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A saturation theorem for combinations of Bernstein–Durrmeyer polynomials

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Abstract. We prove a local saturation theorem in ordinary approximation for combinations of Durrmeyer's integral modification of Bernstein polynomials.

Introduction. The Bernstein–Durrmeyer polynomial of order n is defined by

$$M_n(f,x) = \int_0^1 W(n,x,t)f(t)\,dt\,, \qquad f \in L_1[0,1]\,,$$

where

$$W(n,x,t) = (n+1)\sum_{\nu=0}^{n} p_{n\nu}(x)p_{n\nu}(t)\,,$$

$p_{n\nu}(x)$ being $\binom{n}{\nu}x^\nu(1-x)^{n-\nu}$, $x \in [0,1]$. These operators were introduced by Durrmeyer [5] by replacing $f(\nu/n)$ in $B_n(f,x)$, the Bernstein polynomials, by $(n+1)\int_0^1 p_{n\nu}(t)f(t)\,dt$. Several authors (see [1]–[4], [6], [8], [9]) have studied the operators M_n and obtained direct and inverse results both in sup-norm and L_p-norm. In this paper we study the saturation behaviour of the linear combination $M_n(f,k,x)$ [7]. It turns out that even though Bernstein–Durrmeyer polynomials are not exponential type operators [7] yet their saturation behaviour is similar to that of the operators of exponential type.

The linear combination $M_n(f,k,x)$ of $M_{d_jn}(f,x)$, $j = 0,1,\ldots,k$, is defined by

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)M_{d_jn}(f,x)\,,$$

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where

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}\quad\text{for } k\neq 0\,,\qquad C(0,0)=1\,,$$

and $d_0,d_1,\ldots,d_k$ are $k+1$ arbitrary but fixed distinct positive integers.

Remark. The definition of $C(j,k)$ can also be related to the formula

$$x^k=\sum_{j=0}^{k}C(j,k)\prod_{\substack{i=0\\ i\neq j}}^{k}(x-d_i)\,,$$

which is a simple rephrasing of the interpolation formula

$$x^k=\sum_{j=0}^{k}C(j,k)l_j(x)\,,$$

where $l_0,l_1,\ldots,l_k$ are the Lagrange fundamental polynomials corresponding to the knots $d_0,d_1,\ldots,d_k$. Therefore, by simple computation, we get the relation

$$\sum_{j=0}^{k}C(j,k)=1\,.$$

Throughout this paper, let C_0 denote the set of continuous functions on $[0,1]$ having a compact support and C_0^k the subset of C_0 of k times continuously differentiable functions. The spaces $A.C.[a,b]$ and $L_B[0,1]$ are defined as the class of absolutely continuous functions on $[a,b]$ for every a, b satisfying $0<a<b<1$ and the class of bounded and integrable functions on $[0,1]$ respectively. $\langle a,b\rangle\subset[0,1]$ stands for an open interval containing the closed interval $[a,b]$.

2. Preliminary results. In the following result we obtain an estimate of the degree of approximation by $M_n(\cdot,k,x)$ for smooth functions.

THEOREM 2.1. *Let $0\le p\le 2k+2$, $f\in L_B[0,1]$, and suppose $f^{(p)}$ exists and is continuous on $\langle a,b\rangle\subset[0,1]$. Then for all n sufficiently large*,

$$\|M_n(f,k,\cdot)-f(\cdot)\|_{C[a,b]}\le\max\{C_1n^{-p/2}\omega(f^{(p)},n^{-1/2}),C_2n^{-(k+1)}\}$$

where $C_1=C_1(k,p)$, $C_2=C_2(k,p,f)$ and $\omega(f^{(p)},\delta)$ denotes the modulus of continuity of $f^{(p)}$ on $\langle a,b\rangle$.

Proof. For $x\in[a,b]$, writing

$$F(t,x)=f(t)-\sum_{j=0}^{p}\frac{f^{(j)}(x)}{j!}(t-x)^j\,,$$

we have

$$|F(t,x)| \le \frac{|t-x|^p}{p!}\left(1+\frac{|t-x|}{n^{-1/2}}\right)\omega(f^{(p)}, n^{-1/2})$$

for $t \in \langle a,b\rangle$. Thus if $\chi(t)$ denotes the characteristic function of $\langle a,b\rangle$, by [3, Prop. II.3] and the Schwarz inequality,

$$M_n(|F(t,x)|\chi(t),x) \le C_3 n^{-p/2}\omega(f^{(p)}, n^{-1/2}),$$

where $C_3 = C_3(p)$. Similarly, for some constant C_4 and all n sufficiently large, we have

$$M_n(|F(t,x)|(1-\chi(t)),x) \le C_4[M_n((t-x)^{2(2k+2)},x)]^{1/2} \le C_5 n^{-(k+1)}.$$

Hence,

$$|M_n(|F(t,x)|,x)| \le C_3 n^{-p/2}\omega(f^{(p)}, n^{-1/2}) + C_5 n^{-(k+1)}.$$

But by [6, Prop.]

$$M_n\bigg(\sum_{j=1}^{p}\frac{f^{(j)}(x)}{j!}(t-x)^j, k, x\bigg) = O(n^{-(k+1)})$$

uniformly in $x \in [a,b]$. Hence for all n sufficiently large

$$\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a,b]} \le C_6 n^{-p/2}\omega(f^{(p)}, n^{-1/2}) + C_7 n^{-(k+1)},$$

where C_6 does not depend on f, from which the required result is immediate.

In the following lemma, the inner product $\langle h(\cdot), g(\cdot)\rangle$ is defined as $\int_0^1 h(x)g(x)\,dx$.

LEMMA 2.2. *Let $0 < a < b < 1$. If $f \in C[0,1]$ and $g \in C_0^\infty$ with* $\operatorname{supp} g \subset (a,b)$, *then*

$$|n^{k+1}\langle M_{2n}(f,k,\cdot) - M_n(f,k,\cdot), g(\cdot)\rangle| \le K\|f\|_{C[0,1]},$$

where K is a constant independent of f and n.

Proof. We write

$$M_{2n}(f,k,x) - M_n(f,k,x) = \sum_{j=1}^{2k+2}\alpha(j,k)M_{e_j n}(f,x),$$

where $e_j \in \{d_0, d_1, \ldots, d_k, 2d_0, 2d_1, \ldots, 2d_k\}$. By [7, Lemma 3.5] it follows that

$$\sum_{j=1}^{2k+2}\alpha(j,k)e_j^{-m} = 0, \quad m = 0,1,\ldots,k. \tag{2.1}$$

Next, by using [3, Prop. II.3], we have

$$\begin{aligned}
&n^{k+1}\langle M_{2n}(f,k,\cdot)-M_n(f,k,\cdot),g(\cdot)\rangle\\
&\qquad= n^{k+1}\int_0^1\int_0^1\Big\{\sum_{j=1}^{2k+2}\alpha(j,k)W(e_jn,x,t)f(t)g(x)\Big\}\,dt\,dx\\
&\qquad= n^{k+1}\int_{\operatorname{supp} g}\int_0^1\{\ldots\}\,dt\,dx\\
&\qquad= n^{k+1}\int_{\operatorname{supp} g}\int_a^b\{\ldots\}\,dt\,dx+o(1)\|f\|_{C[0,1]}\\
&\qquad= n^{k+1}\int_0^1\int_a^b\{\ldots\}\,dt\,dx+o(1)\|f\|_{C[0,1]}\,.
\end{aligned}$$

Now, using Fubini's theorem and expanding $g(x)$ by Taylor's theorem, we get

$$\begin{aligned}
(2.2)\quad &n^{k+1}\langle M_{2n}(f,k,\cdot)-M_n(f,k,\cdot),g(\cdot)\rangle\\
&= n^{k+1}\int_a^b\int_0^1\sum_{i=0}^{2k+2}\sum_{j=1}^{2k+2}\frac{\alpha(j,k)}{i!}W(e_jn,x,t)f(t)g^{(i)}(t)(x-t)^i\,dx\,dt\\
&\quad+n^{k+1}\int_a^b\int_0^1\sum_{j=1}^{2k+2}\alpha(j,k)W(e_jn,x,t)f(t)\varepsilon(x,t)(x-t)^{2k+2}\,dx\,dt\\
&\quad+o(1)\|f\|_{C[0,1]}\\
&=\sum_{i=0}^{2k+2}n^{k+1}\int_a^b\int_0^1\sum_{j=1}^{2k+2}\frac{\alpha(j,k)}{i!}W(e_jn,x,t)f(t)g^{(i)}(t)(x-t)^i\,dx\,dt\\
&\quad+n^{k+1}\int_0^1\int_a^b\sum_{j=1}^{2k+2}\alpha(j,k)W(e_jn,x,t)f(t)\varepsilon(x,t)(x-t)^{2k+2}\,dt\,dx\\
&\quad+o(1)\|f\|_{C[0,1]}\\
&=J_1+J_2+o(1)\|f\|_{C[0,1]}\,,\quad\text{say},
\end{aligned}$$

where $\varepsilon(x,t)(x-t)^{2k+2}$ is the remainder term corresponding to the partial Taylor expansion of g.

Since, for ξ lying between t and x,

$$|\varepsilon(x,t)|=\frac{|g^{(2k+2)}(\xi)-g^{(2k+2)}(x)|}{(2k+2)!}\le\frac{2}{(2k+2)!}\|g^{(2k+2)}\|_{C[a,b]}<\infty\,,$$

using [3, Prop. II.3], it follows that $J_2 = O(1)\|f\|_{C[0,1]}$.

To estimate J_1, we proceed as follows: J_1 may be rewritten as

$$J_1 = n^{k+1}\sum_{i=0}^{2k+2}\frac{1}{i!}\sum_{j=1}^{2k+2}\alpha(j,k)\int\limits_a^b\Big(\int\limits_0^1 W(e_jn,x,t)(x-t)^i\,dx\Big)f(t)g^{(i)}(t)\,dt\,.$$

Now, we note that $W(n,x,t) = W(n,t,x)$, therefore using [3, Prop. II.3], after interchanging the variables t and x, together with equation (2.1) it follows that $J_1 = O(1)\|f\|_{C[0,1]}$. Combining the estimates for J_1, J_2 and (2.2), we obtain the required result.

THEOREM 2.3 [2]. *Let* $f \in C[0,1]$, $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$ *and* $0 < \alpha < 2$. *Then*, *in the following*, *the implications* (i)⇒(ii)⇔(iii)⇒(iv) *hold*:

(i) $\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_1,b_1]} = O(n^{-\alpha(k+1)/2})$.

(ii) $f \in \mathrm{Liz}(\alpha, k+1, a_2, b_2)$.

(iii) (a) *For* $m < \alpha(k+1) < m+1$, $m = 0,1\ldots,2k+1$, $f^{(m)}$ *exists and is is in* $\mathrm{Lip}(\alpha(k+1)-m, a_2, b_2)$.

(b) *For* $\alpha(k+1) = m+1$, $m = 0,1,\ldots,2k$, $f^{(m)}$ *exists and is in* $\mathrm{Lip}^*(1, a_2, b_2)$.

(iv) $\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_3,b_3]} = O(n^{-\alpha(k+1)/2})$.

Here $\mathrm{Liz}(\alpha,k,a,b)$ denotes the class of functions for which $\omega_{2k}(f,h,a,b) \le Mh^{\alpha k}$; when $k=1$, $\mathrm{Liz}(\alpha,1)$ reduces to the Zygmund class $\mathrm{Lip}^*\,\alpha$.

3. The saturation result

THEOREM 3.1. *Let* $f \in C[0,1]$ *and* $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$. *Then*, *in the following statements*, *the implications* (i)⇒(ii)⇒(iii) *and* (iv)⇒(v)⇒(vi) *hold true*:

(i) $n^{k+1}\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_1,b_1]} = O(1)$;

(ii) $f^{(2k+1)} \in A.C.[a_2,b_2]$ *and* $f^{(2k+2)} \in L_\infty[a_2,b_2]$;

(iii) $n^{k+1}\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_3,b_3]} = O(1)$;

(iv) $n^{k+1}\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_1,b_1]} = o(1)$;

(v) $f \in C^{2k+2}[a_2,b_2]$ *and* $\displaystyle\sum_{j=1}^{2k+2}\frac{Q(j,k,x)}{j!}f^{(j)}(x) = 0, \quad x \in [a_2,b_2]$,

where $Q(j,k,x)$ *are the polynomials occurring in* [6, *Th.* 2];

(vi) $n^{k+1}\|M_n(f,k,\cdot) - f(\cdot)\|_{C[a_3,b_3]} = o(1)$,

where all $O(1)$ *and* $o(1)$ *terms are with respect to* n, *as* $n \to \infty$.

Proof. First assume (i); then in view of (i)⇒(iii) of Theorem 2.3, it follows that $f^{(2k+1)}$ exists and is continuous on (a_1,b_1). Moreover, the statement

$$\|M_n(f,k,\cdot)-f(\cdot)\|_{C[a_1,b_1]}=O(n^{-(k+1)}) \tag{3.1}$$

is equivalent to

$$\|M_{2n}(f,k,\cdot)-M_n(f,k,\cdot)\|_{C[a_1,b_1]}=O(n^{-(k+1)})\,. \tag{3.2}$$

Indeed, trivially (3.1)⇒(3.2). Also, assuming (3.2), since $\lim_{n\to\infty} M_n(f,k,x) = f(x)$, we can write

$$\begin{aligned} f(x)=M_n(f,k,x)&+[M_{2n}(f,k,x)-M_n(f,k,x)]\\ &+[M_{4n}(f,k,x)-M_{2n}(f,k,x)]+\ldots\\ &+[M_{2^r n}(f,k,x)-M_{2^{r-1}n}(f,k,x)]+\ldots \end{aligned}$$

Hence,

$$\begin{aligned} \|f(\cdot)&-M_n(f,k,\cdot)\|_{C[a_1,b_1]}\\ &\le \|M_{2n}(f,k,\cdot)-M_n(f,k,\cdot)\|_{C[a_1,b_1]}\\ &\quad+\|M_{4n}(f,k,\cdot)-M_{2n}(f,k,\cdot)\|_{C[a_1,b_1]}\\ &\quad+\ldots+\|M_{2^r n}(f,k,\cdot)-M_{2^{r-1}n}(f,k,\cdot)\|_{C[a_1,b_1]}+\ldots\\ &=K_1\left(\frac{1}{n^{k+1}}+\frac{1}{2^{k+1}n^{k+1}}+\ldots+\frac{1}{(2^{r-1})^{k+1}n^{k+1}}+\ldots\right)\\ &=\frac{K_1}{n^{k+1}}\frac{1}{1-2^{-(k+1)}}=\frac{K_2}{n^{k+1}}\,, \end{aligned}$$

where $K_2=K_1/(1-2^{-(k+1)})$, showing that (3.1) holds.

Thus, we may assume that $\{n^{k+1}(M_{2n}(f,k,\cdot)-M_n(f,k,\cdot))\}$ is bounded as a sequence in $C[a_1,b_1]$ and hence in $L_\infty[a_1,b_1]$. Since $L_\infty[a_1,b_1]$ is the dual space of $L_1[a_1,b_1]$, it follows by Alaoglu's theorem that there exists $h\in L_\infty[a_1,b_1]$ such that for some subsequence $\{n_i\}_{i=1}^\infty$ of natural numbers and for every $g\in C_0^\infty$ with $\operatorname{supp} g\subset(a_1,b_1)$

$$\lim_{n_i\to\infty} n_i^{k+1}\langle M_{2n_i}(f,k,\cdot)-M_{n_i}(f,k,\cdot),g(\cdot)\rangle=\langle h(\cdot),g(\cdot)\rangle\,. \tag{3.3}$$

Now, since $C^{2k+2}[a_1,b_1]\cap C[0,1]$ is dense in $C[0,1]$ there exists a sequence $\{f_\sigma\}_{\sigma=1}^\infty$ in $C^{2k+2}[a_1,b_1]\cap C[0,1]$ converging to f in $\|\cdot\|_{C[0,1]}$-norm. Then, for any $g\in C_0^\infty$ with $\operatorname{supp} g\subset(a_1,b_1)$ and each function f_σ, by [6, Th. 2] we have

$$\text{(3.4)}\quad \lim_{n_i\to\infty} n_i^{k+1}\langle M_{2n_i}(f_\sigma,k,\cdot)-M_{n_i}(f_\sigma,k,\cdot),g(\cdot)\rangle$$
$$=\left\langle -(1-2^{-(k+1)})\sum_{j=1}^{2k+2}\frac{Q(j,k,\cdot)}{j!}f_\sigma^{(j)}(\cdot),g(\cdot)\right\rangle$$
$$=\langle P_{2k+2}(D)f_\sigma(\cdot),g(\cdot)\rangle=\langle f_\sigma(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle\,,$$

where $P^*_{2k+2}(D)$ denotes the operator adjoint to $P_{2k+2}(D)$ (in this case, it is simply a result of integration by parts). By Lemma 2.2, we conclude that

$$\text{(3.5)}\quad \lim_{n_i\to\infty} n_i^{k+1}|\langle M_{2n_i}(f-f_\sigma,k,\cdot)-M_{n_i}(f-f_\sigma,k,\cdot),g(\cdot)\rangle| \le K\|f-f_\sigma\|_{C[0,1]}\,.$$

Hence, by (3.5), (3.4) and (3.3) (in that order)

$$\langle f(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle=\lim_{\sigma\to\infty}\langle f_\sigma(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle$$
$$=\lim_{\sigma\to\infty}\{\lim_{n_i\to\infty} n_i^{k+1}\langle M_{2n_i}(f-f_\sigma,k,\cdot)-M_{n_i}(f-f_\sigma,k,\cdot),g(\cdot)\rangle + \langle f_\sigma(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle\}$$
$$=\lim_{n_i\to\infty} n_i^{k+1}\langle M_{2n_i}(f,k,\cdot)-M_{n_i}(f,k,\cdot),g(\cdot)\rangle=\langle h(\cdot),g(\cdot)\rangle\,,$$

for all $g\in C_0^\infty$ with $\operatorname{supp} g\subset(a_1,b_1)$. Thus

$$\text{(3.6)}\qquad P_{2k+2}(D)f(x)=h(x)$$

as generalized functions.

Note that $Q(2k+2,k,x)\ne 0$ by [6, Prop.]. Therefore, regarding (3.6) as a first order linear differential equation for $f^{(2k+1)}$, we deduce that $f^{(2k+1)}\in A.C.[a_2,b_2]$ and hence $f^{(2k+2)}\in L_\infty[a_2,b_2]$. This completes the proof of the implication (i)⇒(ii).

Now assuming (ii), it follows that $f^{(2k+1)}\in \mathrm{Lip}_M(1,a_2,b_2)$ with $M=\|f^{(2k+2)}\|_{L_\infty[a_2,b_2]}$. Hence (iii) follows by Theorem 2.1.

To prove (iv)⇒(v), assuming (iv) and proceeding in the manner of the proof of (i)⇒(ii), we get $P_{2k+2}(D)f(x)=0$, from which in view of the non-vanishing of $Q(2k+2,k,x)$, (v) is clear.

The proof of (v)⇒(vi) follows from [6, Th. 2]. This completes the proof of the theorem.

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References

[1] P. N. Agrawal and V. Gupta, *Simultaneous approximation by linear combination of the modified Bernstein polynomials*, Bull. Soc. Math. Grèce 30 (1989), 21–29 (1990).
[2] —, —, *Inverse theorem for linear combinations of modified Bernstein polynomials*, preprint.
[3] M. M. Derriennic, *Sur l'approximation de fonctions intégrables sur* $[0,1]$ *par des polynômes de Bernstein modifiés*, J. Approx. Theory 31 (1981), 325–343.
[4] Z. Ditzian and K. Ivanov, *Bernstein-type operators and their derivatives*, ibid. 56 (1989), 72–90.
[5] J. L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: Application à la théorie des moments*, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
[6] H. S. Kasana and P. N. Agrawal, *On sharp estimates and linear combinations of modified Bernstein polynomials*, Bull. Soc. Math. Belg. Sér. B 40 (1) (1988), 61–71.
[7] C. P. May, *Saturation and inverse theorems for combinations of a class of exponential type operators*, Canad. J. Math. 28 (1976), 1224–1250.
[8] B. Wood, L_p*-approximation by linear combinations of integral Bernstein-type operators*, Anal. Numér. Théor. Approx. 13 (1) (1984), 65–72.
[9] —, *Uniform approximation by linear combinations of Bernstein-type polynomials*, J. Approx. Theory 41 (1984), 51–55.

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