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A method of construction of an invariant measure

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Abstract. A method of construction of an invariant measure on a function space is presented.

0. Introduction. The problem of existence of a measure invariant with respect to the dynamical system generated by the differential equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = f(u) \tag{1}$$

has been considered by Lasota [4], Rudnicki [5] and the author [1], [2]. At present, there are various theorems on the existence and properties of such measures. In this paper we present a general method of construction of an invariant measure.

1. Formulation of the result. Consider the differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u, \qquad t \geq 0,\ 0 \leq x \leq 1\,, \tag{2}$$

with the initial condition

$$u(0;x) = v(x) \tag{3}$$

and the boundary condition

$$u(t;0) = 0\,. \tag{4}$$

This problem generates a semidynamical system on the space V of all Lipschitz functions on the interval $[0;1]$ vanishing at 0 by the formula

$$T_t v(x) = e^{\lambda t} v(xe^{-t}) \tag{5}$$

Denote by $\|v'\|$ the maximal Lipschitz constant of v, i.e.

$$\|v'\| = \sup_{0\leq x<y\leq 1} \frac{|v(y)-v(x)|}{y-x}\,.$$

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For Lipschitz functions this supremum is finite.

DEFINITION. Let $\lambda > 1$. A measure μ on V satisfies *assumption* A if there exists a sequence $\{\varrho_n\}$ of positive real numbers and a positive constant K such that

(i) $$\frac{\varrho_n}{2^{(\lambda-1)n}} \le K, \qquad n = 1, 2, \ldots,$$

(ii) $$\sum_{n=0}^{\infty} \mu(\{v : \|v'\| \ge \varrho_n\}) < \infty .$$

THEOREM 1. *Let $\lambda > 1$ and suppose μ satisfies assumption* A. *Then we can construct a T_1-invariant measure $\widehat{\mu}$. The construction has the following property: If μ is T_1-invariant, then $\widehat{\mu} = \mu$.*

2. Proof of Theorem 1. Let $A : V \to V$ be given by $(Av)(x) = v(\frac{1}{2}(x+1)) - v(\frac{1}{2})$ and let $\mu'(E) = \mu(A^{-1}(E))$. Clearly μ' also satisfies (i), (ii) (with another sequence $\{\varrho_n\}$). Define $\Phi : V^{\mathbb{N}} \to \mathcal{P}(V)$ by

(6) $$w \in \Phi(v_0, v_1, \ldots) \Leftrightarrow \forall n,\ w(2^{-n-1}(x+1)) - w(2^{-n-1}) = 2^{-\lambda n} v_n(x) \quad \text{for } 0 \le x \le 1 .$$

This condition determines the values of w on the interval $[2^{-n-1}; 2^{-n}]$ up to an additive constant. Since $w(0) = 0$ the condition $w \in \Phi(v_0, v_1, \ldots)$ can be satisfied by at most one function w. Hence $\operatorname{card} \Phi(v_0, v_1, \ldots) \le 1$.

Let $\widehat{\mu}'$ be the product measure on $V^{\mathbb{N}}$. We claim that $\widehat{\mu}'(\{(v_0, v_1, \ldots) : \Phi(v_0, v_1, \ldots) = \emptyset\}) = 0$. First suppose the sequence $\{v_n\}$ satisfies

(7) $$\exists n_0\ \forall n \ge n_0 \quad \|v_n'\| \le \varrho_n .$$

Let $\overline{w}(x) = 2^{-\lambda n} v_n(2^{n+1}x - 1) + C_n$ for $x \in [2^{-n-1}; 2^{-n}]$ where the constant C_0 is arbitrary and $\{C_n\}$ is a sequence such that $\overline{w}$ is continuous on $(0; 1]$. Since, for $n \ge n_0$, $\overline{w} \mid [2^{-n-1}; 2^{-n}]$ satisfies the Lipschitz condition with constant $2\varrho_n 2^{-(\lambda-1)n} \le 2K$, the function $\overline{w}$ also satisfies the Lipschitz condition and in consequence $\lim_{x\to 0} \overline{w}(x)$ exists. Define

$$w(x) = \overline{w}(x) - \lim_{y \to 0} \overline{w}(y)$$

for $x > 0$ and $w(0) = 0$. Hence $w \in \Phi(v_0, v_1, \ldots)$ and in consequence for every sequence $(v_0, v_1, \ldots)$ satisfying (7), $\Phi(v_0, v_1, \ldots)$ is nonempty. Moreover, from the Borel–Cantelli lemma it follows that the set of sequences satisfying condition (7) has full measure.

Now, let $\widehat{\mu}$ be defined by

(8) $$\widehat{\mu}(E) = \widehat{\mu}'(\{(v_0, v_1, \ldots) : \Phi(v_0, v_1, \ldots) \subset E\}) .$$

Clearly, $\widehat{\mu}$ is the transport of $\widehat{\mu}'$ by a map defined on a full-measure set. The invariance and the ergodicity follow from an argument analogous to that in [1]. From the construction it also follows that if μ is T_1-invariant, then $\mu = \widehat{\mu}$.

Remark. By the same method as in [1] a T-invariant measure can be constructed.

3. Properties of the measure $\widehat{\mu}$

THEOREM 2. *The measure $\widehat{\mu}$ is defined on the σ-algebra of Borel sets for the topology of uniform convergence.*

Proof. Let Σ be the σ-algebra on which $\widehat{\mu}$ is defined, i.e. $E \in \Sigma$ if and only if $(\tau\Phi)^{-1}(E)$ is $\widehat{\mu}'$-measurable where $\tau\Phi(v_0, v_1, \ldots)$ denotes the unique element of $\Phi(v_0, v_1, \ldots)$. Since V is a separable space it is sufficient to prove that if $\widetilde{v} \in V$, $\varepsilon > 0$ then $U(\widetilde{v}; \varepsilon) = \{v : \sup_{x\in[0;1]} |v(x) - \widetilde{v}(x)| < \varepsilon\}$. In [2] (Lemma 4) it is proved that the map

$$(v_0, v_1, \ldots) \mapsto \sup_{x\in[0;1]} |\tau\Phi(v_0, v_1, \ldots)(x) - \widetilde{v}(x)|$$

is measurable, which completes the proof.

THEOREM 3. *If μ is positive on open nonempty sets, then so is $\widetilde{\mu}$.*

Proof. First, let $G(n;\varepsilon) = \{v \in V : \forall x \in [0; 2^{-n}], |v(x)| < \varepsilon\}$. From the proof of Lemma 5 of [2] it follows that if $\widehat{\mu}(G(n; 2^{\lambda}\varepsilon)) > 0$, then there exists $\varepsilon' < \varepsilon$ such that $\widehat{\mu}(G(n+1; \varepsilon')) > 0$. Clearly $\widehat{\mu}(G(n;\varepsilon)) = \widehat{\mu}(G(n+1;\varepsilon'))\widehat{\mu}'(\{v \in V : \|v\| < 2^{\lambda n}(\varepsilon - \varepsilon')\}) > 0$. By an argument analogous to that in [2] it follows that $\widehat{\mu}(G(n;\varepsilon)) > 0$. The end of proof is also analogous to [2].

4. Examples

EXAMPLE 1. Let $\{\sigma_n\}$ and $\{p_n\}$ be as in [1] and let $\mu(E) = \sum_{\sigma_n \in E} p_n$. In this situation the measure obtained from μ by the procedure presented in Theorem 1 is the measure considered in [1], [2] (clearly, with another sequence $\{\sigma_n\}$).

EXAMPLE 2. Let μ_W be the Wiener measure on $C[0;1]$ and let $I : C[0;1] \to V$ be defined by the formula

$$(Iv)(x) = \int_0^x v(s)\, ds\,. \tag{9}$$

Let μ be the transport of the Wiener measure by I. Using [4] it can be proved that μ satisfies assumption A. By the presented procedure we can obtain a Gaussian T_1-invariant measure.

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