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Nevanlinna theory on the p-adic plane

by Capi Corrales Rodrigáñez (Madrid)

Abstract. Let $\mathbb{K}$ be a complete and algebraically closed non-Archimedean valued field. Following ideas of Marc Krasner and Philippe Robba, we define K-meromorphic functions from $\mathbb{K}$ to $\mathbb{K}$. We show that the Nevanlinna theory for functions of a single complex variable may be extended to those functions (and consequently to meromorphic functions).

1. Introduction. In non-Archimedean analysis an *entire* function on a complete and algebraically closed non-Archimedean valued field $\mathbb{K}$ is defined as a Taylor series $\sum_{n=0}^{\infty} c_n x^n$ which converges on all of $\mathbb{K}$; an *analytic* function as a Laurent series which converges on a certain domain D, and a *meromorphic* function as the quotient of two entire functions.

Many of the series that appear in non-Archimedean analysis have small domain of convergence. For example,

$$\exp_p(z) = \sum_{n=0}^{\infty} z^n/n! \quad \text{converges for } |z| < p^{-1/(p-1)}\,;$$
$$\log_p(1+z) = \sum (-1)^{n+1} z^n/n \quad \text{converges for } |z| < 1\,.$$

The natural question to ask is whether the domain of convergence of a non-Archimedean function can be extended in a unique way. The technique by means of power series used in the theory of complex functions does not work, since due to the peculiar properties of non-Archimedean valuations, when we change the point of expansion of a non-Archimedean series,

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its radius of convergence will not change. There are, however, other techniques to extend the domain of definition of a non-Archimedean function in a unique way. The older technique is due to Krasner, and quite elaborate. The more modern one ("rigid analytic spaces") is due to Tate and requires sophisticated commutative algebra.

Following ideas of Marc Krasner and Philippe Robba, we define K-meromorphic functions from $\mathbb{K}$ to $\mathbb{K}$. Our goal is to show that the Nevanlinna theory for functions of a single complex variable may be extended to those functions (and consequently to meromorphic functions). The essential ingredient in the classical Nevanlinna theory is the Poisson–Jensen formula. In the p-adic case our formula (3) plays the role of the Poisson–Jensen formula, but it is not achieved in an analogous way. In the process, we show that in the p-adic case a stronger version of Picard's Theorem holds, more specifically, a K-entire function excludes no value, and a K-meromorphic function excludes at most one value. Such theory proves to be a successful tool in the study of the following two problems.

PROBLEM 1. How many fibers determine univocally functions defined on a non-Archimedean valued field $\mathbb{K}$, complete, algebraically closed and with $\operatorname{char}\mathbb{K} = 0$? That is, on the fibers over how many points must two non-Archimedean functions agree so we can guarantee them to be the same function?

In the complex case, a well known result of Rolf Nevanlinna tells us that two non-constant meromorphic functions which agree on the fibers of five distinct values must be identical. On the other hand, two non-constant polynomials defined over an algebraically closed field of characteristic zero are identical if they agree on the fibers over two distinct finite values, and a rational function defined over an algebraically closed field of characteristic zero is determined by the fibers over four distinct values. In many aspects, entire non-Archimedean functions behave more like polynomials than like entire complex functions, and non-Archimedean meromorphic functions more like rational functions than like complex meromorphic ones. In this connection, Adams and Straus [1] showed that

(i) if f and g are two non-constant non-Archimedean entire functions so that for two distinct finite values a, b, we have $f(x) = a \Leftrightarrow g(x) = a$, and $f(x) = b \Leftrightarrow g(x) = b$, then $f \neq g$;

(ii) if f and g are two non-constant non-Archimedean meromorphic functions so that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i$, then $f \neq g$.

In the process of developing our Nevanlinna theory we extend the properties of the maximum modulus function of p-adic series to K-meromorphic functions. Once these properties are obtained, Adams and Straus' proofs can be extended to Krasner functions.

In the complex plane, Nevanlinna theory is being successfully used as a tool for the study of the factorization of meromorphic functions. Some recent results are collected in [14].

PROBLEM 2. Do those results hold for Krasner functions? The answer is yes. Most of the results by I. N. Baker, Fred Gross and C. F. Oswood in [2]–[6] which use the complex analogues to our Theorems 9–14 and their corollaries, follow similarly or with minor alterations in the p-adic plane.

Notation. Let $\mathbb{K}$ be a complete algebraically closed field with respect to a non-Archimedean valuation $|\cdot|$ (for the properties of $\mathbb{K}$ see [7]). Let $D(a,r) = \{x \in \mathbb{K}\,; |x-a| \le r\}$; $S(a,r) = \{x \in \mathbb{K}\,; |x-a| = r\}$; and if E is a subset of $\mathbb{K}$, let $E^{\mathrm{c}} = \{x \in \mathbb{K}\,; x \notin E\}$.

2. *K*-Meromorphic functions and their properties

K-Meromorphic functions. A subset E of $\mathbb{K}$ is called *quasi-connected* if E has at least two points, and given any $a \neq \infty$ in E, and any real $r < \sup\{|y-x|\ ;\ y, x \in E\}$, the set $L = \{|y-a|\ ;\ y \notin E\}$ has only a finite number of elements $\le r$. (Discs, complements of discs, circles, complements of circles, annuli, discs with finitely many concentric circles removed, etc., are quasi-connected sets.)

Given a quasi-connected set E of $\mathbb{K}$, a function $f : E \to \mathbb{K}$ will be called an *analytic element of support* E if it is a uniform limit in E of rational functions having no poles on E.

Given a subset E of $\mathbb{K}$, a function $F : E \to \mathbb{K}$ will be called *K-analytic* on E if there exists a family of chained analytic elements $\{f_i\}$ with respective supports $\{D_i\}$ such that $\bigcup_i D_i$ contains E, and F restricted to D_i is f_i for each i. We say that a function F is *K-meromorphic* on a subset E of $\mathbb{K}$ if F is analytic on E minus a set of points $\{p_i\}$, which we call the poles of F. The set of poles of F is denoted by P_F.

Given an analytic element f, denote by D_f its support. Given a K-meromorphic function F, denote by Φ_F the family of analytic elements defining F, and by S_F the set of real numbers r such that the circumference $S(0,r)$ contains either a zero or a pole of F.

Properties of K-meromorphic functions. The set of functions K-meromorphic on $\mathbb{K}$ forms a field closed under differentiation. This field contains the set of Taylor series converging on $\mathbb{K}$ and their quotients. The following properties of K-meromorphic functions will be needed to develop a Nevanlinna theory on them. Properties (P1)–(P3) are known, and hence appear without proof. For their proofs see [12].

(P1) *K-meromorphic functions satisfy the principle of analytic continuation (i.e., if a K-meromorphic function is zero in a disc, it is identically zero).*

(P2) *Let F be a K-meromorphic function with $0, R \notin S_F$, and let $a_1, \ldots, a_M$ and $b_1, \ldots, b_N$ be, respectively, the zeroes and poles of F in $D(0,R)$, let m_i be the multiplicity of a_i, and n_j the multiplicity of b_j. Then there exists a K-meromorphic function $G(x)$ with no zeroes or poles in $D(0,R)$ such that*

$$F(x) = G(x) \prod_{i=1}^{M} (x - a_i)^{m_i} \Big/ \prod_{j=1}^{N} (x - b_j)^{n_j} . \tag{1}$$

(P3) *Every K-meromorphic function F is the quotient of two analytic functions.*

(P4) *Let G be a K-meromorphic function with no zeroes or poles in $D(0,R)$. Then, for all $x \in D(0,R)$,*

$$|G(x)| = |G(0)| . \tag{2}$$

Proof. We will use the following result, proved by Marc Krasner in [10], pp. 170–171: If G is an analytic function defined by a finite family of chained analytic elements, say $\Phi_G = \{f_1, \ldots, f_s\}$, then G is itself an analytic element with support $D_G = \bigcup_{i=1}^{s} D_i$, where D_i is the support of f_i for each $i = 1, \ldots, s$. Choose $f \in \Phi_G$ such that $0 \notin D_f$. Then, since D_f is quasi-connected, there exist finitely many values of r, $0 < r \le R$, such that $S(0,r) \cap D_f^{\mathrm{c}} \ne \emptyset$. Call them $r_1, \ldots, r_N$.

Claim. *For each r_i, $i = 1, \ldots, N$, there exist finitely many $g \in \Phi_G$, say g_{ij}, with $j = 1, \ldots, M_i$, such that $\bigcup_{j=1}^{M_i} D_j \supset S(0, r_i)$.*

Proof of claim. Suppose we need infinitely many elements of Φ_G to cover $S(0,r_i)$ with their supports. Then we would have an infinite sequence of embedded circles, say $S_1 \supset S_2 \supset \ldots$, with each S_k contained in the complement of D_g, for some $g \in \Phi_G$, and since $\mathbb{K}$ is maximally complete, their non-empty intersection would give us a pole of G in $S(0,r_i)$, which is absurd, since G has no poles in $D(0,R)$. Hence, we can cover $S(0,r_i)$ with finitely many supports of elements of Φ_G.

Now, by the claim, the finite family $\Psi = \{g_{ij}\ ;\ i = 1, \ldots, N,\ j = 1, \ldots, M_i\}$ defines G in $D(0,R)$, and so, by Krasner's result, $G|_{D(0,R)}$ is an analytic element with support $D(0,R)$ and no zeroes in $D(0,R)$. Hence, by the properties of $M(r,f) = \sup_{|x|=r} |f(x)|$ for an analytic element f (see [10], p. 143), $|G(x)| = |G(0)|$ for all $x \in D(0,R)$.

(P5) *If F is a K-meromorphic function with $0, R \notin S_F$, and $a_1, \ldots, a_M$, $b_1, \ldots, b_N$ are the zeroes and poles of F in $D(0,R)$, repeated according to* *their multiplicities, and* x_R *is such that* $|x_R| = R$, *then*

$$|F(x_R)|R^{N-M} = |F(0)|\prod_{j=1}^{N}|b_j|\Big/\prod_{i=1}^{M}|a_i|\,. \tag{3}$$

Proof. This is a consequence of (1) and (2).

(P6) *If* F *is entire and not a constant, then* F *has at least one zero.*

Proof. Suppose F is entire and without zeroes. Then for each $R > 0$, F has no zeroes and poles in $D(0,R)$, and hence $F|_{D(0,R)}$ is an analytic element. Thus, there exists a sequence of rational functions $\{f_{n,R}\}$, which we can assume to have all of their zeroes and poles outside $D(0,R)$, such that $\{f_{n,R}\}$ converges uniformly to F in $D(0,R)$. For $R \to \infty$, the corresponding $f_{n,R}$'s must be constant, and hence so is F.

As consequences of (P6) we obtain:

(P7) *Every K-meromorphic function which is not a constant has at least a zero or a pole.*

(P8) *If* F *is a non-constant entire function,* F *has no excluded values.*

(P9) *If* F *is K-meromorphic, then, for all* $R \notin S_F \cup S_{F'}$, $R > 0$, *we have*

$$|F'(x)/F(x)| \le 1/|x| \quad \text{for } |x| = R\,. \tag{4}$$

Proof. We need the following two lemmas:

Lemma 1. *Let* g *be a rational function. Then*

$$|g'(x)| \le |g(x)|/|x| \quad \text{for } |x| > 0. \tag{5}$$

Proof. Let $g(x) = h(x)/t(x)$, where h and t are polynomials. Then $|h'(x)| \le |h(x)|/|x|$ if $|x| > 0$, and $|t'(x)| \le |t(x)|/|x|$ for $|x| > 0$. Now,

$$\begin{aligned}|g'(x)| &= |h'(x)/t(x) - h(x)t'(x)/t(x)^2| \\ &\le \max\{|h'(x)|/|t(x)|, |h(x)|\cdot|t'(x)|/|t(x)|^2\}\,.\end{aligned}$$

But

$$|h'(x)|/|t(x)| \le |h(x)|/|x|\cdot|t(x)| = |g(x)|/|x|\,,$$

and

$$|h(x)|\cdot|t'(x)|/|t(x)|^2 \le |h(x)|\cdot|t(x)|/|x|\cdot|t(x)|^2 = |g(x)|/|x|\,,$$

hence, $|g'(x)| \le |g(x)|/|x|$.

Lemma 2. *If* G *is K-meromorphic without zeroes or poles in* $D(0,R)$, *then*

$$|G'(x)| \le |G(x)|/|x| \quad \text{for } 0 < |x| \le R\,. \tag{6}$$

Proof. We know that $G(x)$ is an analytic element without zeroes in $D(0,R)$. Let $x \in D(0,R)$. Let $\{g_n\}$ be an approximating sequence for G in $D(0,R)$. Then, by (5),

$$|G'(x)| = \lim_{n\to\infty} |g_n'(x)| \le \lim_{n\to\infty} |g_n(x)|/|x| = |G(x)|/|x|\,.$$

Now, back to the proof of (4), given any positive $R \notin S_F \cup S_{F'}$, let $a_1,\ldots,a_M$ and $b_1,\ldots,b_N$ be the zeroes and poles of $F(x)$ in $D(0,R)$; we then have $F(x) = [\prod_{i=1}^M (x-a_i)/\prod_{j=1}^N (x-b_j)]G(x)$, where $G(x)$ is a K-meromorphic function which has no zeroes and poles in $D(0,R)$.

Set $A = \prod_{i=1}^M (x-a_i)$, $B = \prod_{j=1}^N (x-b_j)$, $A_i = \prod_{k=1,k\ne i}^M (x-a_k)$, $B_j = \prod_{l=1}^N (x-b_l)$. Then

$$\begin{aligned}
&|F'(x)/F(x)|\\
&= \Big|\Big[\Big(B\sum_{i=1}^M A_i - A\sum_{j=1}^N B_j\Big)\Big/B^2\Big]G(x) + (A/B)G'(x)\Big|\Big/|(A/B)G(x)|\\
&= \Big|\Big[B\sum_{i=1}^M A_i - A\sum_{j=1}^N B_j\Big]G(x) + ABG'(x)\Big|\Big/|A|\cdot|B|\cdot|G(x)|\\
&\le \max\Big\{\Big|\sum_{i=1}^M A_i\Big|\Big/|A|, \Big|\sum_{j=1}^N B_j\Big|\Big/|B|, |G'(x)|/|G(x)|\Big\}\,.
\end{aligned}$$

Now, from (6), $|G'(x)|/|G(x)| \le 1/R$, and we easily see that $|\sum_{i=1}^M A_i|/|A| \le 1/R$ and $|\sum_{j=1}^N B_j|/|B| \le 1/R$, so $|F'(x)/F(x)| \le 1/R$ for $|x| = R$.

The function $M(r,f)$. Given a K-meromorphic function F, we define $M(r,F) = \sup_{|x|=r}|F(x)|$. The function $M(r,F)$ satisfies the following properties:

(I) If $r \ne 0,\infty$, then $M(r,F) = 0$ implies $F \equiv 0$.

(II) $M(r,F+G) \le \max[M(r,F), M(r,G)]$.

(III) $M(r,FG) = M(r,F)M(r,G)$.

(IV) If $0 \notin S_F$, then for each $r \ge 0$ such that $r \notin S_F$, $M(r,F) = |F(x_r)|$, with x_r arbitrary such that $|x_r| = r$.

(V) If $r \notin S_F \cup S_{F'}$, $r > 0$, then $M(r,F') \le M(r,F)/r$.

(VI) If F is a non-constant entire function, then $M(r,F) \to \infty$ as $r \to \infty$.

Properties (I)–(III) follow directly from the properties of $M(r,f)$ for f an analytic element (see [10], p. 143); (IV) is easily deduced from (3); (V) follows from (4) and (IV), and finally (VI) is proved using (3) and (IV).

Theorem 3 (Four Points Theorem). *Let F,G be two non-constant K-meromorphic functions on $\mathbb{K}$ so that for distinct a_1,a_2,a_3,a_4 we have $F(x) = a_i \Leftrightarrow G(x) = a_i$, $i = 1,2,3,4$. Then $F \equiv G$.*

Proof. Once we have (V) and (VI) above, Adams and Straus' proof of the 4-points theorem carries over without difficulty (see [1], p. 421). We give a sketch of this proof.

Without loss of generality we may assume $a_3 = 0$, $a_4 = 1$. Let $F = f_1/f_2, G = g_1/g_2$, with f_i, g_i analytic for $i = 1, 2$. Then there exists an entire function H such that

$$(f_1' f_2 - f_1 f_2')(f_1 g_2 - f_2 g_1) = H f_1 f_2 (f_1 - a_1 f_2)(f_1 - a_2 f_2),$$

and for r large enough, using the properties of the function $M(f, r)$, we obtain (see [1]) $M(H, r) \le 1/r$. Consequently, $H \equiv 0$, and since $f_1' f_2 - f_1 f_2' \neq 0$, we must have $F \equiv G$.

3. Nevanlinna theory for meromorphic functions

Definitions. Recall that $\log^+ a = \max\{0, \log a\}$, and let $n(r, F)$ be the number of poles of F in $D(0, r)$. We define, for F K-meromorphic with $0, r \notin S_F$ and arbitrary $x_r \in S(0, r)$,

$$m(r, F) = \log^+ |F(x_r)|, \qquad N(r, F) = \int_0^r (n(t, F)/t)\, dt .$$

Remark. $N(r, F) = \sum_{j=1}^n \log(r/|b_j|)$, since, if $r_1, \ldots, r_n$ are the values of the poles of F in $D(0, r)$ arranged in non-decreasing order, then

$$\sum_{j=1}^n \log(r/|b_j|) = \sum_{j=1}^n \log(r/r_j) = \int_0^r \log(r/t)\, dn(t, F) = (*),$$

which is a real Stieltjes integral. We then integrate by parts to get

$$(*) = n(t, F) \log(r/t)]_0^r + \int_0^r (n(t, F)/t)\, dt = \int_0^r (n(t, F)/t)\, dt .$$

Theorem 4-a (First Fundamental Theorem). *If F is a K-meromorphic function with $0, r \notin S_F$, then* $\log |F(0)| = N(r, F) - N(r, 1/F) + m(r, F) - m(r, 1/F)$.

Proof. By the above definitions, taking logarithms in (3) gives

$$\log |F(x_r)| + (n - m) \log r = \log |F(0)| + \sum_{j=1}^n \log |b_j| - \sum_{i=1}^m \log |a_i| ;$$

and hence, since $\log |a| = \log^+ |a| - \log^+ |1/a|$, we have

$$\log^+ |F(x_r)| + \sum_{j=1}^n \log(r/|b_j|) = \log^+(1/|F(x_r)|) + \sum_{i=1}^m \log(r/|a_i|) + \log |F(0)| .$$

Definition. For F a K-meromorphic function we define $T(r,F) = N(r,F) + m(r,F)$ for all $r \notin S_F$.

Then Theorem 4-a can be rewritten as

Theorem 4-b. *If F is a K-meromorphic function with $0, r \notin S_F$, then*

$$T(r,F) = T(r,1/F) + \log|F(0)|\,. \tag{7}$$

Theorem 5. *If F is a K-meromorphic function with $0, r \notin S_F$, then for all $a \in \mathbb{K}$ with $0, r \notin S_{F-a}$, and for each non-zero $b \in \mathbb{K}$, we have*

(i) $T(r,bF) = T(r,F) + O(1)$,
(ii) $T(r, \sum_{i=1}^n F_i(x)) \le \sum_{i=1}^n T(r, F_i(x))$,
(iii) $T(r, 1/(F-a)) = T(r,F) + O(1)$,
(iv) $T(r, \prod_{i=1}^n F_i(x)) \le \sum_{i=1}^n T(r, F_i(x))$.

Proof. The first equality is clear by definition of $T(r,F)$.

(ii) By definition, $m(r, \sum_{i=1}^n F_i) = \log^+ |\sum_{i=1}^n F_i(x_r)|$, but if $n \in \mathbb{Z}^+$, $a_i \in \mathbb{K}$, then

$$\log^+ \Big|\sum_{i=1}^n a_i\Big| = \max\Big(\log\Big|\sum_{i=1}^n a_i\Big|, 0\Big) \le \max(0, \log \max_{i=1,\ldots,n} |a_i|)$$
$$\le \max(0, \max_{i=1,\ldots,n} \log|a_i|) = \max_{i=1,\ldots,n} \log^+|a_i| \le \sum_{i=1}^n \log^+|a_i|\,,$$

and thus,

$$m\Big(r, \sum_{i=1}^n F_i(x)\Big) \le \sum_{i=1}^n m(r, F_i(x))\,. \tag{8}$$

Also, since the order of a pole of $\sum F_i$ at a point x_0 des not exceed the sum of the orders of the poles of F_i at x_0 (it is at most the maximum of the orders of the zeroes of the F_i at x_0), we have

$$N\Big(r, \sum_{i=1}^n F_i(x)\Big) \le \sum_{i=1}^n N(r, F_i(x))\,. \tag{9}$$

Thus (8) and (9) imply (ii).

(iii) Letting, in (ii), $n = 2$, $F_1(x) = F(x)$ and $F_2(x) = -a$, we get

$$\begin{aligned} T(r, F-a) &\le T(r,F) + T(r,-a) \le T(r,F) + N(r,-a) + m(r,-a) \\ &= T(r,F) + \log^+|-a| + \int_0^r (n(t,-a)/t)\,dt \\ &= T(r,F) + \log^+|-a|\,. \end{aligned}$$

Now (7) implies that $T(r, 1/(F-a)) = T(r, F-a) + \log|F(0)-a|$, and so

$$T(r, 1/(F-a)) \le T(r,F) + \log^+|-a| + \log|F(0)-a| = T(r,F) + O(1)\,.$$

(iv) First for $a_1, \ldots, a_n \in \mathbb{K}$, $n \geq 1$, we have the inequality $\log^+ |\prod_{i=1}^n a_i| = \log^+ \prod_{i=1}^n |a_i| \leq \sum_{i=1}^n \log^+ |a_i|$; hence,

$$m\Big(r, \prod_{i=1}^{n} F_i\Big) \leq \sum_{i=1}^{n} m(r, F_i)\,. \tag{10}$$

Also, if F_i is a K-meromorphic function, $i = 1, \ldots, n$, then since the order of a pole of $\prod F_i(x)$ at a point x_0 is at most the sum of the orders of the poles of the F_i at x_0,

$$N\Big(r, \prod_{i=1}^{n} F_i\Big) \leq \sum_{i=1}^{n} N(r, F_i)\,. \tag{11}$$

Now (10) and (11) imply (iv).

THEOREM 6. *Suppose* $G(x) = [aF(x) + b]/[cF(x) + d]$, *with* $a, b, c, d \in \mathbb{K}$, $ad - bc \neq 0$ *and* F *a* K*-meromorphic function such that* $0, r \notin S_F \cup S_G$. *Then* $T(r, G) = T(r, F) + O(1)$.

Proof. Analogous to the complex case. See [13], p. 174.

LEMMA 7. *If* F *is a meromorphic function with* $0, r \notin S_F \cup S_{F-1}$, *then*

$$T(r, F) = N(r, 1/(F - 1)) + \log^+ |F(0)|\,. \tag{12}$$

Proof. From (1) we have $\log |F(0)| = \log |F(x_r)| - \sum_{i=1}^{m(r)} \log(r/|a_i|) + \sum_{j=1}^{n(r)} \log(r/|b_j|)$, so for $F(z) - 1 = G(z)$,

$$\begin{aligned} \log |F(0) - 1| &= \log |F(x_r) - 1| - N(r, 1/(F - 1)) + N(r, F - 1) \\ &= \log |F(x_r) - 1| - N(r, 1/(F - 1)) + N(r, 1/F)\,. \end{aligned}$$

But since for all $a \in \mathbb{K}$, if $|a| \geq 1$, then $\log |a - 1| = \log |a|$, and if $|a| < 1$ then $\log |a - 1| = 0$, we have $\log |a - 1| = \log^+ |a|$.

Thus, substituting $a = F(0)$ and $a = F(x_r)$, we get

$$\begin{aligned} \log^+ |F(0)| &= \log^+ |F(x_r)| - N(r, 1/(F - 1)) + N(1/r, F) \\ &= m(r, F) - N(r, 1/(F - 1)) + N(r, F) \\ &= T(r, F) - N(r, 1/(F - 1))\,. \end{aligned}$$

THEOREM 8. *If* f *is a rational function on* $\mathbb{K}$ *with numerator of degree* m *and denominator of degree* n, *then* $T(r, f) = \max\{m, n\} \log r + O(1)$.

Proof. By taking r large enough, and considering separately the cases $m > n$, $m < n$, and $m = n$, a long but simple computation gives the result.

THEOREM 9. *If* F *is* K*-meromorphic*, *and not a constant*, *then* $T(r, F) \to \infty$ *as* $r \to \infty$.

Proof. If F is K-meromorphic, and not a constant, we know that F has at least one pole or zero in $\mathbb{K}$. Without loss of generality assume F has at least one pole, say b_1, such that $0 < |b_1| < \infty$. Then we choose any finite R_1 such that $R_1 > |b_1|$, and $R_1 \notin S_F$. Then $b_1 \in D(0,r)$ for all $r \geq R_1$.

Recall that if $a_1, \ldots, a_{M(r)}$ and $b_1, \ldots, b_{N(r)}$ are, respectively, the zeroes and poles of F in $D(0,r)$, and if x_r is such that $|x_r| = r$, then

$$T(r,F) = \log^+ |F(x_r)| + \sum_{j=1}^{N(r)} \log(r/|b_j|) . \tag{13}$$

Claim. $\lim_{r\to\infty} \sum_{j=1}^{N(r)} \log(r/|b_j|) = \infty$.

Proof. Take any $r \geq R_1$, and arrange the poles b_j of F inside $D(0,r)$ so that $|b_1| \leq \ldots \leq |b_{N(R_1)}| \leq \ldots \leq |b_{N(r)}|$. Then $\lim_{r\to\infty} \log(r/|b_1|) = \infty$, and $r/|b_j| \geq 1$ for all $j \geq 2$, so that $\sum_{j=2}^{N(r)} \log(r/|b_j|) \geq 0$. Therefore, for all $r \geq R_1$,

$$\sum_{j=1}^{N(r)} \log(r/|b_j|) = \log(r/|b_1|) + \sum_{j=2}^{N(r)} \log(r/|b_j|) \geq \log(r/|b_1|) ,$$

and hence, $\lim_{r\to\infty} \sum_{j=1}^{N(r)} \log(r/|b_j|) \geq \lim_{r\to\infty} \log(r/|b_1|) = \infty$.

Coming back to (16),

$$T(r,F) = \log^+ |F(x_r)| + \sum_{j=1}^{N(r)} \log(r/|b_j|) \geq \sum_{j=1}^{N(r)} \log(r/|b_j|) ,$$

and so, by the claim, $T(r,F) \to \infty$ as $r \to \infty$.

Theorem 10 (Second Fundamental Theorem). *Suppose F is a non-constant K-meromorphic function with $0, r \notin S_F \cup S_{F'}$. Let $z_1, \ldots, z_q$ (where $q \geq 2$) be distinct numbers in $\mathbb{K}$ such that $|z_i - z_j| \geq \delta$ for $1 \leq i < j \leq q$, $0 < \delta < 1$.*

(I) *We have the inequality*

$$m(r,F) + \sum_{i=1}^{q} m(r, 1/(F - z_i)) \leq 2T(r,F) + N_1(r) + S(r) , \tag{14}$$

where

(II) $N_1(r) = N(r, 1/F') + 2N(r,F) - N(r,F')$ *is non-negative, and*

(III) $S(r) = m(r, \sum_{i=1}^{q} F'/(F - z_i)) + m(r, F'/F) - \log(1/|F'(0)|)$
$+q \log^+(q/\delta) + \log(q/(q-1))$
$= m(r, \sum_{i=1}^{q} F'/(F - z_i)) + m(r, F'/F) + C(q,\delta)$
is an error term with $S(r)/T(r,F) \to 0$ as $r \to \infty$.

Proof. The proofs of (I) and (II) are analogous to the complex case with minor alterations. See [13], p. 188. For (III) we first show

$$S(r) \le (q+1)\log^+(1/r) + C(q,\delta)\,. \tag{15}$$

Indeed, by (4), we know that $|F'(x)/F(x)| \le 1/r$ for $|x| = r$; we thus have

$$m(r, F'/F) = \log^+ |F'(x_r)/F(x_r)| \le \log^+(1/r)\,, \tag{16}$$

$$m\Big(r, \sum_{i=1}^{q} F'/(F-z_i)\Big) \le \sum_{i=1}^{q} m(r, F'/(F-z_i)) \le q\log^+(1/r) \tag{17}$$

(by (8) and (16)).

Now (16) and (17) imply (15).

Finally, since $\log^+(1/r) = 0$ for $r \ge 1$, and by Theorem 9, $T(r,F) \to \infty$ as $r \to \infty$, we have $S(r)/T(r,F) \le [(q+1)\log^+(1/r) + C(q,\delta)]/T(r,F) \to 0$ as $r \to \infty$. This completes the proof of Theorem 10.

Consequences of the Second Fundamental Theorem. As in the complex case, we will cast Theorem 10 in a somewhat different form which contains slightly less information, but which may sometimes be more easily used. To do this, we need some definitions. All the results following the definitions have proofs analogous to those for the complex case.

Definitions. We redefine the function $N(t, 1/(F-a))$ to allow for the possibility that $F(0) = a$. Hence, if F is a K-meromorphic function with $r \notin S_{F-a}$, redefine

$$N(t, 1/(F-a)) = n(0, 1/(F-a))\log r + \int_0^r ([n(t, 1/(F-a)) - n(0, 1/(F-a))]/t)\,dt\,.$$

Let $\widetilde{n}(t,F)$ = number of distinct poles of F in $D(0,t)$ (multiple poles are counted singly). Then, if F is a K-meromorphic function with $r \notin S_{F-a}$, we define

$$\widetilde{N}(r, 1/(F-a)) = \int_0^r ([\widetilde{n}(t, 1/(F-a)) - \widetilde{n}(0, 1/(F-a))]/t)\,dt + \widetilde{n}(0, 1/(F-a))\log r\,,$$

$$\partial(a) = \partial(a,F) = \liminf_{r\to\infty} m(r, 1/(F-a))/T(r,F) = 1 - \limsup_{r\to\infty} N(r, 1/(F-a))/T(r,F)\,.$$

$\partial(a)$ is called the *deficiency* of F at a. If for all $x \in \mathbb{K}$, $F(x) \ne a$, then $N(r, 1/(F-a)) = 0$, and so $\partial(a) = 1$. In any case, since $0 \le m(r, 1/(F-a)) \le T(r,F)$, we have $0 \le \partial(a) \le 1$, and $\partial(a) > 0$ means that there are "relatively few" (though maybe infinitely many) values of x such that $F(x) = a$.

We will see this cannot happen for too many values of a. We also define

$$Q(a) = Q(a,F) = 1 - \limsup_{r\to\infty} \widetilde{N}(r, 1/(F-a))/T(r,F),$$

$$q(a) = q(a,F) = \liminf_{r\to\infty} [N(r,1/(F-a)) - \widetilde{N}(r,1/(F-a))]/T(r,F)\,.$$

The function $q(a)$ is called the *ramification index* or *index of multiplicity* of a; we have $0 \le q(a) \le 1$, and $q(a) > 0$ means that there are "relatively many" multiple roots of the equation $F(x) = a$.

THEOREM 11. *If F is a K-meromorphic function, then the set of values of a for which $q(a) > 0$ is at most countable, and*

$$\sum_{a,Q(a)>0} (\partial(a) + q(a)) \le \sum_{a,Q(a)>0} Q(a) \le 2\,. \tag{18}$$

Proof. This theorem is equivalent to Theorem 10; see [13], p. 206.

Note. If F is entire, then $\partial(\infty) = Q(\infty) = 1$ (since $N(r,F) = \widetilde{N}(r,F) = 0$), and so, from Theorem 11, $\sum_{a \text{ finite}, Q(a)>0} (\partial(a) + q(a)) \le \sum_{a \text{ finite}, Q(a)>0} Q(a) \le 1$; hence $\partial(a) > 1/2$ for at most one finite value of a.

COROLLARY 12 (see [11]). *Let F be a K-meromorphic function.*

(i) *There can be at most two values of a for which $N(r,1/(F-a)) = O(T(r,F))$.*

(ii) *There can be at most two values of a for which $\partial(a) > 2/3$ (i.e. for which $\limsup_{r\to\infty} N(r,1/(F-a))/T(r,F) < 1/3$).*

(iii) *There can be at most four values of a such that every root of $F(x)-a$ is multiple.*

Note. Rolf Nevanlinna proved both Picard's Theorem and the Five Points Theorem as corollaries to Theorem 11. Such results can also be deduced in the p-adic case, but we have already seen both of them in a stronger form as property (P8) of K-meromorphic functions, and Theorem 3, respectively.

THEOREM 13 (J. G. Clunie; see [8]). *Let $G(x)$ be a K-meromorphic function, $F(x)$ entire, and $H(x) = G(F(x))$. Then $T(r,F)/T(r,H) \to 0$ as $r \to \infty$.*

THEOREM 14 (Rolf Nevanlinna; see [8]). *If F is a K-meromorphic function, and $a_1(x), a_2(x), a_3(x)$ are distinct K-meromorphic functions satisfying $T(r,a_i(x)) = o(T(r,F))$ as $r \to \infty$, then*

$$\{1 + o(1)\}T(r,F) \le \sum_{i=1}^{3} \widetilde{N}(r,1/(F-a_i(x))) + S(r,F)\,,$$

where $S(r,F)/T(r,F) \to 0$ as $r \to \infty$.

Remarks and applications. In his paper [9], Ha Huy Khoai develops analogs of the Nevanlinna counting functions for quotients of p-adic series converging on a disc of radius 1. With these he obtains an analog to the First Fundamental Theorem. ("If $T(r, f - a)$ is bounded for some $a \in \mathbb{K}$, it is bounded for all $a \in \mathbb{K}$".) He does not give an analog to the Second Fundamental Theorem, but he proves, by use of interpolation methods, both Picard's Theorem and the Three Points Theorem, always for quotients of series converging on a disc of radius 1.

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DEPARTAMENTO DE ALGEBRA
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD COMPLUTENSE DE MADRID
28040 MADRID, SPAIN

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