$$

holds. In consequence, for the above k,x,y, we have

$$|H(z)| \ge |f(z)| \ge A_1|y|^m \ge A_1\,k^m|x|^m\,. \tag{5}$$

Since $\mathcal{L}_\infty(H) < m$, from (5) we get

$$|H(z)| \ge A_1\,k^{\mathcal{L}_\infty(H)}|x|^{\mathcal{L}_\infty(H)} \quad \text{for } |x| \ge 1,\ |y| \ge k|x|\,. \tag{6}$$

On the other hand, for the above k and $|y| \le k|x|$, we have $|z| \le k|x|$. Then, for $-\infty < \mathcal{L}_\infty(H) < 0$, we get

$$(k|x|)^{\mathcal{L}_\infty(H)} \le |z|^{\mathcal{L}_\infty(H)}\,. \tag{7}$$

From (7) and Proposition 5.1(a), there exist $A, B > 0$ such that for $|x| \ge B$, $|y| \le k|x|$, we get

$$A\,k^{\mathcal{L}_\infty(H)}|x|^{\mathcal{L}_\infty(H)} \le |H(z)|\,.$$

Hence and from (6), for $A_2 = k^{\mathcal{L}_\infty(H)}\min(A,\ A_1)$ and $|x| \ge \max(1,\ B)$, we have

$$A_2|x|^{\mathcal{L}_\infty(H)} \le |H(z)|\,.$$

Then $\mathcal{L}_\infty(H) \in N(H,x)$ and, in consequence, $\mathcal{L}_\infty(H) \le \mathcal{L}_\infty(H,x)$.

We now show that $\mathcal{L}_\infty(H,x) \le \mathcal{L}_\infty(H)$ for $\mathcal{L}_\infty(H) \neq -\infty$. It suffices to prove this for $\mathcal{L}_\infty(H,x) \neq -\infty$. Take $\nu \in N(H,x)$. Then there exist positive numbers A_3, B_3 such that

$$A_3|x|^\nu \le |H(z)| \quad \text{for } |x| > B_3\,. \tag{8}$$

Considering H on the branch Γ from Proposition 5.1(b), we easily conclude, by (8), that $\nu \le \mathcal{L}_\infty(H)$. Since ν was arbitrary, we get $\mathcal{L}_\infty(H,x) \le \mathcal{L}_\infty(H)$.

From this and the above we deduce that (3) holds for $\mathcal{L}_\infty(H) \neq -\infty$.

From Theorem 3.1(iv) it follows that, for $\mathcal{L}_\infty(H) = -\infty$, there exists a sequence $\{z_n\}$, $z_n = (x_n, y_n)$, such that $|x_n| \to \infty$ and $H(z_n) = 0$. Hence $N(H,x) = \emptyset$, which gives $\mathcal{L}_\infty(H,x) = -\infty$. This ends the proof.

7. Proof of Theorem 3.2. This proof is taken, to a considerable extent, from [P_2] by A. Płoski.

First note that $N > 0$. Indeed, this follows from the fact that $Q_0 = \text{const.}$ and $Q_N \neq \text{const.}$ Put $\Delta(Q) = [\max_{1\le i\le N}(\deg Q_i)/i]^{-1}$. Since $Q_N \neq \text{const.}$, therefore, $\Delta(Q) > 0$.

We first show that $\Delta(Q) \le \mathcal{L}_\infty(H)$. From Lemma 2.1 of [P$_2$] it follows that there exist $A, B > 0$ such that

$$\{(w,x) : |w| > B,\ Q(w,x) = 0\} \subset \{(w,x) : |w| > B,\ A|x|^{\Delta(Q)} \le |w|\}.$$

From the properties of the resultant we have $Q(H(z), x) \equiv 0$. Then, by the above, $A|x|^{\Delta(Q)} \le |H(z)|$ for $|H(z)| > B$. Since H is proper (see Prop. 4.2), there exists a constant $B_1 > 0$ such that $|H(z)| > B$ for $|x| > B_1$. Then $A|x|^{\Delta(Q)} \le |H(z)|$ for $|x| > B_1$. This means that $\Delta(Q) \in N(H, x)$. In consequence, $\Delta(Q) \le \mathcal{L}_\infty(H, x)$. Hence and from Proposition 6.1 we get $\Delta(Q) \le \mathcal{L}_\infty(H)$.

We now show that $\mathcal{L}_\infty(H) \le \Delta(Q)$. Take an arbitrary $\nu \in N(H)$. Then there exist positive C, D_1 such that $C|z|^\nu \le |H(z)|$ for $|z| > D_1$. We may assume that $D_1 \ge 1$. Let $E \ge 1$ be a constant such that $|H(z)| \le E|z|^{\deg H}$ for $|z| \ge 1$. Put $D = ED_1^{\deg H}(1+\max_{|z|\le 1}|H(z)|)$. Then, obviously, $|H(z)| > D$ implies $|z| > D_1$. Take now w, x such that $|w| > D$ and $Q(w,x) = 0$. By the properties of the resultant there exists $z = (x,y)$ such that $w = H(z)$. From the above we have $|z| > D_1$ and, in consequence, $C|z|^\nu \le |w|$. Hence, $\nu \le \Delta(Q)$ by Lemma 2.1 of [P$_2$]. Since ν was arbitrary, we get $\mathcal{L}_\infty(H) \le \Delta(Q)$. This ends the proof.

8. Proof of Theorem 3.3. Let the resultant $Q(w,x)$ have the form (1) and $Q_0(0) = \ldots = Q_r(0) = 0$, $Q_{r+1}(0) \ne 0$. Put

(9) $$Q^*(w,t) = Q_0(w) + Q_1(w)\,t + \ldots + Q_{r+1}(w)\,t^{r+1} + \ldots + Q_N(w)\,t^N.$$

By the Weierstrass preparation theorem, there exist $\varrho > 0$ and a distinguished pseudopolynomial $P^*(w,t)$ of the form

(10) $$P^*(w,t) = t^{r+1} + a_r(w)\,t^r + \ldots + a_0(w),$$

such that, for $|w| < \varrho$, $|t| < \varrho$, we have

(11) $$Q^*(w,t) = P^*(w,t)\,R^*(w,t),$$

where $a_r, \ldots, a_0$ are holomorphic functions for $|w| < \varrho$, $a_i(0) = 0$, and R^* is a pseudopolynomial with holomorphic coefficients in $\{w : |w| < \varrho\}$, and $R^*(w,t) \ne 0$ for $|w| < \varrho$, $|t| < \varrho$.

8.1. Lemma. *With the above notations, we have*

(12) $$\min_{0\le i\le r} \frac{\operatorname{ord}_0 Q_i}{r+1-i} = \min_{0\le i\le r} \frac{\operatorname{ord}_0 a_i}{r+1-i}.$$

Proof. Let $R^*(w,t) = b_0(w) + \ldots + b_s(w)\,t^s$ where $s = \max(r, N-r-1)$ and $b_j \equiv 0$ for $j > N-r-1$. Obviously, $b_0(0) \ne 0$. From (10) and (11) we get $Q_l = a_0 b_l + \ldots + a_l b_0$ for $l \in \{0, \ldots, r\}$.

We show inductively that, for any $l \in \{0, \dots, r\}$,

$$\min_{0 \le i \le l} \frac{\operatorname{ord}_0 Q_i}{r+1-i} = \min_{0 \le i \le l} \frac{\operatorname{ord}_0 a_i}{r+1-i}. \tag{13}$$

In fact, this is obvious for $l = 0$. Assume that (13) holds for $l = k$. Consider two cases:

1° $\operatorname{ord}_0 a_{k+1} b_0 < \min_{0 \le i \le k} \operatorname{ord}_0 a_i b_{k+1-i}$,
2° $\operatorname{ord}_0 a_{k+1} b_0 \ge \min_{0 \le i \le k} \operatorname{ord}_0 a_i b_{k+1-i}$.

In case 1°, we have $\operatorname{ord}_0 Q_{k+1} = \operatorname{ord}_0 a_{k+1}$, which, together with the induction hypothesis, gives (13) for $l = k + 1$. In case 2°, after easy estimations we get $\operatorname{ord}_0 Q_{k+1}/(r-k) \ge \min_{0 \le i \le k} \operatorname{ord}_0 Q_i/(r+1-i)$ and $\operatorname{ord}_0 a_{k+1}/(r-k) \ge \min_{0 \le i \le k} \operatorname{ord}_0 a_i/(r+1-i)$, which, together with the induction hypothesis, gives (13) for $l = k + 1$, too.

Putting $l = r$ in (13), we get (12).

Put $\delta(Q) = [-\min_{0 \le i \le r}(\operatorname{ord}_0 Q_i)/(r+1-i)]^{-1}$. Obviously, $-\infty < \delta(Q) < 0$.

8.2. LEMMA. *There exist positive constants A, B such that*

$$\{(w,x) : |x| > B,\ Q(w,x) = 0\} \subset \{(w,x) : |x| > B,\ A|x|^{\delta(Q)} \le |w|\}. \tag{14}$$

Proof. By Proposition 2.2 of [P1] and Lemma 8.1, it follows that there exist $A_1, B_1 > 0$ such that

$$\{(w,t) : |w| < B_1,\ P^*(w,t) = 0\} \subset \{(w,t) : |w| < B_1,\ A_1|t|^{-\delta(Q)} \le |w|\}.$$

Hence and from (11) we get, for $\varrho < B_1$,

$$\{(w,t) : |w| < \varrho,\ |t| < \varrho,\ Q^*(w,t) = 0\} \subset \{(w,t) : |w| < \varrho,\ |t| < \varrho,\ A_1|t|^{-\delta(Q)} \le |w|\}.$$

In consequence, we have

$$\{(w,x) : |w| < \varrho,\ |x| > 1/\varrho,\ Q(w,x) = 0\} \subset \{(w,x) : |w| < \varrho,\ |x| > 1/\varrho,\ A_1|x|^{\delta(Q)} \le |w|\}.$$

This implies that, for $A = \min(A_1, \varrho^{\delta(Q)+1})$ and $B = 1/\varrho$, inclusion (14) holds. This ends the proof.

8.3. LEMMA. *If there exist $C, D > 0$ and $\nu < 0$ such that*

$$\{(w,x) : |x| > D,\ Q(w,x) = 0\} \subset \{(w,x) : |x| > D,\ C|x|^{\nu} \le |w|\}, \tag{15}$$

then $\nu \le \delta(Q)$.

Proof. From (15) we get

$$\{(w,t) : |w| < 1/D,\ |t| < 1/D,\ Q^*(w,t) = 0\} \subset \{(w,t) : |w| < 1/D,\ |t| < 1/D,\ C|t|^{-\nu} \le |w|\}.$$

Hence and from (11), putting $\varrho < 1/D$, we get

$$(16) \quad \{(w,t) : |w| < \varrho,\ |t| < \varrho,\ P^*(w,t) = 0\} \subset \{(w,t) : |w| < \varrho,\ |t| < \varrho,\ C|t|^{-\nu} \le |w|\}.$$

Take a sufficiently small $\varepsilon > 0$ such that all the roots of the equations $P^*(w,t) = 0$ for $|w| < \varepsilon$ lie in the disc $\{t : |t| < \varrho\}$. Then from (16) we get

$$\{(w,t) : |w| < \varepsilon,\ P^*(w,t) = 0\} \subset \{(w,t) : |w| < \varepsilon,\ C|t|^{-\nu} \le |w|\}.$$

Hence, from Proposition 2.2 of [P$_1$] and Lemma 8.1 we get $\nu \le \delta(Q)$. This ends the proof.

Let us pass to the proof of Theorem 3.3.

From the properties of the resultant we have $Q(H(z), x) \equiv 0$. Then, by Lemma 8.2 we get $\delta(Q) \in N(H, x)$. Hence $\delta(Q) \le \mathcal{L}_\infty(H,x)$.

Take now $\nu \in N(H, x)$. Then there exist $C, D > 0$ such that $C|x|^\nu \le |H(z)|$ for $|x| > D$. Take w, x such that $|x| > D$ and $Q(w,x) = 0$. From the properties of the resultant there exists $z = (x,y)$ such that $w = H(z)$. Hence $C|x|^\nu \le |w|$. Then, by Lemma 8.3, $\nu \le \delta(Q)$. Since ν was arbitrary, we get $\mathcal{L}_\infty(H,x) \le \delta(Q)$.

Summing up, $\mathcal{L}_\infty(H,x) = \delta(Q)$. Hence and from Proposition 6.1 we get $\mathcal{L}_\infty(H) = \delta(Q)$, which completes the proof.

9. The Łojasiewicz exponent at infinity for a polynomial. Let $h : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function. Put $H = (h'_x, h'_y)$. Then $\mathcal{L}_\infty(H)$ will be called the *Łojasiewicz exponent at infinity* of h and denoted by $\mathcal{L}_\infty(h)$.

The following simple property holds.

9.1. PROPERTY. *If L is a linear automorphism of $\mathbb{C}^2$, then $\mathcal{L}_\infty(h \circ L) = \mathcal{L}_\infty(h)$.*

Proof. Let $L(x,y) = (ax+by,\ cx+dy)$ and $L^*(x,y) = (ax+cy,\ bx+dy)$. From the invariance of $\mathcal{L}_\infty(H)$ with respect to linear automorphisms of the domain and the codomain of H we have $\mathcal{L}_\infty(h \circ L) = \mathcal{L}_\infty(L^* \circ H \circ L) = \mathcal{L}_\infty(H) = \mathcal{L}_\infty(h)$. This ends the proof.

We now give a theorem following from Theorems 3.1–3.3, which completes the result of Ha (see [H], Theorem 1.4.5).

Let h satisfy $(**)$. Then $H = (h'_x, h'_y)$ satisfies $(*)$. Let $Q(w,x) = \operatorname{Res}_y(h'_x - u, h'_y - v)$ where $w = (u,v)$, and let $Q(w,x) = Q_0(w)x^N + \ldots + Q_N(w)$.

9.2. THEOREM. *Under the above assumptions and notations, we have*

(i) $Q_0 = \text{const.}$ *if and only if* $\mathcal{L}_\infty(h) > 0$,

(ii) $Q_0 \ne \text{const.}$ *and* $Q_0(0) \ne 0$ *if and only if* $\mathcal{L}_\infty(h) = 0$,

(iii) *there exists* r *such that* $Q_0(0) = \ldots = Q_r(0) = 0$ *and* $Q_{r+1}(0) \neq 0$ *if and only if* $-\infty < \mathcal{L}_\infty(h) < 0$,

(iv) $Q_0(0) = \ldots = Q_N(0) = 0$ *if and only if* $\mathcal{L}_\infty(h) = -\infty$.

Moreover,

$$\mathcal{L}_\infty(h) = \left[\max_{1\le i\le N} \frac{\deg Q_i}{i}\right]^{-1} \qquad \textit{in case (i)},$$

$$\mathcal{L}_\infty(h) = \left[-\min_{0\le i\le r} \frac{\operatorname{ord}_0 Q_i}{r+1-i}\right]^{-1} \qquad \textit{in case (iii)}.$$

Let now h satisfy the condition $0 < \deg h = \deg_y h$. Let $\operatorname{Res}_y(h-\lambda, h'_y) = c_0(\lambda)x^M + \ldots + c_M(\lambda)$, $c_0 \neq 0$, and

$$\Lambda(h) = \{\lambda \in \mathbb{C} : c_0(\lambda) = 0\}.$$

The following proposition holds (see [K], Proposition 7.1).

9.3. PROPOSITION. *If* L *is a linear automorphism of* $\mathbb{C}^2$ *such that* $0 < \deg h \circ L = \deg_y h \circ L$, *then* $\Lambda(h \circ L) = \Lambda(h)$.

From this proposition it follows that we can define $\Lambda(h)$ for arbitrary h, $0 < \deg h$. Namely, we put $\Lambda(h) = \Lambda(h\circ L)$, where L is a linear automorphism of $\mathbb{C}^2$ such that $\deg h \circ L = \deg_y h \circ L$.

We now give a result due to Ha (cf. [H], Th. 1.5 and [HN], Th. 1.3.1 and Prop. 1.5.1(iii)). Since it was announced by Ha without proof and we shall apply it in the sequel, we give a simple proof of it.

9.4. THEOREM. *If* $h : \mathbb{C}^2 \to \mathbb{C}$ *is a polynomial function and* $0 < \deg h$, *then*

(a) $\Lambda(h) = \emptyset$ *if and only if* $\mathcal{L}_\infty(h) > -1$,

(b) $\Lambda(h) \neq \emptyset$ *if and only if* $\mathcal{L}_\infty(h) < -1$.

Proof. It is easy to find a linear automorphism of $\mathbb{C}^2$ such that $\deg h \circ L = \deg_x h \circ L = \deg_y h \circ L$. Hence, from Property 9.1 and Proposition 9.3 it follows that we may assume without loss of generality that

$$\deg h = \deg_x h = \deg_y h\,. \tag{17}$$

Let $\operatorname{Res}_x(h - \lambda, h'_x) = d_0(\lambda)\, x^M + \ldots + d_M(\lambda)$, $d_0 \neq 0$, and $\Lambda^*(h) = \{\lambda \in \mathbb{C} : d_0(\lambda) = 0\}$. From (17) and Proposition 9.3 we easily get

$$\Lambda(h) = \Lambda^*(h)\,. \tag{18}$$

From Proposition 6.2 of [CK] we easily get

$$c_0(\lambda) \equiv \text{const.} \Leftrightarrow (\deg(h - \lambda) \circ \Phi_i > 0 \text{ for any } \lambda, i)\,, \tag{19}$$

where $\Phi_1, \ldots, \Phi_r$ are parametrizations of the branches at infinity of the curve $h'_y = 0$. Moreover, from (17) it follows that $\deg \Phi_i = \deg \varphi_{1i} > 0$, where $\Phi_i = (\varphi_{1i}, \varphi_{2i})$, $i = 1, \ldots, r$. Hence, differentiating $(h - \lambda) \circ \Phi_i$, we obtain

$$\deg((h-\lambda)\circ\Phi_i)' = \deg h'_x \circ \Phi_i + \deg \Phi_i - 1.$$

So, if $\deg(h-\lambda)\circ\Phi_i > 0$ or $\deg h'_x \circ \Phi_i + \deg\Phi_i > 0$, then from the above we get

$$\deg(h-\lambda)\circ\Phi_i = \deg h'_x \circ \Phi_i + \deg\Phi_i\,. \tag{20}$$

Assume that $\Lambda(h) = \emptyset$. Then $c_0(\lambda) \equiv \text{const}$. Hence, by (19) and (20) we get

$$\deg h'_x \circ \Phi_i / \deg \Phi_i > -1 \quad \text{for each } i\,. \tag{21}$$

On the other hand, from (18) we have $\Lambda^*(h) = \Lambda(h) = \emptyset$. Then, proceeding analogously we obtain

$$\deg h'_y \circ \Psi_j / \deg \Psi_j > -1 \quad \text{for each } j\,,$$

where $\Psi_1, \ldots, \Psi_s$ are parametrizations of the branches at infinity of the curve $h'_x = 0$. Hence, from (21) and Main Theorem of [CK] we get $\mathcal{L}_\infty(h) > -1$.

Assume now that $\mathcal{L}_\infty(h) > -1$. Then again from Main Theorem of [CK] it follows that (21) holds. Hence and from (20) we get $\deg(h-\lambda)\circ\Phi_i > 0$ for any λ, i. But this, according to (19), implies $c_0(\lambda) \equiv \text{const}$. So, $\Lambda(h) = \emptyset$.

We have shown (a). To prove (b), it suffices to show that $\mathcal{L}_\infty(h) \neq -1$. Assume to the contrary that $\mathcal{L}_\infty(h) = -1$. Then, according to Main Theorem of [CK], there exists a parametrization of a branch at infinity of the curve $h'_x = 0$ or $h'_y = 0$, say Ψ_j, such that $\deg h'_y \circ \Psi_j = -\deg \Psi_j$. Hence

$$\deg(h\circ\Psi_j)' = \deg h'_y \circ \Psi_j + \deg\Psi_j - 1 = -1\,,$$

which is impossible because the degree of the derivative of a Laurent series is different from -1.

10. Proof of Theorem 3.4. We precede the proof of the theorem with two equivalent characterizations of a component of a polynomial automorphism of $\mathbb{C}^2$. The first was Theorem 19.1 of [K] and the second is a simple corollary from the first and from Theorem 9.4.

Let $h : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function, $0 < \deg h$.

10.1. THEOREM. *The function h is a component of a polynomial automorphism of $\mathbb{C}^2$ if and only if* $\operatorname{grad} h = (h'_x, h'_y)$ *vanishes nowhere in $\mathbb{C}^2$, and $\Lambda(h) = \emptyset$.*

10.2. THEOREM. *The function h is a component of a polynomial automorphism of $\mathbb{C}^2$ if and only if* $\operatorname{grad} h$ *vanishes nowhere in $\mathbb{C}^2$, and $\mathcal{L}_\infty(h) > -1$.*

Let us pass to the proof of Theorem 3.4.

Assume first that h is a component of a polynomial automorphism of $\mathbb{C}^2$. Then, by Theorem 10.2, grad h vanishes nowhere in $\mathbb{C}^2$. Hence, from the properties of the resultant we easily get

$$Q_0(0) = 0, \ldots, Q_{N-1}(0) = 0 \quad \text{and} \quad Q_N(0) \neq 0\,.$$

This gives the first part of the assertion. If, additionally, $N > 0$, then Theorem 9.2 implies

$$\mathcal{L}_\infty(h) = \left[- \min_{0 \le i < N} \frac{\operatorname{ord}_0 Q_i}{N - i} \right]^{-1}. \tag{22}$$

On the other hand, by Theorem 10.2, we have $\mathcal{L}_\infty(h) > -1$. Hence and from (22) we get the second part of the assertion.

Assume now that $\operatorname{ord}_0 Q_N = 0$ and, if $N > 0$, that $\operatorname{ord}_0 Q_i > N - i$ for $i \in \{0, \ldots, N-1\}$. Then $\operatorname{Res}_y(h'_x, h'_y) = Q_N(0) \neq 0$. This means that grad h vanishes nowhere in $\mathbb{C}^2$. If $N = 0$, then from Theorem 9.2(ii) we get $\mathcal{L}_\infty(h) = 0 > -1$. If $N > 0$, then, by the second part of Theorem 9.2, we obtain (22). So, from the assumption we easily get $\mathcal{L}_\infty(h) > -1$. Thus, by Theorem 10.2, h is a component of a polynomial automorphism.

11. Concluding remarks

11.1. Remark. Assumption $(*)$ in Theorems 3.1–3.3 can be weakened at the cost of its symmetry. Namely, the theorems are still true if we replace $(*)$ by

$$(*)' \qquad 0 < \deg f = \deg_y f, \quad 0 < \deg g$$

or

$$(*)'' \qquad 0 < \deg f, \quad 0 < \deg g = \deg_y g\,.$$

The proofs are unchanged.

11.2. Remark. Assumption $(**)$ in Theorems 3.4 and 9.2 can also be weakened. Namely, the theorems remain true if we replace $(**)$ by

$$(**)' \qquad 0 < \deg h - 1 = \deg_y h'_x$$

or

$$(**)'' \qquad 0 < \deg h - 1 = \deg_y h'_y, \quad 0 < \deg_y h'_x\,.$$

The first condition in $(**)''$ is equivalent to $1 < \deg h = \deg_y h$. The proofs are unchanged.

11.3. Remark. No further weakening of $(*)$ and $(**)$ is possible. This is shown by the following example. Let $h(x,y) = x^2y^2 + x$. Easy calculations give $\mathcal{L}_\infty(h) = -3$, while $\operatorname{Res}_y(h'_x - u,\, h'_y - v) = 4(1-u)\,x^4 + 2v^2x$.

11.4. Remark. From Theorem 9.2 it follows that $\mathcal{L}_\infty(h)$ is a rational number or $-\infty$. Note that, for each rational number r different from -1, there exists a polynomial function $h : \mathbb{C}^2 \to \mathbb{C}$ such that $\mathcal{L}_\infty(h) = r$. This follows from the following examples:

(a) $\mathcal{L}_\infty(y^p + (x + y^q)^p) = -1 + p/q$ for $1 < p,\ 0 < q$,
(b) $\mathcal{L}_\infty(y + y^{1+q}x^{p-q}) = -p/q$ for $0 < q < p$.

Indeed, from (a) we get any $r > -1$, whereas from (b) any $r < -1$. In both cases, the Łojasiewicz exponent at infinity can easily be found by using Main Theorem of [CK]. Obviously, $\mathcal{L}_\infty(h) \neq -1$ for every h (Theorem 9.4). Example (b) is due to Ha ([H], Remark 1.5.2(ii)).

11.5. Remark. From Theorems 3.1–3.3 it also follows that $\mathcal{L}_\infty(H)$, for every polynomial mapping H, is a rational number or $-\infty$. Note that, for each rational number r, there exists $H : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\mathcal{L}_\infty(H) = r$. This follows from Remark 11.4 and the fact that, for $H(x,y) = (x,\ xy - 1)$, we have $\mathcal{L}_\infty(H) = -1$.

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