ANNALES
POLONICI MATHEMATICI
LVII.2 (1992)

p-Envelopes of non-locally convex F-spaces

by C. M. Eoff (Fayetteville, Ark.)

Abstract. The p-envelope of an F-space is the p-convex analogue of the Fréchet envelope. We show that if an F-space is locally bounded (i.e., a quasi-Banach space) with separating dual, then the p-envelope coincides with the Banach envelope only if the space is already locally convex. By contrast, we give examples of F-spaces with are not locally bounded nor locally convex for which the p-envelope and the Fréchet envelope are the same.

1. Introduction. For a non-locally convex F-space $\mathbb{X}$ (complete, metrizable, linear topological space), the idea of a p-envelope is analogous to that of a Fréchet envelope. Suppose $\mathbb{X}$ has separating dual space; recall that the *Fréchet envelope* of $\mathbb{X}$, denoted by $\widehat{\mathbb{X}}$, is the closure of $\mathbb{X}$ with respect to the Mackey topology, μ. The *Mackey topology* is the strongest locally convex topology on $\mathbb{X}$ for which $\mathbb{X}$ still has dual space $\mathbb{X}^*$. A countable base for the μ-zero neighborhoods $\{\widetilde{V}_n\}$ can be obtained by taking the closure in $\mathbb{X}$ of the absolutely convex hull of each V_n, where $\{V_n\}$ is any countable base for the zero-neighborhoods of $\mathbb{X}$; this description in fact characterizes μ [13]. In general $\widehat{\mathbb{X}}$ is a Fréchet space; for a locally bounded F-space, $\widehat{\mathbb{X}}$ turns out to be a Banach space—the *Banach envelope.* ($S \subset \mathbb{X}$ is *bounded* if given any zero neighborhood U, there exists $n \in \mathbb{N}$ such that $S \subset nU$. $\mathbb{X}$ is *locally bounded* if it has a bounded neighborhood of zero.)

Interest in the containing Fréchet space of a non-locally convex F-space was first sparked by the pioneering work of Duren, Romberg, and Shields, who showed that the Hardy space H^p, $0 < p < 1$, could be densely imbedded in a certain Banach space, the Bergman space B^p, and $(H^p)^* \simeq (B^p)^*$ [4]. Somewhat later Shapiro identified the Banach envelope of H^p directly, using his "convex-hull" characterization of the Mackey topology [13]. This characterization of the Mackey topology provides an important intuitive

1991 *Mathematics Subject Classification*: Primary 46A16, 30D55.
Key words and phrases: p-envelope, non-locally convex F-space, multiplier.

picture, via the example $\ell_p, 0 < p < 1$. The absolutely convex hull of the ℓ_p unit ball is the ℓ_1 unit ball. Thus, the Mackey topology is the ℓ_1 topology and the closure of ℓ_p with respect to this topology is ℓ_1; i.e., ℓ_1 is the Banach envelope of ℓ_p. With its usual metric $d((\alpha_n),0) = \|(\alpha_n)\|_p^p = \sum_{n=0}^{\infty} |\alpha_n|^p$, the sequence space $\ell_p, 0<p<1$, is the prototypical example of a non-locally convex, locally bounded F-space with separating dual (the maps $\phi_k((\alpha_n)) = \alpha_k$ are continuous). In addition, the topology induced by d is p-convex, since the unit ball is (absolutely) p-convex. A set C is *p-convex* if $\sum_{i=1}^n a_i x_i \in C$ whenever $x_i \in C$ and $\sum_{i=1}^n a_i^p = 1$, with $a_i \geq 0$. C is *absolutely p-convex* if $\sum_{i=1}^n a_i x_i \in C$ whenever $x_i \in C$ and $\sum_{i=1}^n |a_i|^p = 1$. The functional $\|(\alpha_n)\|_p = (\sum_{n=0}^{\infty} |\alpha_n|^p)^{1/p}$ is a quasinorm; i.e., it satisfies the requirements for a norm except that the triangle inequality is weakened. For $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$,

$$\|\alpha+\beta\|_p \leq M(\|\alpha\|_p + \|\beta\|_p)$$

for a constant $M \geq 1$. Clearly, $\|\cdot\|_p$ satisfies

$$\|\alpha+\beta\|_p^p \leq \|\alpha\|_p^p + \|\beta\|_p^p\,;$$

a quasinorm with this property is said to be *p-subadditive* and is called a *p-norm*. In general, if an F-space, $\mathbb{X}$, is locally bounded, the metric topology can always be replaced by a quasinorm, in fact by a q-norm for some $0 < q \leq 1$, due to a result of Aoki and Rolewicz; $\mathbb{X}$ is then called a *q-Banach space*. (See [7] or [14] for general facts about non-locally convex F-spaces.)

By analogy with the Fréchet envelope, let $\{V_n\}$ be a countable base for the zero neighborhoods of a non-locally convex F-space, $\mathbb{X}$, with separating dual, and let $\widetilde{V}_n$ be the absolutely p-convex hull of V_n, for some fixed p, $0 < p < 1$. Let $\|\cdot\|_n$ be the Minkowski functional of $\widetilde{V}_n$. For $x, y \in \mathbb{X}$, the functional $\|\cdot\|_n$ satisfies:

(i) $\|x\|_n = 0$ if $x = 0$,
(ii) $\|ax\|_n = |a|\|x\|_n$, $a \in \mathbb{C}$,
(iii) $\|x+y\|_n^p \leq \|x\|_n^p + \|y\|_n^p$.

From (iii) we can deduce that $\|x+y\|_n \leq C(\|x\|_n + \|y\|_n)$. By obvious analogy, we will refer to $\|\cdot\|_n$ as a *p-seminorm*. The family $\{\|\cdot\|_n\}$ generates a p-convex topology on $\mathbb{X}$ weaker than the original topology. We call the closure of $\mathbb{X}$ under the topology induced by $\{\|\ \|_n\}$ the *p-envelope* of $\mathbb{X}$ and denote it by $\widehat{\mathbb{X}}_p$ (cf. [1]). When $\mathbb{X}$ is locally bounded, $\widehat{\mathbb{X}}_p$ is a p-Banach space. $\widehat{\mathbb{X}}_p$ has the property that every continuous linear map $T : \mathbb{X} \to \mathbb{Y}$, $\mathbb{Y}$ a p-Banach space, extends continuously to $\widehat{\mathbb{X}}_p$.

To visualize the situation, let $0 < p < q \leq 1$. The absolutely q-convex hull of the unit ball of ℓ_p is the ℓ_q unit ball, and it follows that ℓ_q is the q-envelope of ℓ_p. (For $0 < p < q < 1$, the q-envelope of H^p was identified by Aleksandrov in [1] and by Coifman and Rochberg in [2].) Now for $0 < p < q \le 1$, ℓ_q is not isomorphic to ℓ_p; however, it can happen that $\widehat{\mathbb{X}}_p$ is isomorphic to $\widehat{\mathbb{X}}_q$ for all $0 < p, q < 1$ (see [7], Chapter 2). However, as we shall prove, the p-envelope, for $0 < p < 1$, can never be isomorphic to the Fréchet (Banach) envelope of a locally bounded, non-locally convex F-space (quasi-Banach space). We accomplish this in §2 by a modification of an argument of Kalton ([7], Theorem 4.13).

For an F-space which is not locally bounded, the situation is much different. We provide a class of examples which have the property that $\widehat{\mathbb{X}}_p = \widehat{\mathbb{X}}$ for $0 < p < 1$. The groundwork is laid in §3; proofs are carried out in §4. Our method of proof will yield various applications along the way.

The author gratefully acknowledges the many helpful comments and suggestions of Nigel Kalton during the preparation of this paper. This material constitutes a portion of the author's Ph.D. thesis, written under the excellent supervision of Professor Kalton, at the University of Missouri, Columbia, Missouri.

2. $\widehat{\mathbb{X}}$ is never isomorphic to $\widehat{\mathbb{X}}_p$ for a quasi-Banach space $\mathbb{X}$. In this section we shall prove that $\widehat{\mathbb{X}}_1 = \widehat{\mathbb{X}}$ (the Banach envelope) is never isomorphic to $\widehat{\mathbb{X}}_p$, $0 < p < 1$, when $\mathbb{X}$ is a non-locally convex quasi-Banach space.

The Aoki–Rolewicz theorem provides every quasi-Banach space with an equivalent p-norm for some p, $0 < p \le 1$. Thus we lose no generality by our formulation of the following proposition.

PROPOSITION 2.1. *Let* $(\mathbb{X}, \| \ \|_{\mathbb{X}})$, $(\mathbb{Y}, \| \ \|_{\mathbb{Y}})$ *be quasi-Banach spaces so that* $\| \ \|_{\mathbb{X}}$ *is an* r*-norm and* $\| \ \|_{\mathbb{Y}}$ *is a* q*-norm for* $0 < r < q \le 1$. *Let* $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < 1\}$. *If* $T : \mathbb{X} \to \mathbb{Y}$ *is a bounded linear map so that* $p\text{-}\overline{\mathrm{co}}\, T(B_{\mathbb{X}})$ *is a neighborhood of the origin for* $0 < r \le p < q \le 1$, *then* T *is an open map.*

Proof (cf. [7], Theorem 4.13). For convenience, let $\| \ \|$ denote the quasinorms for both $\mathbb{X}$ and $\mathbb{Y}$, as well as the operator quasinorm for T. No confusion should arise from this. We assume, with no loss, that $\|T\| = 1$.

There exists $\delta > 0$ so that if $\|y\| < \delta$ then $y \in p\text{-}\overline{\mathrm{co}}\, T(B_{\mathbb{X}})$. It is enough to show that a constant M exists so that if $\|y\| < 1$, there is an $x \in \mathbb{X}$ with $\|x\| \le M$ and $\|Tx - y\| < 1/2$. If this can be done, then we can choose x_n by induction satisfying $\|x_n\| \le 2^{-n} M$, $n = 0, 1, \ldots$, with $\|T(x_0 + \ldots + x_n) - y\| \le 2^{-n-1}$. Then we would have $T(\sum_{n=0}^{\infty} x_n) = y$; the series $\sum_{n=0}^{\infty} x_n$ converges since $\sum_{n=0}^{\infty} \|x_n\|^r < \infty$.

So let $V_m = \{\sum_{i=1}^{m} a_i T(x_i) : \sum_{i=1}^{m} a_i^p \le 1,\ a_i \ge 0,\ \|x_i\| \le 1\}$ and note that $\bigcup_{m=1}^{\infty} V_m = p\text{-}\mathrm{co}\, T(B_{\mathbb{X}})$. For any $w \in V_{2m}$, $w = \sum_{i=1}^{2m} a_i T x_i$, where we label the a_i's so that $a_{i-1} \ge a_i$, $i = 2, 3, \ldots, 2m$. Put $w_0 = \sum_{i=1}^{m} a_i T(x_i)$, $w_0 \in V_m$. Notice that $a_i \le (1/(2m))^{1/p}$ for $2m \ge i \ge m$; whereby,

$$\|w - w_0\|^q = \Big\| \sum_{i=m+1}^{2m} a_i T x_i \Big\|^q \le \sum_{i=m+1}^{2m} |a_i|^q \|T x_i\|^q$$
$$\le m \left(\frac{1}{2m} \right)^{q/p} = C_1 m^{-\alpha} ,$$

with $C_1 = 2^{-q/p}$, $\alpha = q/p - 1 > 0$.

For $w \in V_{2^{m+n}}$, $w = \sum_{i=1}^{2^{m+n}} a_i T(x_i)$, with $\sum_{i=1}^{2^{m+n}} a_i^p \le 1$, put

$$w_j = \sum_{i=1}^{2^{m+j}} a_i T(x_i) \in V_{2^{m+j}} , \qquad j = 0, \ldots, n ;$$

then $w_n = w$. From our previous observation, we deduce that

$$\|w - w_0\|^q \le \sum_{j=1}^{n} \|w_j - w_{j-1}\|^q \le \sum_{j=1}^{n} C_1 (2^{m+j})^{-\alpha}$$
$$= C_1 2^{-m\alpha} \sum_{j=1}^{n} 2^{-j\alpha} \le C_1 2^{-m\alpha} \sum_{j=1}^{\infty} 2^{-j\alpha} = C_2 2^{-m\alpha} ,$$

with $C_2 = C_1 (2^{\alpha} - 1)^{-1}$. Thus for $w \in V_{2^{m+n}}$

$$\operatorname{dist}(w, V_{2^m}) = \inf_{y \in V_{2^m}} \|w - y\| \le \|w - w_0\| \le C_2^{1/q} 2^{-m\beta} ,$$

independent of n, with $\beta = 1/p - 1/q > 0$. In particular, we can choose m_0 so large that if $w \in \bigcup_{n=1}^{\infty} V_n$, then

$$\operatorname{dist}(w, V_{2^{m_0}}) < \delta/(4C) ,$$

where C is the quasinorm constant for $\mathbb{Y}$. Put $2^{m_0} = N$. If $\|y\| < 1$, there exists $z \in \bigcup_{n=1}^{\infty} V_n$ so that $\|\delta y - z\| < \delta/(4C)$. Let $v \in V_N$; we have

$$\|\delta y - v\| \le C(\|\delta y - z\| + \|z - v\|) < \delta/2 ,$$

i.e., $\|y - \delta^{-1} v\| < 1/2$. Now $v = \sum_{n=1}^{N} a_i T x_i$, for $\sum_{n=1}^{N} a_i^p \le 1$, $\|x_i\| \le 1$; put $x = \delta^{-1} \sum_{n=1}^{N} a_i x_i$, so that we obtain

$$\|y - Tx\| < 1/2 \quad \text{and} \quad \|x\| \le N^{1/r} \delta^{-1} = M .$$

This completes the proof.

THEOREM 2.2. *Let $\mathbb{X}$ be a locally bounded F-space which is r-normable for $0 < r < 1$. If $\widehat{\mathbb{X}}_p$ is locally q-convex for $0 < r \le p < q \le 1$, then $\mathbb{X}$ is necessarily q-convex.*

Proof. Let $j : \mathbb{X} \to \widehat{\mathbb{X}}_p$ be the natural inclusion map, so that $p\text{-}\overline{\mathrm{co}}\, j(B_{\mathbb{X}})$ is the closed unit ball of $\widehat{\mathbb{X}}_p$. If $\widehat{\mathbb{X}}_p$ can be endowed with an equivalent

q-convex topology, it follows from Proposition 2.1 that j is an open map; consequently, $\mathbb{X} = \widehat{\mathbb{X}}_p$, so that $\mathbb{X}$ must be q-convex.

COROLLARY 2.3. *Let* $\mathbb{X}$ *be a quasi-Banach space such that* $\widehat{\mathbb{X}}_p$, $0 < p < 1$, *is locally convex. Then* $\mathbb{X}$ *is locally convex*; *i.e.*, $\mathbb{X}$ *is a Banach space.*

3. The classes N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$. Let $\mathbb{D}$ denote the unit disc in the complex plane, $\mathbb{C}$. Recall that a function analytic in the unit disc is said to be *of bounded characteristic*, or of *Nevanlinna class* N, if the integrals

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta$$

are uniformly bounded for $r < 1$. For each function $f \in N$, the nontangential limit $f(e^{i\theta})$ exists for a.e. $\theta \in [-\pi, \pi]$; if a function $f \in N$ further satisfies the condition that

$$\lim_{r \to 1-} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})|\, d\theta$$

then f belongs to the *Smirnov class* N^+ [3]. N^+ has been studied for many years as part of the classical Hardy space theory ([3] is a good general reference), although it was not until the early 70's that N. Yanagihara investigated the linear topological structure of N^+ [16], [17]. He found N^+ to be an F-space, not locally convex nor locally bounded, but still possessing a rich dual space, which he identified. Recently, McCarthy [8] has taken a different approach to the study of N^+, obtaining new results as well as giving new proofs to certain of Yanagihara's results. The structure of N^+ as a topological algebra has been studied in [12], for example. Generalizations of Yanagihara's work to $\mathbb{C}^n$, and even to Banach space valued functions have been carried out by Nawrocki [10], [11].

For $\alpha \ge 1$, define N_+^α to consist of those functions f belonging to N^+ such that

$$\int_{-\pi}^{\pi} [\log^+ |f(e^{i\theta})|]^\alpha\, d\theta < \infty\,.$$

Also, define $\mathcal{N}_+^\alpha(\mathbb{D})$ to be the class of functions analytic in the unit disc which satisfy

$$\int_{\mathbb{D}} [\log^+ |f(z)|]^\alpha\, dA(z) < \infty\,,$$

where dA is normalized area measure. The classes N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$ were introduced by M. Stoll in [15] (with different notation), where he showed that they are non-locally convex F-spaces under their respective metrics, in fact, F-algebras. Also, like N^+, both classes have separating dual spaces since point evaluations are continuous. Further results about the algebraic structure of N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$ have been obtained recently by Mochizuki in [9].

The natural metric for N_+^α is

$$d_\alpha(f,0)=\left\{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}[\log(1+|f(e^{i\theta})|)]^\alpha\,d\theta\right\}^{1/\alpha},$$

and in similar fashion, for $\mathcal{N}_+^\alpha(\mathbb{D})$ the natural metric is

$$\varrho_\alpha(f,0)=\left\{\int\limits_{\mathbb{D}}[\log(1+|f(z)|)]^\alpha\,dA(z)\right\}^{1/\alpha}$$

(see [15]). These metrics are rotation-invariant (a fact which was critical to our arguments in [5]).

For $\beta>0$, F_β consists of those analytic functions on $\mathbb{D}$ such that

$$\lim_{r\to 1^-}(1-r)^\beta\log^+\max_{|z|=r}|f(z)|=0\,.$$

For $f\in F_\beta$, $f(z)=\sum_{n=0}^\infty a_nz^n$, and $c>0$, the functional $\|\cdot\|_c$ defined by

$$\|f\|_c=\sum_{n=0}^\infty|a_n|\exp(-cn^{\beta/(1+\beta)})$$

is a seminorm on F_β. With the topology given by the family $\{\|\cdot\|_c\}_{c>0}$, F_β is a Fréchet space [15], [16], [18]. Yanagihara showed that F_1 is the containing Fréchet space for the Smirnov class [17] (see also [8]). For the general case, Stoll identified the likely candidates for the Fréchet envelopes of N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$ as the spaces $F_{1/\alpha}$ and $F_{2/\alpha}$ [15]; we verified this conjecture in [5]. Let us recall those results from [5] which we will need in §4.

THEOREM 3.1. *For* $\alpha\ge 1$, $F_{1/\alpha}$ *is the Frécht envelope of* N_+^α.

THEOREM 3.2. *For* $\alpha\ge 1$, $F_{2/\alpha}$ *is the Frécht envelope of* $\mathcal{N}_+^\alpha(\mathbb{D})$.

LEMMA 3.3. *Let* $f_k(z)=\exp[c_kr_kz(1-r_kz)^{-3}]$, $r_k,c_k>0$, *with Taylor expansion* $f_k(z)=\sum_{n=0}^\infty b_n^{(k)}z^n$. *Let* V *be any neighborhood of zero in* N_+^α. *Then there exist positive constants* a_1,a_2, *and* a_3 *so that if*

$$r_k=1-a_2k^{-\alpha/(\alpha+1)}\quad\textit{and}\quad c_k=a_3(1-r_k)^{(3\alpha-1)/\alpha},$$

then $a_1f_k\in V$; *moreover*, $(b_k^{(k)})^{-1}=O[\exp(-\eta k^{1/(\alpha+1)})]$ *for some* $\eta>0$.

The idea behind this family of test functions is that for each k, f_k is analytic in the disc $\{z:|z|<1/r_k\}$, with $1/r_k>1$, and thus belongs to both N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$, even though $f(z)=\exp[z(1-z)^{-3}]$ belongs to neither. (Clearly $f\notin F_{2/\alpha}$ and $N_+^\alpha\subseteq\mathcal{N}_+^\alpha(\mathbb{D})\subseteq F_{2/\alpha}$; see [15].) Now for N_+^α, it is straightforward to show that every metric neighborhood of zero contains a set of the form

$$G(r,\varepsilon)=G=\Big\{g\in N_+^\alpha:\int\limits_{-\pi}^{\pi}[\log^+|rg(e^{i\theta})|]^\alpha\,d\theta<\varepsilon\Big\}\quad\text{for some } r,\varepsilon>0\,.$$

For the family $\{f_k\}$, there exists a constant $M>0$ so that

$$\int\limits_{-\pi}^{\pi}[\log^+|f_k(e^{i\theta})|]^\alpha\,d\theta\le c_k^\alpha M(1-r_k)^{1-3\alpha}$$

(see [5], Lemma 3.1). Thus for any neighborhood, V, of zero in N_+^α, there exists $G(r,\varepsilon)=G\subseteq V$; by taking $c_k=M^{-1/\alpha}\varepsilon^{1/\alpha}(1-r_k)^{(3\alpha-1)/\alpha}$, we force the family $\{af_k\}$ to belong to G, for $a=\min\{r^{-1},1\}$. This will be true for any choice of $r_k\uparrow 1$. However, to obtain necessary decay estimates on the Taylor coefficients, we had to be rather judicious as to the choice of the r_k's (see [5], Lemmas 3.1 and 3.2, and Theorem 4.2). The same ideas go through for $\mathcal{N}_+^\alpha(\mathbb{D})$ ([5], Lemmas 3.1 and 3.3, and Theorem 4.3).

LEMMA 3.4. *Let* $f_k(z)=\exp[c_kr_kz(1-r_kz)^{-3}], r_k,c_k>0$, *with Taylor expansion* $f_k(z)=\sum_{n=0}^\infty b_n^{(k)}z^n$. *Let V be any neighborhood of zero in* $\mathcal{N}_+^\alpha(\mathbb{D})$. *Then there exist positive constants a_1,a_2, and a_3 so that if*

$$r_k=1-a_2k^{-\alpha/(\alpha+2)}\quad\textit{and}\quad c_k=a_3(1-r_k)^{(3\alpha-2)/\alpha}$$

then $a_1f_k\in V$; *moreover*, $(b_k^{(k)})^{-1}=O[\exp(-\eta_k^{2/(\alpha+2)})]$ *for some* $\eta>0$.

4. $\widehat{\mathbb{X}}=\widehat{\mathbb{X}}_p$: Examples. We will show that for $\widehat{\mathbb{X}}=N_+^\alpha$ or $\mathcal{N}_+^\alpha(\mathbb{D}),\alpha\ge 1$, we have $\widehat{\mathbb{X}}=\widehat{\mathbb{X}}_p$ for $0<p\le 1$. Our method of proof is somewhat similar to arguments used in [16], but draws on the theory of vector-valued analytic functions as developed in [6]. Also, certain estimates which we obtained in [5] are critical to our proofs. Our approach has the benefit of allowing for a characterization of multipliers from N_+^α or $\mathcal{N}_+^\alpha(\mathbb{D})$ into any p-Banach space (H^p, in particular), as well as a characterization of the dual spaces of N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$. We will omit the proofs for results particular to $\mathcal{N}_+^\alpha(\mathbb{D})$ since they parallel the corresponding arguments for N_+^α.

First, let us briefly recall some facts about vector-valued analytic functions and multipliers which we will need in the sequel. Let $(\mathbb{X},\|\ \|)$ be a p-Banach space. A function $f:\mathbb{D}\to\mathbb{X}$ is said to be *analytic* if f can be expanded in a power series $f(z)=\sum_{n=0}^\infty x_nz^n$ for $x_n\in\mathbb{X}$, $z\in\mathbb{D}$ (see [6]). Let $A(\mathbb{X})$ denote the collection of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, quasinormed by $\|f\|_A=\max\{\|f(z)\|:z\in\overline{\mathbb{D}}\}$. Say that $\Lambda=(x_n)$ is a *multiplier* from N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$) into $A(\mathbb{X})$ if for every $h\in N_+^\alpha$ (respectively, $\mathcal{N}_+^\alpha(\mathbb{D})$) with power series $h(z)=\sum_{n=0}^\infty d_nz^n$, we have $\Lambda h\in A(\mathbb{X})$, where $(\Lambda h)(z)=\sum_{n=0}^\infty x_nd_nz^n$. Since $[\log(1+|f(z)|)]^\alpha$ is subharmonic for $\alpha\ge 1$

it follows that for $z \in \mathbb{D}$, $|z| = r$, and $f \in N_+^\alpha$,

$$|f(z)| \le \exp\left\{ \left(\frac{1+r}{1-r}\right)^{1/\alpha} d_\alpha(f,0)\right\};$$

similarly, if $f \in \mathcal{N}_+^\alpha(\mathbb{D})$, then

$$|f(z)| \le \exp\left\{ \left(\frac{1+r}{1-r}\right)^{2/\alpha} \varrho_\alpha(f,0)\right\}$$

(see [15]). Consequently, if $f_k \to f$ in N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$) then by a standard normal family argument, $f_k \to f$ uniformly on compact subsets of $\mathbb{D}$. Thus if $f_k(z) = \sum_{n=0}^\infty b_n^{(k)} z^n$ and $f(z) = \sum_{n=0}^\infty b_n z^n$ then $b_n^{(k)} \to b_n$ as $k \to \infty$, for each $n = 0, 1, 2, \ldots$ It can be deduced from ([6], Theorem 6.1) that if $g \in A(\mathbb{X})$, $g(z) = \sum_{n=0}^\infty y_n z^n$, $y_n \in \mathbb{X}$, then $\|y_n\| \le Cn^\lambda \|g\|_A$ for some $\lambda, C > 0$. Thus if $g_k \to g$ in $A(\mathbb{X})$, with $g_k(z) = \sum_{n=0}^\infty y_n^{(k)} z^n$, then $y_n^{(k)} \to y_n$ as $k \to \infty$ for each $n = 0, 1, 2, \ldots$ It follows from the Closed Graph Theorem that if $\Lambda = (x_n)$ is a multiplier from N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$) into $A(\mathbb{X})$, then Λ is continuous.

LEMMA 4.1. *Let $f \in N_+^\alpha$ (or $\mathcal{N}_+^\alpha(\mathbb{D})$), and let $f_\zeta(z) = f(\zeta z)$ for $\zeta \in \mathbb{D}$. Then $(f_\zeta)_{\zeta\in\mathbb{D}}$ is a bounded set in N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$).*

Proof. Let $f \in N_+^\alpha$ (or $\mathcal{N}_+^\alpha(\mathbb{D})$), and let d denote either metric, d_α or ϱ_α. Recall that d is rotation-invariant; moreover, $\int_{-\pi}^{\pi} [\log(1 + |f(re^{i\theta})|)]^\alpha \, d\theta$ is an increasing function of r, because $[\log(1+|f|)]^\alpha$ is subharmonic [3]. Thus $d(f_r, 0) \le d(f, 0)$ for each r, $0 < r < 1$. Let V denote a d-neighborhood of zero and $\zeta = re^{i\theta} \in \mathbb{D}$. Since $d(f_\zeta, 0) = d(f_r, 0) \le d(f, 0)$ and scalar multiplication is continuous, there exists $a > 0$ so that $af \in V$, whereby $af_\zeta \in V$ for every $\zeta \in \mathbb{D}$; i.e., $(f_\zeta)_{\zeta\in\mathbb{D}}$ is a bounded set in N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$).

LEMMA 4.2. *Let $f \in N_+^\alpha$ (or $\mathcal{N}_+^\alpha(\mathbb{D})$), and $f_\zeta(z) = f(\zeta z)$ for $z \in \mathbb{D}$, $\zeta \in \overline{\mathbb{D}}$. If $z_n \to z_0 \in \overline{\mathbb{D}}$, then $f_{z_n} \to f_{z_0}$ in N_+^α (or $\mathcal{N}_+^\alpha(\mathbb{D})$).*

Proof. Put $F(z) = f_z$; $F : \overline{\mathbb{D}} \to N_+^\alpha$ ($\mathcal{N}_+^\alpha(\mathbb{D})$). We need only show F is continuous. For each $w \in \overline{\mathbb{D}}$ and $0 < r < 1$, if $z_n \to z_0$,

$$|f_r(z_n w)| \le \sup\{|f_r(\zeta)| : \zeta \in \overline{\mathbb{D}}\},$$

it follows by bounded convergence that $\lim_{n\to\infty} d(f_{rz_n}, f_{rz_0}) = 0$. Thus F_r is continuous for each r, $0 < r < 1$, where $F_r(z) = f_{rz}$. For any $z \in \overline{\mathbb{D}}$, $z = \varrho e^{i\theta}$, $0 \le \varrho \le 1$, we have

$$d(F_r(z), F(z)) = d(f_{rz}, f_z) = d(f_{r\varrho}, f_\varrho) \le d(f_r, f).$$

Since $d(f_r, f) \to 0$ as $r \to 1^-$ ([15]), $F_r \to F$ uniformly in z, whereby F is continuous.

PROPOSITION 4.3. *Let* $\mathbb{X}$ *be a p-Banach space*, $0 < p \le 1$. *A sequence* $\Lambda = (x_k)$, $x_k \in \mathbb{X}$, *is a multiplier from* N_+^α, $\alpha \ge 1$, *into* $A(\mathbb{X})$ *if and only if*

$$\|x_k\| = O[\exp(-\eta k^{1/(\alpha+1)})]$$

for some $\eta > 0$.

Proof. Suppose $\Lambda = (x_k)$ is a multiplier from N_+^α into $A(\mathbb{X})$. Λ is continuous, so there exists a neighborhood V of zero so that if $g \in V$, $g(z) = \sum_{n=0}^\infty a_n z^n$, then $\|\Lambda g\| \le 1$. Now $\Lambda g(z) = \sum_{n=0}^\infty a_n x_n z^n$; there exists $\lambda > 0$ so that for each $g \in V$ (cf. [6], Theorem 6.1)

$$\|x_n a_n\| \le Cn^\lambda \|\Lambda g\| \le Cn^\lambda\,,$$

so that

$$\|x_n\| \le Cn^\lambda |a_n|^{-1}\,.$$

Using Lemma 3.3, there exist $a > 0$, $r_k \uparrow 1$, and $c_k \downarrow 0$ so that $af_k \in V$ for all $k = 1, 2, 3, \ldots$, for $f_k(z) = \exp[c_k r_k z(1 - r_k z)^{-3}]$. Let f_k have Taylor series $\sum_{n=0}^\infty b_n^{(k)} z^n$; again, from Lemma 3.3, there exists $\eta_0 > 0$ such that

$$|b_k^{(k)}|^{-1} = O[\exp(-\eta_0 k^{1/(\alpha+1)})]\,;$$

whence it follows that

$$\|x_k\| \le Ck^\lambda |b_k^{(k)}|^{-1} = O[\exp(-\eta k^{1/(\alpha+1)})]\,,$$

for some η, $\eta_0 > \eta > 0$.

Now suppose that $(x_n) \subseteq \mathbb{X}$ and $\|x_k\| = O[\exp(-\eta k^{1/(\alpha+1)})]$ for some $\eta > 0$. It was shown in [15] that if $g \in N_+^\alpha$, with Taylor series $\sum_{n=0}^\infty a_n z^n$, then the Taylor coefficients of g satisfy

$$|a_n| \le M \exp[\eta_k n^{1/(\alpha+1)}]$$

for some constant $M > 0$ and sequence $\eta_k \downarrow 0$. Thus for $\Lambda g(z) = \sum_{n=0}^\infty x_n a_n z^n$, it follows that

$$\|\Lambda g\|^p \le \sum_{n=0}^\infty \|x_n\|^p |a_n|^p < \infty\,.$$

From this we deduce that $\sum_{n=0}^\infty a_n x_n z^n$ converges uniformly on $\overline{\mathbb{D}}$, whereby Λg is continuous on $\overline{\mathbb{D}}$, analytic in $\mathbb{D}$, i.e., $\Lambda g \in A(\mathbb{X})$. $\Lambda = (x_n)$ is therefore a multiplier from N_+^α into $A(\mathbb{X})$, and the proof is finished.

PROPOSITION 4.4. *Let* $\mathbb{X}$ *be a p-Banach space*, $0 < p \le 1$. *A sequence* $(x_k) \subseteq \mathbb{X}$ *is a multiplier from* $\mathcal{N}_+^\alpha(\mathbb{D})$, $\alpha \ge 1$, *into* $A(\mathbb{X})$ *if and only if*

$$\|x_k\| = O[\exp(-\eta k^{2/(\alpha+2)})]$$

for some $\eta > 0$.

Propositions 4.3 and 4.4 allow us to completely characterize continuous linear maps from N_+^α or $\mathcal{N}_+^\alpha(\mathbb{D})$ into any p-Banach space $\mathbb{X}$, $0 < p \le 1$. In the sequel, let e_n denote the function $e_n(z) = z^n$, for $n = 0, 1, \ldots$

Proposition 4.5. *Let $\mathbb{X}$ be a p-Banach space, $0 < p \le 1$. Let T be a linear map, $T : N_+^\alpha \to \mathbb{X}$, $\alpha \ge 1$, and $T(e_n) = x_n$. T is continuous if and only if for every $f \in N_+^\alpha$, with Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$,*

$$Tf = \sum_{n=0}^\infty a_n x_n\,;$$

moreover, $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$.

Proof. Let $T : N_+^\alpha \to \mathbb{X}$ be a continuous linear map. For $f \in N_+^\alpha$, $f(z) = \sum_{n=0}^\infty a_n z^n$, f is the uniform limit of its Taylor series on each disc $\{z : |z| \le r\}$ with $0 < r < 1$. Let $P_N(z) = \sum_{n=0}^N a_n z^n$, denote the Nth Taylor polynomial, and let $P_{\zeta,N}(z) = \sum_{n=0}^N \zeta^n a_n z^n$, for $|\zeta| < 1$. It follows easily that $\lim_{N\to\infty} d_\alpha(P_{\zeta,N}, f_\zeta) = 0$. Thus for each $\zeta \in \mathbb{D}$,

$$T(f_\zeta) = \lim_{N\to\infty} T(P_{\zeta,N}) = \sum_{n=0}^\infty \zeta^n a_n x_n\,.$$

Setting $F(\zeta) = T(f_\zeta)$, we can deduce from Lemma 4.2 that F is analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, i.e., $f \in A(\mathbb{X})$. Thus (x_n) is a multiplier from N_+^α into $A(\mathbb{X})$, whereby $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$, by Proposition 4.3. As in the proof of Proposition 4.3, it follows that $\sum_{n=0}^\infty a_n x_n \zeta^n$ converges uniformly on $\overline{\mathbb{D}}$, i.e., $\lim_{N\to\infty} T(P_{N,\zeta}) = T(f_\zeta)$, and the convergence is uniform in ζ, $\zeta \in \mathbb{D}$. Thus, since

$$\lim_{r\to 1^-}\lim_{N\to\infty} T(P_{N,r}) = \lim_{r\to 1^-} T(f_r) = T(f)\,,$$

we have

$$\lim_{N\to\infty}\lim_{r\to 1^-} T(P_{N,r}) = \lim_{N\to\infty}\sum_{n=0}^N a_n x_n = \sum_{n=0}^\infty a_n x_n = T(f)\,.$$

Next we suppose $T(e_n) = x_n$, with $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$, for some $\eta > 0$. From Proposition 4.3 we see that $\Lambda = (x_n)$ is a multiplier from N_+^α into $A(\mathbb{X})$. Recall that multipliers from N_+^α into $A(\mathbb{X})$ are continuous, so if $f_k \to f$ in N_+^α, then $\Lambda f_k \to \Lambda f$ in $A(\mathbb{X})$; i.e.,

$$\sup_{z\in\overline{\mathbb{D}}}\Big\|\sum_{n=0}^\infty a_n^{(k)} x_n z^n - \sum_{n=0}^\infty a_n x_n z^n\Big\| \to 0$$

as $k \to \infty$; in particular, for $z = 1$ we have $\|Tf_k - Tf\| \to 0$. T is therefore continuous.

PROPOSITION 4.6. *Let* $\mathbb{X}$ *be a p-Banach space*, $0 < p \leq 1$. *Let* T *be a linear map*, $T : \mathcal{N}_+^\alpha(\mathbb{D}) \to \mathbb{X}$, $\alpha \geq 1$, *and* $T(e_n) = x_n$. T *is continuous if and only if for every* $f \in \mathcal{N}_+^\alpha(\mathbb{D})$ *with Taylor series* $f(z) = \sum_{n=0}^\infty a_n z^n$,

$$Tf = \sum_{n=0}^\infty a_n x_n\,;$$

moreover, $\|x_n\| = O[\exp(-\eta n^{2/(\alpha+2)})]$ *for some* $\eta > 0$.

The argument from Proposition 4.5 and its counterpart for Proposition 4.6 yield straightforward characterizations of the dual spaces of N_+^α and $\mathcal{N}_+^\alpha(\mathbb{D})$. For convenience, let $\mathcal{A}$ denote those analytic functions on $\mathbb{D}$ which are also continuous on $\overline{\mathbb{D}}$.

PROPOSITION 4.7 (cf. [16], Theorem 3). *Let* $\phi \in (N_+^\alpha)^*$. *There is a unique* $g \in \mathcal{A}$, $g(z) = \sum_{n=0}^\infty b_n z^n$, *so that*

$$\phi(f) = \sum_{n=0}^\infty a_n b_n$$

for each $f \in N_+^\alpha$, *with Taylor series* $f(z) = \sum_{n=0}^\infty a_n z^n$. *The series* $\sum_{n=0}^\infty a_n b_n$ *converges absolutely. Moreover*, *the Taylor coefficients of g satisfy*

$$(*) \qquad |b_n| = O[\exp(-\eta n^{1/(\alpha+1)})]$$

for some $\eta > 0$. *Conversely*, *every* $g \in \mathcal{A}$ *whose Taylor coefficients* (b_n) *satisfy* $(*)$ *defines a continuous linear functional* ϕ_g *on* N_+^α.

Proof. Let $\phi \in (N_+^\alpha)^*$, and let $\phi(e_n) = b_n$. Proposition 4.5 implies that $\phi(f) = \sum_{n=0}^\infty a_n b_n$ for $f \in N_+^\alpha$ with Taylor series $\sum_{n=0}^\infty a_n z^n$. Moreover, $|b_n| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$, so that $g(z) = \sum_{n=0}^\infty b_n z^n$ converges uniformly and absolutely on $\overline{\mathbb{D}}$; thus $g \in \mathcal{A}$.

On the other hand, if $g \in \mathcal{A}$, $g(z) = \sum_{n=0}^\infty b_n z^n$, with b_n satisfying $(*)$, we may define

$$\phi_g(f) = \sum_{n=0}^\infty a_n b_n$$

for $f \in N_+^\alpha$, $f(z) = \sum_{n=0}^\infty a_n z^n$. Since $N_+^\alpha \subseteq F_{1/\alpha}$ ([15]), ϕ_g is a well-defined linear functional on N_+^α, and the series $\sum_{n=0}^\infty a_n b_n$ converges absolutely. Since (b_n) satisfies $(*)$, Proposition 4.5 implies that ϕ_g is continuous, and the proof is complete.

PROPOSITION 4.8. *Let* $\phi \in (\mathcal{N}_+^\alpha(\mathbb{D}))^*$. *There is a unique* $g \in \mathcal{A}$, $g(z) = \sum_{n=0}^\infty b_n z^n$, *so that*

$$\phi(f) = \sum_{n=0}^\infty a_n b_n$$

for each $f \in \mathcal{N}_+^\alpha(\mathbb{D})$, *with Taylor series* $f(z) = \sum_{n=0}^\infty a_n z^n$. *The series* $\sum_{n=0}^\infty a_n b_n$ *converges absolutely. Moreover, the Taylor coefficients of* g *satisfy*

$$(**) \qquad |b_n| = O[\exp(-\eta n^{2/(\alpha+2)})]$$

for some $\eta > 0$. *Conversely, every* $g \in \mathcal{A}$ *whose Taylor coefficients* (b_n) *satisfy* $(**)$ *defines a continuous linear functional* ϕ_g *on* $\mathcal{N}_+^\alpha(\mathbb{D})$.

Propositions 4.3 and 4.4 may be used to characterize multipliers from N_+^α or $\mathcal{N}_+^\alpha(\mathbb{D})$ into the Hardy spaces H^p, $0 < p$. Suppose for example that $\lambda_n \subseteq \mathbb{C}$ is a multiplier from N_+^α into H^p; since convergence in N_+^α or H^p implies uniform convergence on compact subsets of $\mathbb{D}$, it follows as a consequence of the Closed Graph Theorem that $\Lambda = (\lambda_n)$ is continuous. Propositions 4.3 and 4.4 yield the following (cf. [16], Theorem 2):

PROPOSITION 4.9. (i) $\Lambda = (\lambda_n)$ *is a multiplier from* N_+^α, $\alpha \ge 1$, *into* H^p, $0 < p$, *if and only if* $|\lambda_n| = O[\exp(-\eta n^{1/(\alpha+1)})]$ *for some* $\eta > 0$.

(ii) $\Lambda = (\lambda_n)$ *is a multiplier from* $\mathcal{N}_+^\alpha(\mathbb{D})$, $\alpha \ge 1$, *into* H^p, $0 < p$, *if and only if* $|\lambda_n| = O[\exp(-\eta n^{2/(\alpha+2)})]$ *for some* $\eta > 0$.

For an arbitrary F-space, $\mathbb{X}$, the topology induced by the p-envelope is stronger than that induced by the Fréchet envelope, $0 < p \le 1$. Let $\mathbb{X} = N_+^\alpha$ or $\mathcal{N}_+^\alpha(\mathbb{D})$, and $d = d_\alpha$ or ϱ_α. If we can show that the $\widehat{\mathbb{X}}$ topology is stronger than the $\widehat{\mathbb{X}}_p$ topology on $\mathbb{X}$, then necessarily $\widehat{\mathbb{X}} = \widehat{\mathbb{X}}_p$. Let V be a d-ball of radius $1/n$, $n = 1, 2, \ldots$, and let $\|\cdot\|_{p,n}$ be the Minkowski functional of the p-co V_n. Recall that the family $\{\|\cdot\|_{p,n}\}$ induces the $\widehat{\mathbb{X}}_p$ topology on $\mathbb{X}$. For $f \in \mathbb{X}$, if $\|f\|_{p,n} = 0$, then since $\|f\|_{p,n} \ge \|f\|_{1,n}$ it must follow that $f \equiv 0$. Thus each $\|\cdot\|_{p,n}$ is actually a p-norm on $\mathbb{X}$ and the completion of $\mathbb{X}$ with respect to $\|\cdot\|_{p,n}$ is a p-Banach space. This observation will be utilized in the proof of the following theorem.

THEOREM 4.10. *For* $0 < p \le 1$, *the* p-*envelope of* N_+^α, $\alpha \ge 1$, *is* $F_{1/\alpha}$.

Proof. Let $\|\cdot\|$ be any one of the p-norms $\|\cdot\|_{p,n}$, $n = 1, 2, \ldots$, and let $\mathbb{Y}$ be the completion of N_+^α with respect to $\|\cdot\|$. Let T be the natural inclusion map $T : N_+^\alpha \to \mathbb{Y}$; T is continuous and linear. From Proposition 4.5 we have $Tf = \sum_{n=0}^\infty a_n e_n$ for $f \in N_+^\alpha$, $f(z) = \sum_{n=0}^\infty a_n z^n$, and, in addition, $\|e_n\| \le M \exp(-\eta n^{1/(\alpha+1)})$ for some η, $M > 0$. Let $\eta_1, \eta_2 > 0$ be such that $\eta_1 + \eta_2 = \eta$ and let $q > 0$ be such that $p + q = 1$. (If $q = 0$, then the result is simply a restatement of Theorem 3.1.) For $f \in N_+^\alpha$, we have

$$\|Tf\|^p \le \sum_{n=0}^\infty |a|^p \|e_n\|^p \le \sum_{n=0}^\infty |a_n|^p [M \exp(-\eta n^{1/(\alpha+1)})]^p$$

$$= \sum_{n=0}^{\infty} |a_n|^p [M \exp(-\eta_1 n^{1/(\alpha+1)})]^p [\exp(-p\eta_2 n^{1/(\alpha+1)})]$$

$$\le \Big\{ M \sum_{n=0}^{\infty} |a_n| \exp(-\eta_1 n^{1/(\alpha+1)}) \Big\}^p \Big\{ \sum_{n=0}^{\infty} \exp\Big(-\frac{p}{q}\eta_2 n^{1/(\alpha+1)}\Big) \Big\}^q ;$$

consequently, for a constant $C > 0$,

$$\|Tf\| \le C \sum_{n=0}^{\infty} |a_n| \exp(-\eta_1 n^{1/(\alpha+1)}) = C\|f\|_{\eta_1} .$$

The $F_{1/\alpha}$ topology is therefore stronger than the p-envelope topology on N_+^α, and the proof is complete.

THEOREM 4.11. *For* $0 < p \le 1$, *the p-envelope of* $\mathcal{N}_+^\alpha(\mathbb{D})$, $\alpha \ge 1$, *is* $F_{2/\alpha}$.

References

[1] A. B. Aleksandrov, *Essays on non-locally convex Hardy classes*, in: Complex Analysis and Spectral Theory, Lecture Notes in Math. 864, Springer, Berlin 1981, 1–89.

[2] R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in* L^p, Astérisque 77 (1980), 11–66.

[3] P. L. Duren, *Theory of* H^p *Spaces*, Academic Press, New York 1970.

[4] P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on* H^p *spaces with* $0 < p < 1$, J. Reine Angew. Math. 238 (1969), 32–60.

[5] C. M. Eoff, *Fréchet envelopes of certain algebras of analytic functions*, Michigan Math. J. 35 (1988), 413–426.

[6] N. J. Kalton, *Analytic functions in non-locally convex spaces and applications*, Studia Math. 83 (1986), 275–303.

[7] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, 1984.

[8] J. E. McCarthy, *Topologies on the Smirnov class*, to appear.

[9] N. Mochizuki, *Algebras of holomorphic functions between* H^p *and* N_*, Proc. Amer. Math. Soc. 105 (1989), 898–902.

[10] M. Nawrocki, *Linear functionals on the Smirnov class of the unit ball in* $\mathbb{C}^n$, Ann. Acad. Sci. Fenn. AI 14 (1989), 369–379.

[11] —, *The Fréchet envelopes of vector-valued Smirnov classes*, Studia Math. 94 (1989), 163–177.

[12] J. W. Roberts and M. Stoll, *Prime and principal ideals in the algebra* N^+, Arch. Math. (Basel) 27 (1976), 387–393; Correction, ibid. 30 (1978), 672.

[13] J. H. Shapiro, *Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J. 43 (1976), 187–202.

[14] —, *Remarks on F-spaces of analytic functions*, in: Banach Spaces of Analytic Functions, Lecture Notes in Math. 604, Springer, Berlin 1977, 107–124.

[15] M. Stoll, *Mean growth and Taylor coefficients of some topological algebras of analytic functions*, Ann. Polon. Math. 35 (1977), 139–158.

[16] N. Yanagihara, *Multipliers and linear functionals for the class N^+*, Trans. Amer. Math. Soc. 180 (1973), 449–461.
[17] —, *The containing Fréchet space for the class N^+*, Duke Math. J. 40 (1973), 93–103.
[18] A. I. Zayed, *Topological vector spaces of analytic functions*, Complex Variables 2 (1983), 27–50.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, ARKANSAS 72701
U.S.A.

Reçu par la Rédaction le 22.10.1990