Abstract. The $p$-envelope of an $F$-space is the $p$-convex analogue of the Fréchet envelope. We show that if an $F$-space is locally bounded (i.e., a quasi-Banach space) with separating dual, then the $p$-envelope coincides with the Banach envelope only if the space is already locally convex. By contrast, we give examples of $F$-spaces with are not locally bounded nor locally convex for which the $p$-envelope and the Fréchet envelope are the same.

1. Introduction. For a non-locally convex $F$-space $X$ (complete, metrizable, linear topological space), the idea of a $p$-envelope is analogous to that of a Fréchet envelope. Suppose $X$ has separating dual space; recall that the Fréchet envelope of $X$, denoted by $\hat{X}$, is the closure of $X$ with respect to the Mackey topology, $\mu$. The Mackey topology is the strongest locally convex topology on $X$ for which $X$ still has dual space $X^*$. A countable base for the $\mu$-zero neighborhoods $\{\tilde{V}_n\}$ can be obtained by taking the closure in $X$ of the absolutely convex hull of each $V_n$, where $\{V_n\}$ is any countable base for the zero-neighborhoods of $X$; this description in fact characterizes $\mu$ [13]. In general $\hat{X}$ is a Fréchet space; for a locally bounded $F$-space, $\hat{X}$ turns out to be a Banach space—the Banach envelope. ($S \subset X$ is bounded if given any zero neighborhood $U$, there exists $n \in \mathbb{N}$ such that $S \subset nU$. $X$ is locally bounded if it has a bounded neighborhood of zero.)

Interest in the containing Fréchet space of a non-locally convex $F$-space was first sparked by the pioneering work of Duren, Romberg, and Shields, who showed that the Hardy space $H^p$, $0 < p < 1$, could be densely imbedded in a certain Banach space, the Bergman space $B^p$, and $(H^p)^* \simeq (B^p)^*$ [4]. Somewhat later Shapiro identified the Banach envelope of $H^p$ directly, using his “convex-hull” characterization of the Mackey topology [13]. This characterization of the Mackey topology provides an important intuitive

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picture, via the example $\ell_p, 0 < p < 1$. The absolutely convex hull of the $\ell_p$ unit ball is the $\ell_1$ unit ball. Thus, the Mackey topology is the $\ell_1$ topology and the closure of $\ell_p$, with respect to this topology is $\ell_1$; i.e., $\ell_1$ is the Banach envelope of $\ell_p$. With its usual metric $d((\alpha_n), 0) = \|(\alpha_n)\|_p = \sum_{n=0}^{\infty} |\alpha_n|^p$, the sequence space $\ell_p, 0 < p < 1$, is the prototypical example of a non-locally convex, locally bounded $F$-space with separating dual (the maps $\phi_n((\alpha_n)) = \alpha_n$ are continuous). In addition, the topology induced by $d$ is $p$-convex, since the unit ball is (absolutely) $p$-convex. A set $C$ is $p$-convex if $\sum_{i=1}^{n} a_i x_i \in C$ whenever $x_i \in C$ and $\sum_{i=1}^{n} a_i = 1$, with $a_i \geq 0$. $C$ is absolutely $p$-convex if $\sum_{i=1}^{n} a_i x_i \in C$ whenever $x_i \in C$ and $\sum_{i=1}^{n} |a_i|^p = 1$. The functional $\|(\alpha_n)\|_p = (\sum_{n=0}^{\infty} |\alpha_n|^p)^{1/p}$ is a quasinorm; i.e., it satisfies the requirements for a norm except that the triangle inequality is weakened. For $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$,

$$\|\alpha + \beta\|_p \leq M(\|\alpha\|_p + \|\beta\|_p)$$

for a constant $M \geq 1$. Clearly, $\|\cdot\|_p$ satisfies

$$\|\alpha + \beta\|_p^p \leq \|\alpha\|_p^p + \|\beta\|_p^p;$$

a quasinorm with this property is said to be $p$-subadditive and is called a $p$-norm. In general, if an $F$-space, $X$, is locally bounded, the metric topology can always be replaced by a quasinorm, in fact by a $q$-norm for some $0 < q \leq 1$, due to a result of Aoki and Rolewicz; $X$ is then called a $q$-Banach space. (See [7] or [14] for general facts about non-locally convex $F$-spaces.)

By analogy with the Fréchet envelope, let $\{V_a\}$ be a countable base for the zero neighborhoods of a non-locally convex $F$-space, $X$, with separating dual, and let $\tilde{V}_a$ be the absolutely $p$-convex hull of $V_a$, for some fixed $p$, $0 < p < 1$. Let $\|\cdot\|_n$ be the Minkowski functional of $\tilde{V}_a$. For $x, y \in X$, the functional $\|\cdot\|_n$ satisfies:

(i) $\|x\|_n = 0$ if $x = 0$,
(ii) $\|ax\|_n = |a|\|x\|_n, \ a \in \mathbb{C},$
(iii) $\|x + y\|_n^p \leq \|x\|_n^p + \|y\|_n^p.$

From (iii) we can deduce that $\|x + y\|_n \leq C(\|x\|_n + \|y\|_n).$ By obvious analogy, we will refer to $\|\cdot\|_n$ as a $p$-seminorm. The family $\{\|\cdot\|_n\}$ generates a $p$-convex topology on $X$ weaker than the original topology. We call the closure of $X$ under the topology induced by $\{\|\cdot\|_n\}$ the $p$-envelope of $X$ and denote it by $\tilde{X}_p$ (cf. [1]). When $X$ is locally bounded, $\tilde{X}_p$ is a $p$-Banach space. $\tilde{X}_p$ has the property that every continuous linear map $T : X \to Y$, $Y$ a $p$-Banach space, extends continuously to $\tilde{X}_p$.

To visualize the situation, let $0 < p < q \leq 1$. The absolutely $q$-convex hull of the unit ball of $\ell_p$ is the $\ell_q$ unit ball, and it follows that $\ell_q$ is the $q$-envelope of $\ell_p$. (For $0 < p < q < 1$, the $q$-envelope of $H^p$ was identified
by Aleksandrov in [1] and by Coifman and Rochberg in [2].) Now for $0 < p < q < 1$, $\ell_q$ is not isomorphic to $\ell_p$; however, it can happen that $\tilde{X}_p$ is isomorphic to $\tilde{X}_q$ for all $0 < p, q < 1$ (see [7], Chapter 2). However, as we shall prove, the $p$-envelope, for $0 < p < 1$, can never be isomorphic to the Fréchet (Banach) envelope of a locally bounded, non-locally convex $F$-space (quasi-Banach space). We accomplish this in §2 by a modification of an argument of Kalton ([7], Theorem 4.13).

For an $F$-space which is not locally bounded, the situation is much different. We provide a class of examples which have the property that $\tilde{X}_p = \tilde{X}$ for $0 < p < 1$. The groundwork is laid in §3; proofs are carried out in §4. Our method of proof will yield various applications along the way.

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2. $\tilde{X}$ is never isomorphic to $\tilde{X}_p$ for a quasi-Banach space $X$.
In this section we shall prove that $\tilde{X}_1 = \tilde{X}$ (the Banach envelope) is never isomorphic to $\tilde{X}_p$, $0 < p < 1$, when $X$ is a non-locally convex quasi-Banach space.

The Aoki–Rolewicz theorem provides every quasi-Banach space with an equivalent $p$-norm for some $p$, $0 < p \leq 1$. Thus we lose no generality by our formulation of the following proposition.

**Proposition 2.1.** Let $(X, \| \cdot \|_X)$, $(Y, \| \cdot \|_Y)$ be quasi-Banach spaces so that $\| \cdot \|_X$ is an $r$-norm and $\| \cdot \|_Y$ is a $q$-norm for $0 < r < q \leq 1$. Let $B_X = \{ x \in X : \| x \|_X < 1 \}$. If $T : X \to Y$ is a bounded linear map so that $p\text{-co}T(B_X)$ is a neighborhood of the origin for $0 < r \leq p < q \leq 1$, then $T$ is an open map.

**Proof** (cf. [7], Theorem 4.13). For convenience, let $\| \cdot \|$ denote the quasinorms for both $X$ and $Y$, as well as the operator quasinorm for $T$. No confusion should arise from this. We assume, with no loss, that $\| T \| = 1$.

There exists $\delta > 0$ so that if $\| y \| < \delta$ then $y \in p\text{-co}T(B_X)$. It is enough to show that a constant $M$ exists so that if $\| y \| < 1$, there is an $x \in X$ with $\| x \| \leq M$ and $\| Tx - y \| < 1/2$. If this can be done, then we can choose $x_n$ by induction satisfying $\| x_n \| \leq 2^{-n}M$, $n = 0, 1, \ldots$, with $\| T(x_0 + \ldots + x_n) - y \| \leq 2^{-n}$. Then we would have $T(\sum_{n=0}^{\infty} x_n) = y$; the series $\sum_{n=0}^{\infty} x_n$ converges since $\sum_{n=0}^{\infty} \| x_n \| < \infty$.

So let $V_m = \{ \sum_{i=1}^{m} a_i T(x_i) : \sum_{i=1}^{m} a_i^p \leq 1, a_i \geq 0, \| x_i \| \leq 1 \}$ and note that $\bigcup_{m=1}^{\infty} V_m = p\text{-co}T(B_X)$. For any $w \in V_{2m}$, $w = \sum_{i=1}^{2m} a_i T x_i$, where we
label the $a_i$’s so that $a_{i-1} \geq a_i$, $i = \ldots$. Put $w_0 = \sum_{i=1}^{2m} a_i T(x_i)$, 
$w_0 \in V_m$. Notice that $a_i \leq (1/(2m))^{1/p}$ for $2m \geq i \geq m$; whereby,

$$\|w - w_0\|^q = \left\| \sum_{i=m+1}^{2m} a_i T(x_i) \right\|^q \leq \sum_{i=m+1}^{2m} |a_i|^q \|T(x_i)\|^q$$

$$\leq m \left( \frac{1}{2m} \right)^{q/p} = C_1 m^{-\alpha},$$

with $C_1 = 2^{-q/p}$, $\alpha = q/p - 1 > 0$.

For $w \in V_{2m+n}$, $w = \sum_{i=1}^{2m+n} a_i T(x_i)$, with $\sum_{i=1}^{2m+n} a_i^p \leq 1$, put

$$w_j = \sum_{i=1}^{2m+n} a_i T(x_i) \in V_{2m+j}, \quad j = 0, \ldots, n;$$

then $w_n = w$. From our previous observation, we deduce that

$$\|w - w_0\|^q \leq \sum_{j=1}^{n} \|w_j - w_{j-1}\|^q \leq \sum_{j=1}^{n} C_1(2^{m+j})^{-\alpha}$$

$$= C_1 2^{-ma} \sum_{j=1}^{n} 2^{-ja} \leq C_1 2^{-ma} \sum_{j=1}^{\infty} 2^{-ja} = C_2 2^{-ma},$$

with $C_2 = C_1(2^\alpha - 1)^{-1}$. Thus for $w \in V_{2m+n}$

$$\text{dist}(w, V_{2m}) = \inf_{y \in V_{2m}} \|w - y\| \leq \|w - w_0\| \leq C_2 1/2^{-m\beta},$$

independent of $n$, with $\beta = 1/p - 1/q > 0$. In particular, we can choose $m_0$ so large that if $w \in \bigcup_{n=1}^{\infty} V_n$, then

$$\text{dist}(w, V_{2m_0}) < \delta/(4C),$$

where $C$ is the quasinorm constant for $Y$. Put $2^{m_0} = N$. If $\|y\| < 1$, there exists $z \in \bigcup_{n=1}^{\infty} V_n$ so that $\|y - z\| < \delta/(4C)$. Let $v \in V_N$; we have

$$\|y - v\| \leq C(\|y - z\| + \|z - v\|) < \delta/2,$$

i.e., $\|y - \delta^{-1} v\| < 1/2$. Now $v = \sum_{n=1}^{N} a_n T(x_i)$, for $\sum_{n=1}^{N} a_n^p \leq 1$; put $x = \delta^{-1} \sum_{n=1}^{N} a_n x_i$, so that we obtain

$$\|y - Tx\| < 1/2 \quad \text{and} \quad \|x\| \leq N^{1/r} \delta^{-1} = M.$$

This completes the proof.

**Theorem 2.2.** Let $X$ be a locally bounded $F$-space which is $r$-normable for $0 < r < 1$. If $\hat{X}_p$ is locally $q$-convex for $0 < q \leq p \leq 1$, then $X$ is necessarily $q$-convex.

**Proof.** Let $j : X \rightarrow \hat{X}_p$ be the natural inclusion map, so that $p \overline{\text{ord}} j(Bx)$ is the closed unit ball of $\hat{X}_p$. If $\hat{X}_p$ can be endowed with an equivalent
$q$-convex topology, it follows from Proposition 2.1 that $j$ is an open map; consequently, $X = \hat{X}_p$, so that $X$ must be $q$-convex.

**Corollary 2.3.** Let $X$ be a quasi-Banach space such that $\hat{X}_p, 0 < p < 1$, is locally convex. Then $X$ is locally convex; i.e., $X$ is a Banach space.

**3. The classes $N^+_\alpha$ and $N^+_\alpha(D)$.** Let $D$ denote the unit disc in the complex plane, $\mathbb{C}$. Recall that a function analytic in the unit disc is said to be of bounded characteristic, or of Nevanlinna class $N$, if the integrals

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

are uniformly bounded for $r < 1$. For each function $f \in N$, the nontangential limit $f(e^{i\theta})$ exists for a.e. $\theta \in [-\pi, \pi]$; if a function $f \in N$ further satisfies the condition that

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| \, d\theta$$

then $f$ belongs to the Smirnov class $N^+$ [3]. $N^+$ has been studied for many years as part of the classical Hardy space theory ([3] is a good general reference), although it was not until the early 70’s that N. Yanagihara investigated the linear topological structure of $N^+$ [16], [17]. He found $N^+$ to be an $F$-space, not locally convex nor locally bounded, but still possessing a rich dual space, which he identified. Recently, McCarthy [8] has taken a different approach to the study of $N^+$, obtaining new results as well as giving new proofs to certain of Yanagihara’s results. The structure of $N^+$ as a topological algebra has been studied in [12], for example. Generalizations of Yanagihara’s work to $\mathbb{C}^n$, and even to Banach space valued functions have been carried out by Nawrocki [10], [11].

For $\alpha \geq 1$, define $N^+_{\alpha}$ to consist of those functions $f$ belonging to $N^+$ such that

$$\int_{-\pi}^{\pi} |\log^+ |f(e^{i\theta})||^\alpha \, d\theta < \infty.$$ 

Also, define $N^+_\alpha(D)$ to be the class of functions analytic in the unit disc which satisfy

$$\int_{D} |\log^+ |f(z)||^\alpha \, dA(z) < \infty,$$

where $dA$ is normalized area measure. The classes $N^+_\alpha$ and $N^+_\alpha(D)$ were introduced by M. Stoll in [15] (with different notation), where he showed that they are non-locally convex $F$-spaces under their respective metrics, in fact, $F$-algebras. Also, like $N^+$, both classes have separating dual spaces.
since point evaluations are continuous. Further results about the algebraic structure of \( N_+^\alpha \) and \( N_+^\alpha(\mathbb{D}) \) have been obtained recently by Mochizuki in [9].

The natural metric for \( N_+^\alpha \) is
\[
d_{\alpha}(f, 0) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \| \log(1 + |f(e^{i\theta})|) \|^\alpha \, d\theta \right\}^{1/\alpha},
\]
and in similar fashion, for \( N_+^\alpha(\mathbb{D}) \) the natural metric is
\[
g_{\alpha}(f, 0) = \left\{ \int_{\mathbb{D}} \| \log(1 + |f(z)|) \|^\alpha \, dA(z) \right\}^{1/\alpha}
\]
(see [15]). These metrics are rotation-invariant (a fact which was critical to our arguments in [5]).

For \( \beta > 0 \), \( F_\beta \) consists of those analytic functions on \( \mathbb{D} \) such that
\[
\lim_{r \to 1^-} (1 - r)^\beta \log^+ \max_{|z|=r} |f(z)| = 0.
\]
For \( f \in F_\beta \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and \( c > 0 \), the functional \( \| \cdot \|_c \) defined by
\[
\|f\|_c = \sum_{n=0}^{\infty} |a_n| \exp(-cn^{\beta/(1+\beta)})
\]
is a seminorm on \( F_\beta \). With the topology given by the family \( \{ \| \cdot \|_c \}_{c>0} \), \( F_\beta \) is a Fréchet space [15], [16], [18]. Yanagihara showed that \( F_1 \) is the containing Fréchet space for the Smirnov class [17] (see also [8]). For the general case, Stoll identified the likely candidates for the Fréchet envelopes of \( N_+^\alpha \) and \( N_+^\alpha(\mathbb{D}) \) as the spaces \( F_{1/\alpha} \) and \( F_{2/\alpha} \) [15]; we verified this conjecture in [5]. Let us recall those results from [5] which we will need in §4.

**Theorem 3.1.** For \( \alpha \geq 1 \), \( F_{1/\alpha} \) is the Fréchet envelope of \( N_+^\alpha \).

**Theorem 3.2.** For \( \alpha \geq 1 \), \( F_{2/\alpha} \) is the Fréchet envelope of \( N_+^\alpha(\mathbb{D}) \).

**Lemma 3.3.** Let \( f_k(z) = \exp[c_k r_k z (1 - r_k z)^{-3}] \), \( r_k, c_k > 0 \), with Taylor expansion \( f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n \). Let \( V \) be any neighborhood of zero in \( N_+^\alpha \). Then there exist positive constants \( a_1, a_2, \) and \( a_3 \) so that if
\[
r_k = 1 - a_2 k^{-\alpha/(\alpha+1)} \quad \text{and} \quad c_k = a_3 (1 - r_k)^{(3\alpha - 1)/\alpha},
\]
then \( a_1 f_k \in V \); moreover, \( (b_k^{(k)})^{-1} = O[\exp(-\eta k^{1/(\alpha+1)})] \) for some \( \eta > 0 \).

The idea behind this family of test functions is that for each \( k \), \( f_k \) is analytic in the disc \( \{ z : |z| < 1/r_k \} \), with \( 1/r_k > 1 \), and thus belongs to both \( N_+^\alpha \) and \( N_+^\alpha(\mathbb{D}) \), even though \( f(z) = \exp[z(1 - z)^{-3}] \) belongs to neither. (Clearly \( f \not\in F_{2/\alpha} \) and \( N_+^\alpha \subseteq N_+^\alpha(\mathbb{D}) \subseteq F_{2/\alpha} \); see [15].) Now for \( N_+^\alpha \), it is straightforward to show that every metric neighborhood of zero contains a
set of the form
\[ G(r, \varepsilon) = G = \left\{ g \in N^\alpha_0 : \int_{-\pi}^{\pi} |\log^+ |rg(e^{i\theta})||^\alpha \, d\theta < \varepsilon \right\} \quad \text{for some } r, \varepsilon > 0. \]

For the family \{f_k\}, there exists a constant \( M > 0 \) so that
\[ \int_{-\pi}^{\pi} |\log^+ |f_k(e^{i\theta})||^\alpha \, d\theta \leq c_k^2 M(1 - r_k)^{1 - 3\alpha} \]
(see [5], Lemma 3.1). Thus for any neighborhood, \( V \), of zero in \( N^\alpha_0 \), there exists \( G(r, \varepsilon) = G \subseteq V \); by taking \( c_k = M^{-1/\alpha} e^{1/\alpha}(1 - r_k)^{(3\alpha - 1)/\alpha} \), we force the family \( \{a f_k\} \) to belong to \( G \), for \( a = \min\{r^{-1}, 1\} \). This will be true for any choice of \( r_k \uparrow 1 \). However, to obtain necessary decay estimates on the Taylor coefficients, we had to be rather judicious as to the choice of the \( r_k \)'s (see [5], Lemmas 3.1 and 3.2, and Theorem 4.2). The same ideas go through for \( N^\alpha_+ (\mathbb{D}) \) ([5], Lemmas 3.1 and 3.3, and Theorem 4.3).

**Lemma 3.4.** Let \( f_k(z) = \exp[c_k r_k z (1 - r_k z)^{-3}] \), \( r_k, c_k > 0 \), with Taylor expansion \( f_k(z) = \sum_{n=0}^{\infty} h(k)^n z^n \). Let \( V \) be any neighborhood of zero in \( N^\alpha_+ (\mathbb{D}) \). Then there exist positive constants \( a_1, a_2 \), and \( a_3 \) so that if
\[ r_k = 1 - a_2 k^{-\alpha/(\alpha + 2)} \quad \text{and} \quad c_k = a_3 (1 - r_k)^{(3\alpha - 2)/\alpha} \]
then \( a_1 f_k \in V \); moreover, \( (h(k))^{-1} = O[\exp(-\eta_k^{2/(\alpha + 2)})] \) for some \( \eta > 0 \).

**4. \( \hat{X} = \hat{X}_p \): Examples.** We will show that for \( \hat{X} = N^\alpha_0 \) or \( N^\alpha_+ (\mathbb{D}) \), \( \alpha \geq 1 \), we have \( \hat{X} = \hat{X}_p \) for \( 0 < p \leq 1 \). Our method of proof is somewhat similar to arguments used in [16], but draws on the theory of vector-valued analytic functions as developed in [6]. Also, certain estimates which we obtained in [5] are critical to our proofs. Our approach has the benefit of allowing for a characterization of multipliers from \( N^\alpha_0 \) or \( N^\alpha_+ (\mathbb{D}) \) into any \( p \)-Banach space \( (H^p, \text{in particular}) \), as well as a characterization of the dual spaces of \( N^\alpha_0 \) and \( N^\alpha_+ (\mathbb{D}) \). We will omit the proofs for results particular to \( N^\alpha_+ (\mathbb{D}) \) since they parallel the corresponding arguments for \( N^\alpha_0 \).

First, let us briefly recall some facts about vector-valued analytic functions and multipliers which we will need in the sequel. Let \( (\mathbb{X}, \Vert \cdot \Vert) \) be a \( p \)-Banach space. A function \( f : \mathbb{D} \to \mathbb{X} \) is said to be analytic if \( f \) can be expanded in a power series \( f(z) = \sum_{n=0}^{\infty} x_n z^n \) for \( x_n \in \mathbb{X}, z \in \mathbb{D} \) (see [6]). Let \( A(\mathbb{X}) \) denote the collection of functions analytic in \( \mathbb{D} \) and continuous on \( \overline{\mathbb{D}} \), quasinormed by \( \| f \|_A = \max \{ \| f(z) \| : z \in \overline{\mathbb{D}} \} \). Say that \( A = (x_n) \) is a multiplier from \( N^\alpha_0 \) (or \( N^\alpha_+ (\mathbb{D}) \)) into \( A(\mathbb{X}) \) if for every \( h \in N^\alpha_0 \) (respectively, \( N^\alpha_+ (\mathbb{D}) \)) with power series \( h(z) = \sum_{n=0}^{\infty} d_n z^n \), we have \( Ah \in A(\mathbb{X}) \), where \( (Ah)(z) = \sum_{n=0}^{\infty} x_n d_n z^n \). Since \( \log(1 + |f(z)|)^\alpha \) is subharmonic for \( \alpha \geq 1 \)
it follows that for \( z \in \mathbb{D} \), \( |z| = r \), and \( f \in N_+^0 \),
\[
|f(z)| \leq \exp\left(\frac{(1+r)}{1-r} \alpha \right) d_\alpha(f,0);
\]
similarly, if \( f \in N_+^0(\mathbb{D}) \), then
\[
|f(z)| \leq \exp\left(\frac{(1+r)}{1-r} \right)^{2/\alpha} g_\alpha(f,0)
\]
(see [15]). Consequently, if \( f_k \to f \) in \( N_+^0 \) (or \( N_+^0(\mathbb{D}) \)) then by a standard normal family argument, \( f_k \to f \) uniformly on compact subsets of \( \mathbb{D} \). Thus if \( f_k(z) = \sum_{n=0}^\infty b_n^{(k)} z^n \) and \( f(z) = \sum_{n=0}^\infty b_n z^n \) then \( b_n^{(k)} \to b_n \) as \( k \to \infty \), for each \( n = 0,1,2,\ldots \). It can be deduced from ([6], Theorem 6.1) that if \( g \in A(\mathbb{X}) \), \( g(z) = \sum_{n=0}^\infty y_n z^n \), \( y_n \in \mathbb{X} \), then \( ||y_n|| \leq C n^\lambda \|g\|_\lambda \) for some \( \lambda, C > 0 \). Thus if \( g_k \to g \) in \( A(\mathbb{X}) \), with \( g_k(z) = \sum_{n=0}^\infty y_n^{(k)} z^n \), then \( y_n^{(k)} \to y_n \) as \( k \to \infty \) for each \( n = 0,1,2,\ldots \). It follows from the Closed Graph Theorem that if \( \Lambda = (x_n) \) is a multiplier from \( N_+^0 \) (or \( N_+^0(\mathbb{D}) \)) into \( A(\mathbb{X}) \), then \( \Lambda \) is continuous.

**Lemma 4.1.** Let \( f \in N_+^0 \) (or \( N_+^0(\mathbb{D}) \)), and let \( f_\zeta(z) = f(\zeta z) \) for \( \zeta \in \mathbb{D} \). Then \( (f_\zeta)_{\zeta \in \mathbb{D}} \) is a bounded set in \( N_+^0 \) (or \( N_+^0(\mathbb{D}) \)).

**Proof.** Let \( f \in N_+^0 \) (or \( N_+^0(\mathbb{D}) \)), and let \( d \) denote either metric, \( d_\alpha \) or \( g_\alpha \). Recall that \( d \) is rotation-invariant; moreover, \( \int_0^\infty [\log(1+|f(re^{i\theta})|)]^\alpha d\theta \) is an increasing function of \( \alpha \), because \( \log(1+|f|)^\alpha \) is subharmonic [3]. Thus \( d(f_r,0) \leq d(f,0) \) for each \( r,0 < r < 1 \). Let \( V \) denote a \( d \)-neighborhood of zero and \( \zeta = re^{i\theta} \in \mathbb{D} \). Since \( d(f_\zeta,0) = d(f_r,0) \leq d(f,0) \) and scalar multiplication is continuous, there exists \( a > 0 \) so that \( af \in V \), whereby \( af \in V \) for every \( \zeta \in \mathbb{D} \); i.e., \( (f_\zeta)_{\zeta \in \mathbb{D}} \) is a bounded set in \( N_+^0 \) (or \( N_+^0(\mathbb{D}) \)).

**Lemma 4.2.** Let \( f \in N_+^0 \) (or \( N_+^0(\mathbb{D}) \)), and \( f(\zeta z) = f(\zeta \bar{z}) \) for \( z \in \mathbb{D} \), \( \zeta \in \mathbb{D} \). If \( z_n \to z_0 \in \overline{\mathbb{D}} \), then \( f_{z_n} \to f_{z_0} \) in \( N_+^0 \) (or \( N_+^0(\mathbb{D}) \)).

**Proof.** Put \( F(z) = f_1 : \overline{\mathbb{D}} \to N_+^{0} \). We need only show \( F \) is continuous. For each \( w \in \overline{\mathbb{D}} \) and \( 0 < r < 1 \), if \( z_n \to z_0 \),
\[
|f_r(z_n,w)| \leq \sup \{|f_r(\zeta) : \zeta \in \overline{\mathbb{D}}\},
\]
it follows by bounded convergence that \( \lim_{n \to \infty} d(f_{rz_n},hf_{r_0}) = 0 \). Thus \( F_r \) is continuous for each \( r,0 < r < 1 \), where \( F_r(z) = f_{rz} \). For any \( z \in \overline{\mathbb{D}} \), \( z = ge^{i\theta}, 0 \leq g \leq 1 \), we have
\[
d(f_r(z),F(z)) = d(f_{rz},f) = d(f_{r_0},f) \leq d(f_r, f).
\]
Since \( d(f_r,f) \to 0 \) as \( r \to 1^- \) ([15]), \( F_r \to F \) uniformly in \( z \), whereby \( F \) is continuous.
Using Lemma 3.3, there exist a continuous, so there exists a neighborhood $V$, all $\eta > 0$.

**Proof.** Suppose $A = (x_k)$ is a multiplier from $\mathcal{N}^\alpha$ into $A(\mathbb{X})$. $A$ is continuous, so there exists a neighborhood $V$ of zero so that if $g \in V$, $g(z) = \sum_{n=0}^\infty a_n z^n$, then $\|Ag\| \leq 1$. Now $Ag(z) = \sum_{n=0}^\infty a_n x_n z^n$; there exists $\lambda > 0$ so that for each $g \in V$ (cf. [6], Theorem 6.1)

$$\|x_n a_n\| \leq Cn^\lambda \|Ag\| \leq Cn^\lambda,$$

so that

$$\|x_n\| \leq Cn^\lambda |a_n|^{-1}.$$ 

Using Lemma 3.3, there exist $a > 0$, $r_k \uparrow 1$, and $c_k \downarrow 0$ so that $a f_k \in V$ for all $k = 1, 2, 3, \ldots$, for $f_k(z) = \exp(c_k r_k z - r_k z^2)$. Let $f_k$ have Taylor series $\sum_{n=0}^\infty b_n^{(k)} z^n$; again, from Lemma 3.3, there exists $\eta_0 > 0$ such that

$$|b_n^{(k)}|^{-1} = O[\exp(-\eta n^{1/(\alpha+1)})];$$

whence it follows that

$$\|x_k\| \leq C k^\lambda |b_n^{(k)}|^{-1} = O[\exp(-\eta k^{1/(\alpha+1)})],$$

for some $\eta, \eta_0 > \eta > 0$.

Now suppose that $(x_n) \subseteq \mathbb{X}$ and $\|x_k\| = O[\exp(-\eta k^{1/(\alpha+1)})]$ for some $\eta > 0$. It was shown in [15] that if $g \in \mathcal{N}^\alpha$, with Taylor series $\sum_{n=0}^\infty a_n z^n$, then the Taylor coefficients of $g$ satisfy

$$|a_n| \leq M \exp[\eta n^{1/(\alpha+1)}]$$

for some constant $M > 0$ and sequence $\eta_k \downarrow 0$.

Thus for $Ag(z) = \sum_{n=0}^\infty x_n a_n z^n$, it follows that

$$\|Ag\|^p \leq \sum_{n=0}^\infty \|x_n\|^p |a_n|^p < \infty.$$ 

From this we deduce that $\sum_{n=0}^\infty a_n x_n z^n$ converges uniformly on $\overline{D}$, whereby $Ag$ is continuous on $\overline{D}$, analytic in $D$, i.e., $Ag \in A(\mathbb{X})$. $A = (x_n)$ is therefore a multiplier from $\mathcal{N}^\alpha$ into $A(\mathbb{X})$, and the proof is finished.

**Proposition 4.4.** Let $\mathbb{X}$ be a $p$-Banach space, $0 < \alpha \leq 1$. A sequence $(x_k) \subseteq \mathbb{X}$ is a multiplier from $\mathcal{N}^\alpha(D)$, $\alpha \geq 1$, into $A(\mathbb{X})$ if and only if

$$\|x_k\| = O[\exp(-\eta k^{2/(\alpha+2)})]$$

for some $\eta > 0$. 


Propositions 4.3 and 4.4 allow us to completely characterize continuous linear maps from $N_+^a$ or $N_+^a(D)$ into any $p$-Banach space $X$, $0 < p \leq 1$. In the sequel, let $e_n$ denote the function $e_n(z) = z^n$, for $n = 0, 1, \ldots$.

**Proposition 4.5.** Let $X$ be a $p$-Banach space, $0 < p \leq 1$. Let $T$ be a linear map, $T : N_+^a \to X$, $\alpha \geq 1$, and $T(e_n) = x_n$. $T$ is continuous if and only if for every $f \in N_+^a$, with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$Tf = \sum_{n=0}^{\infty} a_n x_n;$$

moreover, $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$.

**Proof.** Let $T : N_+^a \to X$ be a continuous linear map. For $f \in N_+^a$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f$ is the uniform limit of its Taylor series on each disc $\{z : |z| \leq r\}$ with $0 < r < 1$. Let $P_N(z) = \sum_{n=0}^{N} a_n z^n$, denote the $N$th Taylor polynomial, and let $P_{\zeta,N}(z) = \sum_{n=0}^{N} \zeta^n a_n z^n$, for $|\zeta| < 1$. It follows easily that $\lim_{N \to \infty} d_\alpha(P_{\zeta,N}, f_\zeta) = 0$. Thus for each $\zeta \in \mathbb{D}$,

$$T(f_\zeta) = \lim_{N \to \infty} T(P_{\zeta,N}) = \sum_{n=0}^{\infty} \zeta^n a_n x_n.$$

Setting $F(\zeta) = T(f_\zeta)$, we can deduce from Lemma 4.2 that $F$ is analytic on $D$ and continuous on $\overline{D}$, i.e., $f \in A(X)$. Thus $(x_n)$ is a multiplier from $N_+^a$ into $A(X)$, whereby $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$, by Proposition 4.3. As in the proof of Proposition 4.3, it follows that $\sum_{n=0}^{\infty} a_n x_n \zeta^n$ converges uniformly on $D$, i.e., $\lim_{N \to \infty} T(P_{N,\zeta}) = T(f_\zeta)$, and the convergence is uniform in $\zeta$, $\zeta \in D$. Thus, since

$$\lim_{r \to 1-} \lim_{N \to \infty} T(P_{N,r}) = \lim_{r \to 1-} T(f_r) = T(f),$$

we have

$$\lim_{N \to \infty} \lim_{r \to 1-} T(P_{N,r}) = \lim_{r \to 1-} \sum_{n=0}^{N} a_n x_n = \sum_{n=0}^{\infty} a_n x_n = T(f).$$

Next we suppose $T(e_n) = x_n$, with $\|x_n\| = O[\exp(-\eta n^{1/(\alpha+1)})]$, for some $\eta > 0$. From Proposition 4.3 we see that $A = (x_n)$ is a multiplier from $N_+^a$ into $A(X)$. Recall that multipliers from $N_+^a$ into $A(X)$ are continuous, so if $f_k \to f$ in $N_+^a$, then $Af_k \to Af$ in $A(X)$; i.e.,

$$\sup_{z \in D} \left\| \sum_{n=0}^{\infty} a_n^{(k)} x_n z^n - \sum_{n=0}^{\infty} a_n x_n z^n \right\| \to 0$$

as $k \to \infty$; in particular, for $z = 1$ we have $\|Tf_k - Tf\| \to 0$. $T$ is therefore continuous.
Proposition 4.6. Let $X$ be a $p$-Banach space, $0 < p \leq 1$. Let $T$ be a linear map, $T : N_+^\alpha(\mathbb{D}) \to X$, $\alpha \geq 1$, and $T(e_n) = x_n$. $T$ is continuous if and only if for every $f \in N_+^\alpha(\mathbb{D})$ with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$Tf = \sum_{n=0}^{\infty} a_n x_n;$$

moreover, $\|x_n\| = O[\exp(-m^{2/(\alpha+2)})]$ for some $\eta > 0$.

The argument from Proposition 4.5 and its counterpart for Proposition 4.6 yield straightforward characterizations of the dual spaces of $N_+^\alpha$ and $N_+^\alpha(\mathbb{D})$. For convenience, let $A$ denote those analytic functions on $\mathbb{D}$ which are also continuous on $\overline{\mathbb{D}}$.

Proposition 4.7 (cf. [16], Theorem 3). Let $\phi \in (N_+^\alpha)^*$. There is a unique $g \in A$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, so that

$$\phi(f) = \sum_{n=0}^{\infty} a_n b_n$$

for each $f \in N_+^\alpha$, with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The series $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely. Moreover, the Taylor coefficients of $g$ satisfy

$$(*) \quad |b_n| = O[\exp(-\eta n^{1/(\alpha+1)})]$$

for some $\eta > 0$. Conversely, every $g \in A$ whose Taylor coefficients $(b_n)$ satisfy $(*)$ defines a continuous linear functional $\phi_g$ on $N_+^\alpha$.

Proof. Let $\phi \in (N_+^\alpha)^*$, and let $\phi(e_n) = b_n$. Proposition 4.5 implies that $\phi(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f \in N_+^\alpha$ with Taylor series $\sum_{n=0}^{\infty} a_n z^n$. Moreover, $|b_n| = O[\exp(-\eta n^{1/(\alpha+1)})]$ for some $\eta > 0$, so that $g(z) = \sum_{n=0}^{\infty} b_n z^n$ converges uniformly and absolutely on $\overline{\mathbb{D}}$; thus $g \in A$.

On the other hand, if $g \in A$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, with $b_n$ satisfying $(*)$, we may define

$$\phi_g(f) = \sum_{n=0}^{\infty} a_n b_n$$

for $f \in N_+^\alpha$. Since $N_+^\alpha \subseteq F_{1/\alpha}$ ([15]), $\phi_g$ is a well-defined linear functional on $N_+^\alpha$, and the series $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely. Since $(b_n)$ satisfies $(*)$, Proposition 4.5 implies that $\phi_g$ is continuous, and the proof is complete.

Proposition 4.8. Let $\phi \in (N_+^\alpha(\mathbb{D}))^*$. There is a unique $g \in A$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, so that

$$\phi(f) = \sum_{n=0}^{\infty} a_n b_n$$
for each \( f \in N_+^\alpha(D) \), with Taylor series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). The series 
\[
\sum_{n=0}^{\infty} a_n b_n
\]
converges absolutely. Moreover, the Taylor coefficients of \( g \) satisfy
\[
(\ast) |b_n| = O[\exp(-\eta n^{2/(\alpha+2)})]
\]
for some \( \eta > 0 \). Conversely, every \( g \in A \) whose Taylor coefficients \( (b_n) \) satisfy \((\ast)\) defines a continuous linear functional \( \phi_g \) on \( N_+^\alpha(D) \).

Propositions 4.3 and 4.4 may be used to characterize multipliers from
\( N_+^\alpha \) or \( N_+^\alpha(D) \) into the Hardy spaces \( H^p \), \( 0 < p \). Suppose for example that \( \lambda_n \subseteq C \) is a multiplier from \( N_+^\alpha \) into \( H^p \); since convergence in \( N_+^\alpha \) or \( H^p \) implies uniform convergence on compact subsets of \( D \), it follows as a consequence of the Closed Graph Theorem that \( A = (\lambda_n) \) is continuous.

Propositions 4.3 and 4.4 yield the following (cf. [16], Theorem 2):

**Proposition 4.9.** (i) \( A = (\lambda_n) \) is a multiplier from \( N_+^\alpha \), \( \alpha \geq 1 \), into \( H^p \), \( 0 < p \), if and only if \( |\lambda_n| = O[\exp(-\eta n^{1/(\alpha+1)})] \) for some \( \eta > 0 \).

(ii) \( A = (\lambda_n) \) is a multiplier from \( N_+^\alpha(D) \), \( \alpha \geq 1 \), into \( H^p \), \( 0 < p \), if and only if \( |\lambda_n| = O[\exp(-\eta n^{2/(\alpha+2)})] \) for some \( \eta > 0 \).

For an arbitrary \( F \)-space, \( X \), the topology induced by the \( p \)-envelope is stronger than that induced by the Fréchet envelope, \( 0 < p \leq 1 \). Let \( X = N_+^\alpha \) or \( N_+^\alpha(D) \), and \( d = d_0 \) or \( \rho_0 \). If we can show that the \( \hat{X} \) topology is stronger than the \( \hat{X}_p \) topology on \( X \), then necessarily \( \hat{X} = \hat{X}_p \). Let \( V \) be a \( d \)-ball of radius \( 1/n \), \( n = 1, 2, \ldots \), and let \( \| \cdot \|_{p,n} \) be the Minkowski functional of the \( p \)-co \( V_n \). Recall that the family \( \{\| \cdot \|_{p,n}\} \) induces the \( \hat{X}_p \) topology on \( X \). For \( f \in X \), if \( \|f\|_{p,n} = 0 \), then since \( \|f\|_{p,n} \geq \|f\|_{1,n} \) it must follow that \( f \equiv 0 \). Thus each \( \| \cdot \|_{p,n} \) is actually a \( p \)-norm on \( X \) and the completion of \( X \) with respect to \( \| \cdot \|_{p,n} \) is a \( p \)-Banach space. This observation will be utilized in the proof of the following theorem.

**Theorem 4.10.** For \( 0 < p \leq 1 \), the \( p \)-envelope of \( N_+^\alpha \), \( \alpha \geq 1 \), is \( F_{1/\alpha} \).

**Proof.** Let \( \| \cdot \| \) be any one of the \( p \)-norms \( \| \cdot \|_{p,n} \), \( n = 1, 2, \ldots \), and let \( Y \) be the completion of \( N_+^\alpha \) with respect to \( \| \cdot \| \). Let \( T \) be the natural inclusion map \( T: N_+^\alpha \rightarrow Y \); \( T \) is continuous and linear. From Proposition 4.5 we have \( Tf = \sum_{n=0}^{\infty} a_n e_n \) for \( f \in N_+^\alpha \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and, in addition, \( \|e_n\| \leq M \exp(-\eta n^{1/(\alpha+1)}) \) for some \( \eta, M > 0 \). Let \( \eta_1, \eta_2 > 0 \) be such that \( \eta_1 + \eta_2 = \eta \) and let \( q > 0 \) be such that \( p + q = 1 \). (If \( q = 0 \), then the result is simply a restatement of Theorem 3.1.) For \( f \in N_+^\alpha \), we have
\[
\|Tf\|^p \leq \sum_{n=0}^{\infty} |a_n|^p \|e_n\|^p \leq \sum_{n=0}^{\infty} |a_n|^p [M \exp(-\eta n^{1/(\alpha+1)})]^p
\]
\[
\sum_{n=0}^{\infty} |a_n|^{p} [M \exp(-\eta n^{1/(\alpha+1)})]^{p} [\exp(-p\eta n^{1/(\alpha+1)})]
\]
\[
\leq \left\{ M \sum_{n=0}^{\infty} |a_n| \exp(-\eta n^{1/(\alpha+1)}) \right\}^{p} \left\{ \sum_{n=0}^{\infty} \exp \left( -\frac{p}{q} \eta n^{1/(\alpha+1)} \right) \right\}^{q} ;
\]
consequently, for a constant \( C > 0 \),
\[
\|Tf\| \leq C \sum_{n=0}^{\infty} |a_n| \exp(-\eta n^{1/(\alpha+1)}) = C \|f\|_{\eta_1} .
\]
The \( F_{1/\alpha} \) topology is therefore stronger than the \( p \)-envelope topology on \( N^\alpha_+ \), and the proof is complete.

**Theorem 4.11.** For \( 0 < p \leq 1 \), the \( p \)-envelope of \( N^\alpha_+ (\mathbb{D}) \), \( \alpha \geq 1 \), is \( F_{2/\alpha} \).

**References**


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