Generalized solutions to boundary value problems
for quasilinear hyperbolic systems
of partial differential-functional equations

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Abstract. Generalized solutions to quasilinear hyperbolic systems in the second canonical form are investigated. A theorem on existence, uniqueness and continuous dependence upon the boundary data is given. The proof is based on the methods due to L. Cesari and P. Bassanini for systems which are not functional.

1. Introduction. We denote by $C(X, Y)$ the set of all continuous functions from X to Y, where X, Y are any metric spaces. Let $a_0 > 0$, $B = [-b_0, 0] \times [-b, b]$, where $b_0 \in \mathbb{R}_+$, $b = (b_1, \ldots, b_r) \in \mathbb{R}_+^r$, $\mathbb{R}_+ = [0, \infty)$. For $z : [-b_0, a_0] \times \mathbb{R}^r \to \mathbb{R}^m$ and $(x, y) = (x, y_1, \ldots, y_r) \in [0, a_0] \times \mathbb{R}^r$, define $z_{(x,y)} : B \to \mathbb{R}^m$ by $z_{(x,y)}(t, s) = z(x + t, y + s)$, $(t, s) \in B$. Put $\Omega = [0, a_0] \times \mathbb{R}^r \times C(B, \mathbb{R}^m)$, and denote by $M(m, r)$ the set of all real $m \times r$ matrices.

Let
$$A : \Omega \to M(m, m), \quad A = [A_{ij}], \ i, \ j = 1, \ldots, m,$$
$$\varrho : \Omega \to M(m, r), \quad \varrho = [\varrho_{ij}], \ i = 1, \ldots, m, \ j = 1, \ldots, r,$$
$$f : \Omega \to M(m, 1), \quad f = [f_1, \ldots, f_m]^T,$$

be given functions of the variables (x, y, w), $y = (y_1, \ldots, y_r)$, $w = (w_1, \ldots, w_r)$, and T the transpose symbol. Note that if $(x, y) \in [0, a_0] \times \mathbb{R}^r$ is fixed then $A(x, y, \cdot)$, $\varrho(x, y, \cdot)$, $f(x, y, \cdot)$ are operators on the function space $C(B, \mathbb{R}^m)$. Furthermore, let

$$B_l : \mathbb{R}^r \to M(m, m), \quad B_l = [B_{lij}], \ i, j = 1, \ldots, m,$$
$$\psi : \mathbb{R}^r \to \mathbb{R}^r, \quad \psi = (\psi_1, \ldots, \psi_r),$$

where $l = 1, \ldots, N$, $m \leq N$, be given functions of the variable $y = (y_1, \ldots, y_r)$.

We consider the following hyperbolic differential-functional system in the second canonical form:

$$(1) \quad \sum_{j=1}^{m} A_{ij}(x, y, z_{(x,y)}) \Big[D_x z_j(x, y) + \sum_{k=1}^{r} \varrho_{ik}(x, y, z_{(x,y)}) D_{y_k} z_j(x, y) \Big]$$
$$= f_i(x, y, z_{(x,y)}), \quad i = 1, \ldots, m,$$

with the boundary data on N arbitrary (not necessarily distinct) hyperplanes $x = a_l$, $a_l \in [0, a_0]$, $l = 1, \ldots, N$,

$$(2) \qquad \sum_{l=1}^{N} B_l(y) z(a_l, y) = \psi(y), \quad y \in \mathbb{R}^r.$$

A function $z \in C(I_a, \mathbb{R}^m)$, $I_a = [0, a] \times \mathbb{R}^r$, where $a \in (0, a_0]$, is a *solution* of (1), (2) if z has partial derivatives $D_x z$, $D_{y_k} z$, $k = 1, \ldots, r$, a.e. on I_a, satisfies (1) a.e. on I_a and (2) for all $y \in \mathbb{R}^r$.

If $b_0 = 0$ and $b = 0$ then the differential-functional system (1) reduces to a differential system in the second canonical form. Generalized (a.e.) solutions of systems of that form have been investigated in a large number of papers. We mention here the papers of P. Bassanini [1]–[3] and L. Cesari [5], [6]. Quasilinear hyperbolic systems with a retarded argument have been studied by Z. Kamont and J. Turo [12]–[15]. In [19]–[22] J. Turo has also considered the existence and uniqueness of generalized solutions for differential-functional hyperbolic systems with operators of Volterra type. In that case the given functions are superpositions of functions defined on subsets of a finite-dimensional Euclidean space with operators of Volterra type. The main assumptions for the operators have been formulated in the form of linear inequalities given on some function space. Classical solutions to nonlinear equations with the same model of functional dependence have been investigated in [11], [18].

In this paper we consider a new model of differential-functional systems which has been used in [9] for initial value problems. For each $(x, y) \in [-b_0, a_0] \times \mathbb{R}^r$ and for any $z : [-b_0, a_0] \times \mathbb{R}^r \to \mathbb{R}^m$ we denote by $z_{(x,y)}$ the translation of the restriction of z to B. In other words, the graph of $z_{(x,y)}$ is the graph of $z : [x - b_0, x] \times [y - b, y + b] \to \mathbb{R}^m$ shifted to B. Since the given functions are now operators on $z_{(x,y)}$ we no longer need the assumption that the right-hand side of the system is the superposition of some function and an operator of Volterra type. This simple model is well known for ordinary differential-functional equations (see [10], [16]). It is also very general since systems of differential equations with retarded argument ([12]–[15]), differential-integral systems ([17]) and differential-functional systems with operators of Volterra type ([19]–[22]) can be obtained from (1) by specializing the given functions. Our formulation of the problem enables us to get

new examples of differential-functional systems which cannot be obtained from the results cited above. More detailed comparison between the model of functional dependence proposed in this paper and that used in [19]–[22] is presented in [7], [8]. The method which we use is based on the Banach fixed point theorem in a product space and it is close to that used in [14].

2. Assumptions and notations. Let $\|\eta\|_m = \max_{1 \leq i \leq m} |\eta_i|$ in $\mathbb{R}^m$ and $\|U\|_{m,k} = \max_{1 \leq i \leq m} \sum_{j=1}^m |U_{ij}|$ in $M(m,k)$. For short we write $\|\eta\|$, $\|U\|$. For $w \in C(B, \mathbb{R}^m)$ put $\|w\|_0 = \sup\{\|w(t,s)\| : (t,s) \in B\}$. Let $C_{0+L}(B, \mathbb{R}^m)$ denote the set of all functions $w \in C(B, \mathbb{R}^m)$ such that

$$\|w\|_L = \sup\{[|t - \bar{t}| + \|w - \overline{w}\|]^{-1} \|w(t,s) - w(\bar{t}, \bar{s})\| : (t,s), (\bar{t}, \bar{s}) \in B\} < \infty.$$

We equip this set with the norm $\|w\|_{0+L} = \|w\|_0 + \|w\|_L$, $w \in C_{0+L}(B, \mathbb{R}^m)$. Furthermore, for any $q \in \mathbb{R}_+$ we set $C(B, \mathbb{R}^m, q) = \{w \in C(B, \mathbb{R}^m) : \|w\|_0 \leq q\}$, $C_{0+L}(B, \mathbb{R}^m, q) = \{w \in C_{0+L}(B, \mathbb{R}^m) : \|w\|_{0+L} \leq q\}$.

ASSUMPTION H_1. 1° $A \in C(\Omega, M(m,m))$ and there is a constant $\nu > 0$ such that $\det A(x,y,w) \geq \nu$ for any $(x,y,w) \in \Omega$.

2° There are nondecreasing functions $n, l : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $q \in \mathbb{R}_+$ we have

  (i) $\|A(x,y,w)\| \leq n(q)$, $(x,y,w) \in [0, a_0] \times \mathbb{R}^r \times C(B, \mathbb{R}^m, q)$,
  (ii) $\|A(x,y,w) - A(\bar{x}, \bar{y}, \overline{w})\| \leq l(q)[|x - \bar{x}| + \|y - \bar{y}\| + \|w - \overline{w}\|_0]$, $(x,y,w), (\bar{x}, \bar{y}, \overline{w}) \in [0, a_0] \times \mathbb{R}^r \times C_{0+L}(B, \mathbb{R}^m, q)$.

R e m a r k 1. Assumption H_1 yields that $A^{-1}(x,y,w)$ exists for any $(x,y,w) \in \Omega$. Furthermore, there are nondecreasing $n', l' : \mathbb{R}_+ \to \mathbb{R}_+$ such that condition 2° of Assumption H_1 holds with A, n, l replaced by A^{-1}, n', l', respectively.

Let θ denote the set of all functions $l : [0, a_0] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $l(\cdot, q) : [0, a_0] \to \mathbb{R}_+$ is measurable for any $q \in \mathbb{R}_+$, and $l(x, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing for a.e. $x \in [0, a_0]$.

ASSUMPTION H_2. 1° $\varrho(\cdot, y, w) : [0, a_0] \to M(m, r)$ is measurable for all $(y, w) \in \mathbb{R}^r \times C(B, \mathbb{R}^m)$.

2° $\varrho(x, \cdot) : \mathbb{R}^r \times C(B, \mathbb{R}^m) \to M(m, r)$ is continuous for a.e. $x \in [0, a_0]$.

3° There are a nondecreasing function $n_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $l_1 \in \theta$ such that for any $q \in \mathbb{R}_+$ we have

  (i) $\|\varrho(x,y,w)\| \leq n_1(q)$, $(x,y,w) \in [0, a_0] \times \mathbb{R}^r \times C(B, \mathbb{R}^m, q)$,
  (ii) $\|\varrho(x,y,w) - \varrho(x, \bar{y}, \overline{w})\| \leq l_1(x,q)[\|y - \bar{y}\| + \|w - \overline{w}\|_0]$ for all (y, w), $(\bar{y}, \overline{w}) \in \mathbb{R}^r \times C_{0+L}(B, \mathbb{R}^m, q)$ and for a.e. $x \in [0, a_0]$.

ASSUMPTION H_3. 1° $f(\cdot, y, w) : [0, a_0] \to M(m, 1)$ is measurable for all $(y, w) \in \mathbb{R}^r \times C(B, \mathbb{R}^m)$.

2° $f(x, \cdot) : \mathbb{R}^r \times C(B, \mathbb{R}^m) \to M(m, 1)$ is continuous for a.e. $x \in [0, a_0]$.

$3°$ There are a nondecreasing function $n_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and $l_2 \in \theta$ such that for any $q \in \mathbb{R}_+$ we have

(i) $\|f(x,y,w)\| \leq n_2(q)$, $(x,y,w) \in [0,a_0] \times \mathbb{R}^r \times C(B, \mathbb{R}^m, q)$,
(ii) $\|f(x,y,w) - f(x,\overline{y},\overline{w})\| \leq l_2(x,q)[\|y - \overline{y}\| + \|w - \overline{w}\|_0]$ for all (y,w), $(\overline{y},\overline{w}) \in \mathbb{R}^r \times C_{0+\mathrm{L}}(B, \mathbb{R}^m, q)$ and for a.e. $x \in [0, a_0]$.

For $(x,y,w) \in \Omega$, $l = 1,\ldots,N$, we put

(3)
$$A(x,y,w) = E + \widetilde{A}(x,y,w), \quad A^{-1}(x,y,w) = E + \bar{A}(x,y,w),$$
$$B_l(y) = E_l + \widetilde{B}_l(y), \quad E = [\delta_{ij}], \quad E_l = [\delta_{li}\delta_{ij}], \quad i,j = 1,\ldots,m,$$

and

$$\sigma_0 = \sup \Big\{ \sum_{l=1}^{N} \|\widetilde{B}_l(y)\| : y \in \mathbb{R}^r \Big\},$$

$$\sigma_1 = \sup\{\|\widetilde{A}(x,y,w)\| : (x,y,w) \in \Omega\},$$

$$\sigma_2 = \sup\{\|\bar{A}(x,y,w)\| : (x,y,w) \in \Omega\}.$$

ASSUMPTION H_4. $1°$ $\psi \in C(\mathbb{R}^r, \mathbb{R}^m)$ and there are constants $\Gamma, \Lambda \in \mathbb{R}_+$ such that for all $y, \overline{y} \in \mathbb{R}^r$ we have

$$\|\psi(y)\| \leq \Gamma, \quad \|\psi(y) - \psi(\overline{y})\| \leq \Lambda\|y - \overline{y}\|.$$

$2°$ $B_l \in C(\mathbb{R}^r, M(m,m))$, $\det B_l(y) \neq 0$ for all $l = 1,\ldots,N$, $y \in \mathbb{R}^r$, and there is a constant $G \in \mathbb{R}_+$ such that

$$\sum_{l=1}^{N} \|B_l(y) - B_l(\overline{y})\| \leq G\|y - \overline{y}\|, \quad y, \overline{y} \in \mathbb{R}^r.$$

$3°$ $(\sigma_0 + \sigma_1)(1 + \sigma_2) < 1$.

We denote by $B_1(a, Q, Q_1, Q_2)$, where $a \in (0, a_0]$, $Q, Q_1, Q_2 \in \mathbb{R}_+$, the set of all functions $z \in C(I_a, \mathbb{R}^m)$ such that for all $(x,y), (\overline{x}, \overline{y}) \in I_a$ we have

(i) $\|z(x,y)\| \leq Q$,
(ii) $\|z(x,y) - z(\overline{x}, \overline{y})\| \leq Q_1|x - \overline{x}| + Q_2\|y - \overline{y}\|$.

Note that $B_1(a, Q, Q_1, Q_2)$ is a closed subset of the Banach space of all continuous and bounded vector functions $z : I_a \to \mathbb{R}^m$ with the norm $\|z\|_{I_a} = \sup\{\|z(x,y)\| : (x,y) \in I_a\}$.

For $a \in (0, a_0]$ define the constants $L_{ia}(Q, Q_1, Q_2) = \int_0^a l_i(t, Q + Q_1 + Q_2)\,dt$, $i = 1, 2$. In the sequel we write L_{ia} for short.

Let $a \in (0, a_0]$ be so small that

(4)
$$L_{1a}(1 + Q_2) < 1,$$

and let $Q, Q_1, Q_2 \in \mathbb{R}_+$, $p \in (0, 1)$. Then we denote by $B_2(a, Q, Q_1, Q_2, p)$ the set of all functions $h \in C(\Delta_a, M(m,r))$, $\Delta_a = [0, a] \times [0, a] \times \mathbb{R}^r$, such that for all $(x, x, y), (\xi, x, y), (\bar{\xi}, \overline{x}, \overline{y}) \in \Delta_a$, $i = 1, \ldots, m$, we have

(i) $h_i(x, x, y) = 0$,

(ii) $\|h_i(\xi, x, y) - h_i(\bar{\xi}, \bar{x}, \bar{y})\| \leq n_1(Q)|\xi - \bar{\xi}| + \gamma_a n_1(Q)|x - \bar{x}| + p\|y - \bar{y}\|$,

where $\gamma_a = [1 - L_{1a}(1 + Q_2)]^{-1}$, $h_i = (h_{i1}, \ldots, h_{ir})$. It is easily seen that $B_2(a, Q, Q_1, Q_2, p)$ is a closed subset of the Banach space of all continuous and bounded matrix functions $h : \Delta_a \to M(m, r)$ with the norm $\|h\|_{\Delta_a} = \max_{1 \leq i \leq m} \sup\{\|h_i(\xi, x, y)\| : (\xi, x, y) \in \Delta_a\}$. Indeed, for all $(\xi, x, y) \in \Delta_a$, $i = 1, \ldots, m$, we have

$$\|h_i(\xi, x, y)\| = \|h_i(\xi, x, y) - h_i(x, x, y)\| \leq n_1(Q)|\xi - x| \leq n_1(Q)a,$$

which yields $\|h\|_{\Delta_a} \leq n_1(Q)a$.

For any $h \in B_2(a, Q, Q_1, Q_2, p)$ let $g : \Delta_a \to M(m, r)$ be defined by

(5) $\qquad g_i(\xi, x, y) = y + h_i(\xi, x, y), \qquad (\xi, x, y) \in \Delta_a, \ i = 1, \ldots, m,$

where $g_i = (g_{i1}, \ldots, g_{ir})$, $h_i = (h_{i1}, \ldots, h_{ir})$. Then for all (x, x, y), (ξ, x, y), $(\bar{\xi}, \bar{x}, \bar{y}) \in \Delta_a$, $i = 1, \ldots, m$, we have

(i) $g_i(x, x, y) = y$,

(ii) $\|g_i(\xi, x, y) - g_i(\bar{\xi}, \bar{x}, \bar{y})\| \leq n_1(Q)|\xi - \bar{\xi}| + \gamma_a n_1(Q)|x - \bar{x}| + (1 + p)\|y - \bar{y}\|$.

3. The operators $V^{(1)}, V^{(2)}$ and their properties. Let $\langle \cdot, \cdot \rangle$ be a scalar product in $\mathbb{R}^m$, let $g_i(t, x, y)$ be defined by (5) and set

$$z_i^*(t, x, y) = (z_1(t, g_i(t, x, y)), \ldots, z_m(t, g_i(t, x, y))),$$

$$A_i^*(t, x, y)$$
$$= (A_{i1}(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}), \ldots, A_{im}(t, g_i(t, x, y), z_{(t, g_i(t, x, y))})).$$

We consider the operators $Z = V^{(1)}[z, h]$, $H = V^{(2)}[z, h]$ defined for $z \in B_1(a, Q, Q_1, Q_2)$, $h \in B_2(a, Q, Q_1, Q_2, p)$ as follows:

$$Z(x, y) = A^{-1}(x, y, z_{(x,y)})(\Delta_1(x, y) + \Delta_2(x, y) + \Delta_3(x, y)),$$
$$\Delta_k(x, y) = (\Delta_{k1}(x, y), \ldots, \Delta_{km}(x, y)), \qquad k = 1, 2, 3,$$
$$\Delta_{1i}(x, y) = \psi_i(g_i(a_i, x, y)) + \int_{a_i}^{x} f_i(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}) \, dt,$$

(6) $\qquad \Delta_{2i}(x, y) = \int_{a_i}^{x} \langle D_t A_i^*(t, x, y), z_i^*(t, x, y) \rangle \, dt$

$$+ \langle A_i^*(a_i, x, y), z_i^*(a_i, x, y) \rangle,$$

$$\Delta_{3i}(x, y) = -\sum_{l=1}^{N} \langle B_{li}(g_i(a_i, x, y)), z(a_l, g_i(a_i, x, y)) \rangle,$$

$$i = 1, \ldots, m, \ (x, y) \in I_a;$$

and
$$H(\xi, x, y) = [H_{ij}(\xi, x, y)], \quad i = 1, \ldots, m, \; j = 1, \ldots, r,$$

(7) $\qquad H_i(\xi, x, y) = \int\limits_{x}^{\xi} \varrho_i(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}) \, dt,$

$$i = 1, \ldots, m, \; (\xi, x, y) \in \Delta_a.$$

R e m a r k 2. By (3) we may simultaneously replace $D_t A_i^*, A_i, B_{li}$ in the above equalities by $D_t \widetilde{A}_i^*, \widetilde{A}_i, \widetilde{B}_{li}$, respectively.

LEMMA 1. *Suppose that Assumptions* H_1–H_4 *hold and that*

(8) $\qquad (1 + \sigma_2)\Gamma + (1 + \sigma_2)(\sigma_0 + \sigma_1)Q < Q,$

(9) $\qquad l'(Q)(1 + Q_1)[\Gamma + (\sigma_0 + \sigma_1)Q] + (1 + \sigma_2)[n_1(Q)(\Lambda + GQ)$

$$+ l(Q)Q(2 + n_1(Q) + 2Q_1 + Q_2 n_1(Q))]$$

$$+ n_1(Q)(1 + \sigma_2)(\sigma_0 + \sigma_1)Q_2 < Q_1,$$

(10) $\quad l'(Q)(1 + Q_2)[\Gamma + (\sigma_0 + \sigma_1)Q] + (1 + \sigma_2)[(1 + p)(\Lambda + GQ) + l(Q)Q(1 + Q_2)]$

$$+ (1 + p)(1 + \sigma_2)(\sigma_0 + \sigma_1)Q_2 < Q_2.$$

Then for $a \in (0, a_0]$ *sufficiently small the operator* $V^{(1)}$ *maps* $B_1(a, Q, Q_1, Q_2)$ $\times B_2(a, Q, Q_1, Q_2, p)$ *into* $B_1(a, Q, Q_1, Q_2)$.

P r o o f. Let $Z = V^{(1)}[z, h]$, $z \in B_1(a, Q, Q_1, Q_2)$, $h \in B_2(a, Q, Q_1, Q_2, p)$. By the estimates

$$\|\Delta_1(x, y)\| \le \max_{1 \le i \le m} \left\{ |\psi_i(g_i(a_i, x, y))| \right.$$

$$+ \left| \int\limits_{a_i}^{x} f_i(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}) \, dt \right| \right\}$$

$$\le \Gamma + \int\limits_{a_i}^{x} n_2(Q) \, dt \le \Gamma + n_2(Q)a,$$

$$\|\Delta_2(x, y)\| \le \max_{1 \le i \le m} \left\{ \left| \int\limits_{a_i}^{x} \langle D_t \widetilde{A}_i^*(t, x, y), z_i^*(t, x, y) \rangle \, dt \right| \right.$$

$$+ |\langle \widetilde{A}_i^*(a_i, x, y), z_i^*(a_i, x, y) \rangle| \right\}$$

$$\le \int\limits_{a_i}^{x} l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Q \, dt + \sigma_1 Q$$

$$\le l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Qa + \sigma_1 Q,$$

$$\|\Delta_3(x, y)\| \le \max_{1 \le i \le m} \left| -\sum_{l=1}^{N} \langle \widetilde{B}_{li}(g_i(a_i, x, y)), z(a_l, g_i(a_i, x, y)) \rangle \right| \le \sigma_0 Q,$$

we obtain

$$\|Z(x,y)\|$$
$$\leq (1+\sigma_2)[\Gamma + n_2(Q)a + l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Qa + \sigma_1 Q + \sigma_0 Q],$$

$(x,y) \in I_a$. By (8) we can choose a so small that

$$(11) \quad (1+\sigma_2)[\Gamma + (\sigma_1 + \sigma_0)Q]$$
$$+ (1+\sigma_2)[n_2(Q) + l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Q]a \leq Q,$$

and thus for $(x,y) \in I_a$ we obtain

$$(12) \qquad\qquad\qquad \|Z(x,y)\| \leq Q.$$

For any $(x,y), (\overline{x},\overline{y}) \in I_a$, we have

$$Z(x,y) - Z(\overline{x},\overline{y}) = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3,$$

where

$$\alpha_0 = (A^{-1}(x,y,z_{(x,y)}) - A^{-1}(\overline{x},\overline{y},z_{(\overline{x},\overline{y})}))[\Delta_1(x,y) + \Delta_2(x,y) + \Delta_3(x,y)],$$
$$\alpha_i = A^{-1}(\overline{x},\overline{y},z_{(\overline{x},\overline{y})})[\Delta_i(x,y) - \Delta_i(\overline{x},\overline{y})], \quad i = 1,2,3.$$

We can estimate the above terms as follows:

$$\|\alpha_0\| \leq [l'(Q)(1+Q_2)|x - \overline{x}| + l'(Q)(1+Q_2)\|y - \overline{y}\|]$$
$$\times [\|\Delta_1(x,y)\| + \|\Delta_2(x,y)\| + \|\Delta_3(x,y)\|]$$
$$\leq [\Gamma + n_2(Q)a + l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Qa$$
$$+ \sigma_1 Q + \sigma_0 Q]l'(Q)(1+Q_1)|x - \overline{x}|$$
$$+ [\Gamma + n_2(Q)a + l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Qa$$
$$+ \sigma_1 Q + \sigma_0 Q]l'(Q)(1+Q_1)\|y - \overline{y}\|,$$

$$\|\alpha_1\| \leq (1+\sigma_2) \max_{1 \leq i \leq m} \Big\{ |\psi_i(g_i(a_i,x,y)) - \psi_i(g_i(a_i,\overline{x},\overline{y}))|$$
$$+ \Big| \int_{a_i}^{x} [f_i(t, g_i(t,x,y), z_{(t,g_i(t,x,y))})$$
$$- f_i(t, g_i(t,\overline{x},\overline{y}), z_{(t,g_i(t,\overline{x},\overline{y}))})] \, dt \Big|$$
$$+ \Big| \int_{\overline{x}}^{x} f_i(t, g_i(t,\overline{x},\overline{y}), z_{(t,g_i(t,\overline{x},\overline{y}))}) \, dt \Big| \Big\}$$
$$\leq (1+\sigma_2)[\Lambda\gamma_a + L_{2a}(1+Q_2)\gamma_a + n_2(Q)a]|x - \overline{x}|$$
$$+ (1+\sigma_2)[\Lambda(1+p) + L_{2a}(1+Q_2)(1+p)]\|y - \overline{y}\|,$$

$$\|\alpha_2\| \leq (1+\sigma_2) \max_{1 \leq i \leq m} \Big\{ \Big| \int_{a_i}^{x} \langle \widetilde{A}_i^*(t,x,y) - \widetilde{A}_i^*(t,\overline{x},\overline{y}), D_t z_i^*(t,x,y) \rangle \, dt \Big|$$

$$+ \Big| \int\limits_{a_i}^{x} \langle D_t \widetilde{A}_i^*(t,\overline{x},\overline{y}), z_i^*(t,x,y) - z_i^*(t,\overline{x},\overline{y}) \rangle \, dt \Big|$$

$$+ \Big| \int\limits_{\overline{x}}^{x} \langle D_t \widetilde{A}_i^*(t,\overline{x},\overline{y}), z_i^*(t,x,y) \rangle \, dt \Big|$$

$$+ |\langle \widetilde{A}_i^*(x,x,y) - \widetilde{A}_i^*(\overline{x},\overline{x},\overline{y}), z_i^*(x,x,y) \rangle|$$

$$+ |\langle \widetilde{A}_i^*(a_i,\overline{x},\overline{y}), z_i^*(a_i,x,y) - z_i^*(a_i,\overline{x},\overline{y}) \rangle| \Big\}$$

$$\leq (1+\sigma_2)\{l(Q)(1+Q_2)\gamma_a(Q_1 + Q_2 n_1(Q))a$$
$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Q_2 \gamma_a a$$
$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Q$$
$$+ l(Q)(1+Q_1)Q + \sigma_1 Q_2 \gamma_a\}|x - \overline{x}|$$
$$+ (1+\sigma_2)\{l(Q)(1+Q_2)(1+p)(Q_1 + Q_2 n_1(Q))a$$
$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Q_2(1+p)a$$
$$+ l(Q)(1+Q_2)Q + \sigma_1 Q_2(1+p)\}\|y - \overline{y}\|,$$

$$\|\alpha_3\| \leq (1+\sigma_2)$$

$$\times \max_{1 \leq i \leq m} \Big\{ \Big| \sum_{l=1}^{N} \langle \widetilde{B}_{li}(g_i(a_i,x,y)) - \widetilde{B}_{li}(g_i(a_i,\overline{x},\overline{y})), z(a_l, g_i(a_i,x,y)) \rangle \Big|$$

$$+ \Big| \sum_{l=1}^{N} \langle \widetilde{B}_{li}(g_i(a_i,\overline{x},\overline{y})), z(a_l, g_i(a_i,x,y)) - z(a_l, g_i(a_i,\overline{x},\overline{y})) \rangle \Big| \Big\}$$

$$\leq (1+\sigma_2)\gamma_a[GQ + \sigma_0 Q_2]|x - \overline{x}| + (1+\sigma_2)(1+p)[GQ + \sigma_0 Q_2]\|y - \overline{y}\|.$$

In estimating α_2 we have used integration by parts. From the above inequalities we obtain

$$\|Z(x,y) - Z(\overline{x},\overline{y})\| \leq W_{1a}|x - \overline{x}| + W_{2a}\|y - \overline{y}\|,$$

where

$$W_{1a} = l'(Q)(1+Q_1)[\Gamma + (\sigma_1 + \sigma_0)]$$
$$+ (1+\sigma_2)[\gamma_a(\Lambda + GQ) + l(Q)Q(2 + n_1(Q) + 2Q_1 + Q_2 n_1(Q))]$$
$$+ \gamma_a(1+\sigma_2)(\sigma_1 + \sigma_0)Q_2 + W_{1a}',$$
$$W_{2a} = l'(Q)(1+Q_2)[\Gamma + (\sigma_1 + \sigma_0)]$$
$$+ (1+\sigma_2)[(1+p)(\Lambda + GQ) + l(Q)Q(1+Q_2)]$$
$$+ (1+p)(1+\sigma_2)(\sigma_1 + \sigma_0)Q_2 + W_{2a}',$$

and $W_{1a}' \geq 0$, $W_{2a}' \geq 0$ are some constants such that $\lim_{a\to 0^+} W_{1a}' = \lim_{a\to 0^+} W_{2a}' = 0$. By (9), (10) we can choose a so small that

$$(13) \qquad W_{1a} \leq Q_1, \quad W_{2a} \leq Q_2.$$

Hence, for any $(x, y), (\overline{x}, \overline{y}) \in I_a$ we get

$$(14) \qquad \|Z(x, y) - Z(\overline{x}, \overline{y})\| \leq Q_1|x - \overline{x}| + Q_2\|y - \overline{y}\|.$$

Finally, if $a \in (0, a_0]$ is so small that inequalities (4), (11), (13) hold, then by (12), (14) we have $Z \in B_1(a, Q, Q_1, Q_2)$. This completes the proof.

LEMMA 2. *Suppose that Assumption* H_2 *holds*, $Q, Q_1, Q_2 \in \mathbb{R}_+$, $p \in (0, 1)$ *and* $a \in (0, a_0]$ *is so small that*

$$(15) \qquad L_{1a}(1 + p)(1 + Q) \leq p.$$

Then the operator $V^{(2)}$ *maps* $B_1(a, Q, Q_1, Q_2) \times B_2(a, Q, Q_1, Q_2, p)$ *into* $B_2(a, Q, Q_1, Q_2)$.

P r o o f. Let $H = V^{(2)}[z, h]$, $z \in B_1(a, Q, Q_1, Q_2)$, $h \in B_2(a, Q, Q_1, Q_2, p)$. It is easy to see that for any $(\xi, x, y), (\overline{\xi}, x, y), (\xi, x, \overline{y}) \in \Delta_a$, $i = 1, \ldots, m$, we have

$$H_i(x, x, y) = 0, \quad \|H_i(\xi, x, y) - H_i(\overline{\xi}, x, y)\| \leq n_1(Q)|\xi - \overline{\xi}|,$$

$$\|H_i(\xi, x, y) - H_i(\xi, x, \overline{y})\|$$

$$\leq \left| \int_x^\xi \|\varrho_i(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}) - \varrho_i(t, g_i(t, x, \overline{y}), z_{(t, g_i(t, x, \overline{y}))})\| \, dt \right|$$

$$\leq \left| \int_x^\xi l_1(t, P + Q)(1 + Q)\|g_i(t, x, y) - g_i(t, x, \overline{y})\| \, dt \right|$$

$$\leq L_{1a}(1 + Q)(1 + p)\|y - \overline{y}\| \leq p\|y - \overline{y}\|$$

by (15). Note that (15) implies (4), and thus for any $(\xi, x, y), (\xi, \overline{x}, y) \in \Delta_a$, $i = 1, \ldots, m$, we have

$$\|H_i(\xi, x, y) - H_i(\xi, \overline{x}, y)\| \leq \left| \int_x^{\overline{x}} \|\varrho_i(t, g_i(t, \overline{x}, y), z_{(t, g_i(t, \overline{x}, y))})\| \, dt \right|$$

$$+ \left| \int_x^\xi \|\varrho_i(t, g_i(t, x, y), z_{(t, g_i(t, x, y))}) - \varrho_i(t, g_i(t, \overline{x}, y), z_{(t, g_i(t, \overline{x}, y))})\| \, dt \right|$$

$$\leq [n_1(Q) + L_{1a}(1 + Q)\gamma_a]|x - \overline{x}| = \gamma_a|x - \overline{x}|.$$

From the above inequalities we derive $H \in B_2(a, Q, Q_1, Q_2, p)$. This is our claim.

For $a \in (0, a_0]$ define constants

$$E_{1a} = l'(Q)[\Gamma + n_2(Q)a + l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))Qa + \sigma_1 Q$$

$$+ \sigma_0 Q] + (1 + \sigma_2)[L_{2a} + l(Q)(Q_1 + Q_2 n_1(Q))a$$

$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))a + l(Q)Q + \sigma_1 + \sigma_0],$$

$$E_{2a} = (1 + \sigma_2)[\Lambda + L_{2a}(1 + Q_2) + l(Q)(1 + Q_2)(Q_1 + Q_2 n_1(Q))a$$
$$+ l(Q)Q_1(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))a + \sigma_1 Q_2 + GQ + \sigma_0 Q_2].$$

LEMMA 3. *Let Assumptions* H_1–H_4 *hold*, $Q, Q_1, Q_2 \in \mathbb{R}_+$, $p \in (0, 1)$ *and* $a \in (0, a_0]$ *be so small that inequality* (4) *holds. Then for any* $z, z' \in B_1(a, Q, Q_1, Q_2)$, $h, h' \in B_2(a, Q, Q_1, Q_2, p)$, *we have*

(16) $\|V^{(1)}[z, h] - V^{(1)}[z', h']\|_{I_a} \le E_{1a}\|z - z'\|_{I_a} + E_{2a}\|h - h'\|_{\Delta_a}$,

(17) $\|V^{(2)}[z, h] - V^{(2)}[z', h']\|_{I_a} \le L_{1a}\|z - z'\|_{I_a} + L_{1a}(1 + Q)\|h - h'\|_{\Delta_a}$.

P r o o f. Let $z, z' \in B_1(a, Q, Q_1, Q_2)$ and $h, h' \in B_2(a, Q, Q_1, Q_2, p)$. For any $(x, y) \in I_a$ we have

$$V^{(1)}[z, h](x, y) - V^{(1)}[z', h'](x, y) = \beta_0 + \beta_1 + \beta_2 + \beta_3,$$

where

$$\beta_0 = (A^{-1}(x, y, z_{(x,y)}) - A^{-1}(x, y, z'_{(x,y)}))[\Delta_1(x, y) + \Delta_2(x, y) + \Delta_3(x, y)],$$
$$\beta_i = A^{-1}(x, y, z'_{(x,y)})[\Delta_i(x, y) - \Delta'_i(x, y)], \quad i = 1, 2, 3,$$

and formulas for $\Delta'_1, \Delta'_2, \Delta'_3$ arise from (6) by replacing z, h by z', h', respectively. Analogously to Lemma 1 we have

$$\|\beta_0\| \le l'(Q)[\Gamma + n_2(Q)a + l(Q)(1 + n_1(Q)$$
$$+ Q_1 + Q_2 n_1(Q))Qa + \sigma_1 Q + \sigma_0 Q]\|z - z'\|_{I_a},$$
$$\|\beta_1\| \le (1 + \sigma_2)L_{2a}\|z - z'\|_{I_a} + (1 + \sigma_2)[\Lambda + L_{2a}(1 + Q_2)]\|h - h'\|_{\Delta_a},$$
$$\|\beta_2\| \le (1 + \sigma_2)[l(Q)(Q_1 + Q_2 n_1(Q))a$$
$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))a + l(Q)Q + \sigma_1]\|z - z'\|_{I_a}$$
$$+ (1 + \sigma_2)[l(Q)(1 + Q_2)(Q_1 + Q_2 n(Q))a$$
$$+ l(Q)(1 + n_1(Q) + Q_1 + Q_2 n_1(Q))a + \sigma_1 Q]\|h - h'\|_{\Delta_a},$$
$$\|\beta_3\| \le (1 + \sigma_2)\sigma_0\|z - z'\|_{I_a} + (1 + \sigma_2)[GQ + \sigma_0 Q_2]\|h - h'\|_{\Delta_a}.$$

From the above estimates we get (16). In a similar way we derive (17), which completes the proof.

4. The main theorem

THEOREM. *Suppose that Assumptions* H_1–H_4, *conditions* (8)–(10) *and the inequality*

(18) $l'(Q)[\Gamma + Q(\sigma_1 + \sigma_0)] + (1 + \sigma_2)l(Q)Q + (1 + \sigma_2)(\sigma_1 + \sigma_0) < \overline{k}$,

are satisfied, where $\overline{k} \in (0, 1)$ *is a constant such that* $\overline{k} > (\sigma_1 + \sigma_0)(1 + \sigma_2)$. *Then for* $a \in (0, a_0]$ *sufficiently small and for any system of numbers* $a_l \in [0, a]$, $l = 1, \ldots, N$, *there is a function* $z \in B_1(a, Q, Q_1, Q_2)$ *which is a*

unique solution of the problem (1), (2) *in the class* $B_1(a, Q, Q_1, Q_2)$. *Furthermore, if* z, z' *are solutions of* (1), (2) *with functions* ψ, ψ' *respectively then*

(19) $$\|z - z'\|_{I_a} \le (1 - \overline{k})^{-1}(1 + \sigma_2)\|\psi - \psi'\|_0 \,,$$

where $\|\psi - \psi'\|_0 = \sup\{\|\psi(y) - \psi'(y)\| : y \in \mathbb{R}^r\}$.

P r o o f. Let $V = (V^{(1)}, V^{(2)})$, where $V^{(1)}, V^{(2)}$ are defined by (6), (7). From Lemmas 1–3 it follows that

$$V : B_1(a, Q, Q_1, Q_2) \times B_2(a, Q, Q_1, Q_2, p)$$
$$\to B_1(a, Q, Q_1, Q_2) \times B_2(a, Q, Q_1, Q_2, p)$$

is continuous provided that a is so small that (11), (13) and (15) hold. By (18) we may additionally assume that a is small enough that

(20) $\quad E_{1a} < \overline{k}, \quad L_{1a}(1 + Q_2) < \overline{k}, \quad E_{2a}L_{1a} < (\overline{k} - E_{1a})(\overline{k} - L_a(1 + Q_2))$.

Let $\alpha, \beta > 0$ satisfy $L_{1a}(\overline{k} - E_{1a})^{-1} \le \alpha/\beta \le (\overline{k} - L_{1a}(1 + Q_2))E_{2a}^{-1}$. Analogously to [4] we define in $B_1(a, Q, Q_1, Q_2) \times B_2(a, Q, Q_1, Q_2, p)$ the following weighted norm:

(21) $$\|w\|_* = \alpha\|z\|_{I_a} + \beta\|h\|_{\Delta_a}, \quad w = (z, h)\,,$$

where $z \in B_1(a, Q, Q_1, Q_2)$, $h \in B_2(a, Q, Q_1, Q_2, p)$. It is easy to check (cf. [4]) that V is a contraction with constant $\overline{k}$ with respect to this norm. Thus V has a unique fixed point $\overline{w} = V\overline{w}$, $\overline{w} = (\overline{z}, \overline{h}) \in B_1(a, Q, Q_1, Q_2) \times B_2(a, Q, Q_1, Q_2, p)$. Let us prove that $\overline{z}$ is a solution of (1), (2). From (6) by integration by parts we obtain

(22) $\quad 0 = A^{-1}(x, y, \overline{z}_{(x,y)})[\Delta_1(x, y) + \widetilde{\Delta}_2(x, y) + \Delta_3(x, y)], \quad (x, y) \in I_a\,,$

where Δ_1, Δ_3 are defined by (6) with $z = \overline{z}$, $h = \overline{h}$ and $\widetilde{\Delta}_2(x, y) = (\widetilde{\Delta}_{21}(x, y), \ldots, \widetilde{\Delta}_{2m}(x, y))$ is defined by

$$\widetilde{\Delta}_{2i}(x, y) = -\int_{a_i}^{x} \langle A_i^*(t, x, y), D_t\overline{z}_i^*(t, x, y)\rangle \, dt, \quad i = 1, \ldots, m\,.$$

Multiplying (22) by $A(x, y, \overline{z}_{(x,y)})$ we obtain

$$\Delta_{1i}(x, y) + \widetilde{\Delta}_{2i}(x, y) + \Delta_{3i}(x, y) = 0, \quad (x, y) \in I_a, \ i = 1, \ldots, m\,,$$

which for $x = a_i$ yields that $\overline{z}$ satisfies the boundary condition (2). Thus (22) reduces to

$$\int_{a_i}^{x} [f_i(t, \overline{g}_i(t, x, y), \overline{z}_{(t, \overline{g}_i(t,x,y))}) - \langle A_i^*(t, x, y), D_t\overline{z}_i^*(t, x, y)\rangle] \, dt = 0\,,$$

$(x, y) \in I_a$, $i = 1, \ldots, m$, where $\overline{g}$ is defined by (5) for $h = \overline{h}$. By the same considerations as in [6] (in particular by using the group property for $\overline{g}$) we

see that $\overline{z}$ satisfies (1) a.e. in I_a. By the reverse considerations and by the uniqueness of the fixed point of V we conclude that $\overline{z}$ is a unique solution of (1), (2).

It remains to prove (19). Let $w = (z, h) = Vw$, $w' = (z', h') = Vw'$ be the fixed points of $V = (V^{(1)}, V^{(2)})$ defined by (6), (7) with functions ψ, ψ' respectively. We then have

$$\|z - z'\|_{I_a} \leq (1 + \sigma_2)\|\psi - \psi'\|_0 + E_{1a}\|z - z'\|_{I_a} + E_{2a}\|h - h'\|_{\Delta_a},$$
$$\|h - h'\|_{\Delta_a} \leq L_{1a}\|z - z'\|_{I_a} + L_{1a}(1 + Q_2)\|h - h'\|_{\Delta_a}.$$

From this we obtain $(1 - E_{1a} - E_{2a}L_{1a}(1 - L_{1a}(1+Q_2))^{-1})\|z - z'\|_{I_a} \leq (1 + \sigma_2)\|\psi - \psi'\|_0$. By (20) we have $1 - \overline{k} \leq 1 - E_{1a} - E_{2a}L_{1a}(1 - L_{1a}(1+Q_2))^{-1}$, and from the last inequality we derive (19). This ends the proof.

5. Examples. Put $\widehat{\Omega} = [0, a_0] \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$.

ASSUMPTION H_5. 1° $\widehat{A} \in C(\widehat{\Omega}, M(m, m))$ and there is a constant $\widehat{\nu} > 0$ such that $\det \widehat{A}(x, y, u, v) \geq \widehat{\nu}$ for any $(x, y, u, v) \in \widehat{\Omega}$.

2° There are constants $n, l \in \mathbb{R}_+$ such that for all $(x, y, u, v), (\overline{x}, \overline{y}, \overline{u}, \overline{v}) \in \widehat{\Omega}$ we have

$$\|\widehat{A}(x, y, u, v)\| \leq n,$$
$$\|\widehat{A}(x, y, u, v) - \widehat{A}(\overline{x}, \overline{y}, \overline{u}, \overline{v})\| \leq l[|x - \overline{x}| + \|y - \overline{y}\| + \|u - \overline{u}\| + \|v - \overline{v}\|].$$

3° $\widehat{\varrho}(\cdot, y, u, v) : [0, a_0] \to M(m, r), \widehat{f}(\cdot, y, u, v) : [0, a_0] \to M(m, 1)$ are measurable for any $(y, u, v) \in \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ and $\widehat{\varrho}(x, \cdot) : \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m \to M(m, r), \widehat{f}(x, \cdot) : \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m \to M(m, 1)$ are continuous for a.e. $x \in [0, a_0]$.

4° There are Lebesgue integrable functions $\widehat{l}_1, \widehat{l}_2 : [0, a_0] \to \mathbb{R}_+$ and constants $n_1, n_2 \in \mathbb{R}_+$ such that for a.e. $x \in [0, a_0]$ and for all $(y, u, v), (\overline{y}, \overline{u}, \overline{v}) \in \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ we have

$$\|\widehat{\varrho}(x, y, u, v)\| \leq n_1, \quad \|\widehat{f}(x, y, u, v)\| \leq n_2,$$
$$\|\widehat{\varrho}(x, y, u, v) - \widehat{\varrho}(x, \overline{y}, \overline{u}, \overline{v})\| \leq l_1(x)[\|y - \overline{y}\| + \|u - \overline{u}\| + \|v - \overline{v}\|],$$
$$\|\widehat{f}(x, y, u, v) - \widehat{f}(x, \overline{y}, \overline{u}, \overline{v})\| \leq l_2(x)[\|y - \overline{y}\| + \|u - \overline{u}\| + \|v - \overline{v}\|].$$

Let $\alpha, \beta, \gamma : [0, a_0] \times \mathbb{R}^r \to \mathbb{R}^{1+r}$, where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_r)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_r)$, $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_r)$, be given functions such that $-b_0 \leq \alpha_0(x, y), \beta_0(x, y), \gamma_0(x, y) \leq x$ for all $(x, y) \in [0, a_0] \times \mathbb{R}^r$. Suppose that α is Lipschitz continuous on $[0, a_0] \times \mathbb{R}^r$ and that β, γ are Lipschitz continuous with respect to the second variable on $\mathbb{R}^r$. Then, if we define

$$A(x, y, w) = \widehat{A}(x, y, w(0, 0), w(\alpha(x, y) - (x, y))),$$
$$\varrho(x, y, w) = \widehat{\varrho}(x, y, w(0, 0), w(\beta(x, y) - (x, y))),$$
$$f(x, y, w) = \widehat{f}(x, y, w(0, 0), w(\gamma(x, y) - (x, y))),$$

and if Assumption H_5 holds then the functions A, ϱ, f satisfy Assumptions H_1–H_3 respectively. Thus the hyperbolic system with retarded argument

$$(23) \quad \sum_{j=1}^{m} \widehat{A}_{ij}(x, y, z(x, y), z(\alpha(x, y)))\Big[D_x z_j(x, y)$$

$$+ \sum_{k=1}^{r} \widehat{\varrho}(x, y, z(x, y), z(\beta(x, y))) D_{y_k} z_j(x, y)\Big]$$

$$= \widehat{f}(x, y, z(x, y), z(\gamma(x, y))), \quad i = 1, \dots, m,$$

is a particular case of (1).

The differential-integral system

$$(24) \quad \sum_{j=1}^{m} \widehat{A}_{ij}\Big(x, y, z(x, y), \int_{\alpha(x,y)}^{\tilde{\alpha}(x,y)} z(t, s) K(t, s, x, y)\, dt\, ds\Big)\Big[D_x z_j(x, y)$$

$$+ \sum_{k=1}^{r} \widehat{\varrho}_{ik}\Big(x, y, z(x, y), \int_{\beta(x,y)}^{\tilde{\beta}(x,y)} z(t, s) K_1(t, s, x, y)\, dt\, ds\Big) D_{y_k} z_j(x, y)\Big]$$

$$= \widehat{f}_i\Big(x, y, z(x, y), \int_{\gamma(x,y)}^{\tilde{\gamma}(x,y)} z(t, s) K_2(t, s, x, y)\, dt\, ds\Big), \quad i = 1, \dots, m,$$

where $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, \gamma, \tilde{\gamma} : [0, a_0] \times \mathbb{R}^r \to \mathbb{R}^{1+r}$ and $K, K_1, K_2 : [0, a_0] \times \mathbb{R}^r \times [0, a_0] \times \mathbb{R}^r \to M(m, m)$, is also a particular case of (1). In this case if we make suitable assumptions on $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, \gamma, \tilde{\gamma}, K, K_1, K_2$ and if Assumption H_5 holds then $A(x, y, w)$ defined by

$$A(x, y, w) = \widehat{A}\Big(x, y, w(0, 0), \int_{\alpha(x,y)}^{\tilde{\alpha}(x,y)} w(t - x, s - y) K(t, s, x, y)\, dt\, ds\Big)$$

satisfies Assumption H_1, and ϱ, f similarly defined satisfy Assumptions H_2, H_3 respectively.

Our last example is the system

$$(25) \quad \sum_{j=1}^{m} \widehat{A}_{ij}(x, y, z(x, y), z(\alpha(x, y, z_{(x,y)})))\Big[D_x z_j(x, y)$$

$$+ \sum_{k=1}^{r} \widehat{\varrho}_{ik}(x, y, z(x, y), z(\beta(x, y, z_{(x,y)}))) D_{y_k} z_j(x, y)\Big]$$

$$= \widehat{f}_i(x, y, z(x, y), z(\gamma(x, y, z_{(x,y)}))),$$

$i = 1, \dots, m, \alpha, \beta, \gamma : \Omega \to \mathbb{R}^{1+r}$.

Lemma 4. *Suppose that*

1° *Assumption* H_1 *holds*;

2° $\alpha(\cdot, y, w), \beta(\cdot, y, w), \gamma(\cdot, y, w) : [0, a_0] \to \mathbb{R}^{1+r}$ *are measurable for any* $(y, w) \in \mathbb{R}^r \times C(B, \mathbb{R}^m)$, *and* $\alpha(x, \cdot), \beta(x, \cdot), \gamma(x, \cdot) : \mathbb{R}^r \times C(B, \mathbb{R}^m) \to \mathbb{R}^{1+r}$ *are continuous for a.e.* $x \in [0, a_0]$;

3° *for any* $(x, y, w) \in \Omega$ *we have*

$$\alpha(x, y, w) \in B, \quad \beta(x, y, w) \in B, \quad \gamma(x, y, w) \in B\,;$$

4° *there is a nondecreasing function* $D : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for all* $q \in \mathbb{R}_+$, $(x, y, w), (\overline{x}, \overline{y}, \overline{w}) \in [0, a_0] \times \mathbb{R}^r \times C_{0+\mathrm{L}}(B, \mathbb{R}^m, q)$ *we have*

$$\|\alpha(x, y, w) - \alpha(\overline{x}, \overline{y}, \overline{w})\| \leq D(q)[|x - \overline{x}| + \|y - \overline{y}\| + \|w - \overline{w}\|_0]\,;$$

5° *there are nondecreasing functions* $D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for all* $q \in \mathbb{R}_+$ *and* $(y, w), (\overline{y}, \overline{w}) \in \mathbb{R}^r \times C_{0+\mathrm{L}}(B, \mathbb{R}^m, q)$ *and for a.e.* $x \in [0, a_0]$ *we have*

$$\|\beta(x, y, w) - \beta(x, \overline{y}, \overline{w})\| \leq D_1(q)[\|y - \overline{y}\| + \|w - \overline{w}\|_0]\,,$$
$$\|\gamma(x, y, w) - \gamma(x, \overline{y}, \overline{w})\| \leq D_2(q)[\|y - \overline{y}\| + \|w - \overline{w}\|_0]\,.$$

Then the functions A, ϱ, f *defined by*

$$A(x, y, w) = \widehat{A}(x, y, w(0, 0), w(\alpha(x, y, w) - (x, y)))\,,$$
$$\varrho(x, y, w) = \widehat{\varrho}(x, y, w(0, 0), w(\beta(x, y, w) - (x, y)))\,,$$
$$f(x, y, w) = \widehat{f}(x, y, w(0, 0), w(\gamma(x, y, w) - (x, y)))\,,$$

satisfy Assumptions H_1, H_2, H_3 *respectively.*

R e m a r k 3. Systems (23), (24) can also be obtained from the theory of differential-functional equations with operators of Volterra type [19]–[22]. System (25) cannot be obtained from that theory.

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