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Continuity of projections of natural bundles

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Abstract. This paper is a contribution to the axiomatic approach to geometric objects. A collection of a manifold M, a topological space N, a group homomorphism $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$ and a function $\pi : N \to M$ is called a quasi-natural bundle if (1) $\pi \circ E(f) = f \circ \pi$ for every $f \in \mathrm{Diff}(M)$ and (2) if $f, g \in \mathrm{Diff}(M)$ are two diffeomorphisms such that $f|U = g|U$ for some open subset U of M, then $E(f)|\pi^{-1}(U) = E(g)|\pi^{-1}(U)$. We give conditions which ensure that $\pi : N \to M$ is continuous. In particular, if (M, N, E, π) is a quasi-natural bundle with N Hausdorff, then π is continuous. Using this result, we classify (quasi) prolongation functors with compact fibres.

0. Introduction. Throughout this paper manifolds are assumed to be paracompact, finite-dimensional and without boundary.

The concept of a natural bundle was introduced by A. Nijenhuis [10] as a modern approach to the classical theory of geometrical objects (see [1]). A *natural bundle* (over n-dimensional manifolds) is a covariant functor F from the category of n-dimensional C^∞ manifolds and C^∞ embeddings into the category of C^∞ locally trivial fibre bundles and C^∞ bundle mappings such that:

(1) for every n-dimensional C^∞ manifold M, FM is a locally trivial fibre bundle over M;

(2) for every C^∞ embedding $\varphi : M \to N$ of n-dimensional manifolds, $F\varphi : FM \to FN$ covers φ and for any $x \in M$, $F\varphi$ maps diffeomorphically the fibre F_xM onto the fibre $F_{\varphi(x)}N$;

(3) F is regular in the following sense: If $\varphi : U \times M \to N$ is a C^∞ mapping (where U is an open subset of $\mathbb{R}^k$) such that for every $t \in U$, $\varphi_t : M \to N$, $\varphi_t(x) = \varphi(t, x)$, is an embedding, then the mapping

$$U \times FM \ni (t, y) \to F\varphi_t(y) \in FN$$

is of class C^∞.

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The functor T associating with each n-manifold M the tangent bundle TM and with each embedding $\varphi : M \to N$ the differential $d\varphi : TM \to TN$ is an example of a natural bundle. An equivalent formulation of an interesting result of Palais and Terng [11] is that every natural bundle is isomorphic to a fibre bundle associated with an r-frame bundle and whose standard fibre is an L_n^r space (cf. [4, p. 13]).

The definition of a locally determined associated space was introduced by Epstein and Thurston [2] as a generalization of natural bundles. Let $n \geq 1$ be a fixed natural number and let $q \geq 1$ be a fixed integer or infinity. A *locally determined associated space* is a functor E associating with each n-dimensional C^q manifold M a topological space $EM \neq \emptyset$ and a continuous mapping $\pi_M : EM \to M$ and with each C^q embedding $\varphi : M \to N$ of n-manifolds a homeomorphism between EM and $\pi_N^{-1}(\mathrm{im}(\varphi))$ such that $\pi_N \circ E(\varphi) = \varphi \circ \pi_M$.

In [2], the authors proved that if E is a locally determined associated space such that $E\mathbb{R}^n$ is a locally compact second countable Hausdorff space, then E is isomorphic to a topological fibre bundle associated with an r-frame bundle and whose standard fibre is an L_n^r space. (In [8], we extended the result of Epstein and Thurston to the situation when $E\mathbb{R}^n$ is not locally compact.) The result of Epstein and Thurston implies that the regularity condition in the definition of a natural bundle is a consequence of the other conditions in the definition.

The purpose of this paper is to give an answer to the following question: Is the condition "π_M is continuous" in the definition of locally determined associated spaces a consequence of the other conditions of the definition (see Problem in [2, p. 236])? If EM is a Hausdorff space, the answer is affirmative. In fact, we obtain a more general result.

Let us begin with the definition of a quasi-natural bundle.

Definition 0.1. Let $r \geq 1$ be an integer or infinity or $r = \omega$ and let $n \geq 1$ be a natural number. A collection (M, N, E, π, r, n) of r, n and of

— an n-dimensional C^r manifold M,

— a topological space $N \neq \emptyset$,

— a group homomorphism $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$ of the group of all global C^r diffeomorphisms of M onto M into the group of all homeomorphisms of N onto N,

— a function $\pi : N \to M$,

is called a *quasi-natural bundle* if the following two conditions are satisfied:

(1) for every $\varphi \in \mathrm{Diff}(M)$, $\pi \circ E(\varphi) = \varphi \circ \pi$,

(2) if $f, g \in \mathrm{Diff}(M)$ are such that $f|U = g|U$ for some open subset U of M, then $E(f)|\pi^{-1}(U) = E(g)|\pi^{-1}(U)$.

We have the following obvious example of quasi-natural bundles.

EXAMPLE 0.1. Let E be a locally determined associated space over n-dimensional C^r manifolds and let M be an n-dimensional C^r manifold. It is clear that $(M, EM, E|\mathrm{Diff}\,(M), \pi_M, r, n)$ is a quasi-natural bundle.

In Section 1, conditions are given under which $\pi : N \to M$ is continuous, where (M, N, E, π, r, n) is a quasi-natural bundle. In particular, we deduce the following two theorems.

THEOREM 0.1. *Let (M, N, E, π, r, n) be a quasi-natural bundle. Suppose that N is a Hausdorff space and $r \geq 1$ is an integer or infinity. Then π is continuous.*

THEOREM 0.2. *Let (M, N, E, π, r, n) be a quasi-natural bundle. If $r \geq 1$ is an integer or infinity, then the set $\{v \in N : \pi$ is continuous at $v\}$ is open in N.*

In Example 2.1, we present a quasi-natural bundle $(\mathbb{R}, N, E, \pi, \omega, 1)$ such that: (i) N is a metric space and (ii) π is not continuous. Therefore, Theorem 0.1 is not true in the analytic situation.

If N is a C^r manifold, we have the following natural question: Is $\pi : N \to M$ of class C^r? It is a difficult question and I have only been able to give a partial answer. We introduce the following definition.

DEFINITION 0.2. Let $(M, N, E, \pi, \infty, n)$ be a quasi-natural bundle. We say it is *regular* if the following three conditions are satisfied:

(1) N is a manifold of class C^∞;

(2) if $\varphi \in \mathrm{Diff}(M)$, then $E(\varphi) : N \to N$ is of class C^∞;

(3) if $\varphi : U \times M \to M$ is a C^∞ mapping (where U is an open subset of $\mathbb{R}^k$) such that for every t in U, $\varphi_t : M \to M$, $\varphi_t(x) = \varphi(t, x)$, is a diffeomorphism, then the mapping

$$U \times N \ni (t, y) \to E(\varphi_t)(y) \in N$$

is of class C^∞.

In Section 3, we prove the following theorem.

THEOREM 0.3. *Let $(M, N, E, \pi, \infty, n)$ be a regular quasi-enatural bundle. Assume that M and N are connected. Then the set $\{v \in N :$ there exists a neighbourhood V of v such that $\pi|V$ is of class $C^\infty\}$ is dense in N.*

Similarly to the definition of prolongation functors in the sense of [6] we introduce the following definitions of quasi-prolongation functors.

DEFINITION 0.3. Let $r \geq 1$ be an integer or infinity or $r = \omega$. A *quasi-prolongation functor* over (positive-dimensional) C^r manifolds is a covariant functor F associating with each positive-dimensional C^r manifold M a topological space $FM \neq \emptyset$ and a function $\pi_M : FM \to M$ and with each C^r mapping $f : M \to N$ a continuous mapping $Ff : FM \to FN$ such that:

(1) for every C^r mapping $f : M \to N$, $\pi_N \circ Ff = f \circ \pi_M$;

(2) if $\varphi : M \to N$ is a C^r diffeomorphism onto an open subset of N, then $F\varphi : FM \to \pi_N^{-1}(\mathrm{im}(\varphi))$ is a homeomorphism.

Definition 0.4. Let F be a quasi-prolongation functor over C^∞ manifolds. We say that F is *regular* if the following conditions are satisfied:

(1) for every C^∞ manifold M, FM is a C^∞ manifold;

(2) for every C^∞ mapping $f : M \to N$ the mapping $Ff : FM \to FN$ is of class C^∞;

(3) if $\varphi : M \to N$ is a C^∞ diffeomorphism onto an open subset, then $F\varphi : FM \to \pi_N^{-1}(\mathrm{im}(\varphi))$ is also a diffeomorphism. (By Theorem 0.1, π_N is continuous, and so $\pi_N^{-1}(\mathrm{im}(\varphi))$ is open in FN.)

In Section 4, we prove the following theorem.

Theorem 0.4. (I) *Let F be a quasi-prolongation functor over C^r manifolds. Suppose that*:

(1) *r is a natural number or infinity*;

(2) *for each n, $F\mathbb{R}^n$ has a countable basis*;

(3) *for each C^r manifold M, FM is Hausdorff*;

(4) *for each n the fibre $\pi_{\mathbb{R}^n}^{-1}(0)$ is compact.*

Then there exists a compact set G such that F is isomorphic to the trivial quasi-prolongation functor $(\ ) \times G$ which associates with each C^r manifold M the space $M \times G$ and the mapping $p_M : M \times G \to M$, $p_M(x, y) = x$, and with each C^r mapping f the mapping $f \times \mathrm{id}_G$.

(II) *Let F be a regular quasi-prolongation functor. Assume that*:

(1) *for each n, $F\mathbb{R}^n$ has a countable basis*;

(2) *for each n, $\pi_{\mathbb{R}^n}^{-1}(0)$ is compact.*

Then there exists a C^∞ compact manifold G such that F is C^∞ isomorphic to the trivial regular quasi-prolongation functor $(\ ) \times G$ over C^∞ manifolds.

Remark. We say that quasi-prolongation functors F^1, F^2 (resp. regular quasi-prolongation functors F^1, F^2) over C^r (resp. C^∞) manifolds are *isomorphic* (C^∞ *isomorphic*) if for every C^r (resp. C^∞) manifold M there exists a homeomorphism (resp. a C^∞ diffeomorphism) $I_M : F^1M \to F^2M$ such that:

(1) for every C^r (resp. C^∞) manifold M, $\pi_M^2 \circ I_M = \pi_M^1$, where $\pi_M^i : F^iM \to M$ $(i = 1, 2)$ are the projections;

(2) for each C^r (resp. C^∞) mapping $f : M \to N$ of C^r (resp. C^∞) manifolds, $I_N \circ F^1 f = F^2 f \circ I_M$.

1. Continuity of projections of quasi-natural bundles. In this section we will prove Theorems 0.1 and 0.2.

Let X be a topological space and let $Y \subset X$. We say that Y is *Hausdorff in* X iff any two distinct points in Y have disjoint open neighbourhoods in X. It is clear that if $h : X \to X$ is a homeomorphism and Y is Hausdorff in X, then $h(Y)$ is also Hausdorff in X.

Let (M, N, E, π, r, n) be a quasi-natural bundle. Throughout this section we use the following notations:

$$\mathrm{Orb}(v) = \{E(f)(v) : f \in \mathrm{Diff}(M)\}, \qquad v \in N;$$

for every $x \in M$,

$$D^-(x) = \{f \in \mathrm{Diff}(M) : f(x) = x \text{ and there exists a chart } (U, g, x) \\ \text{such that } g(x) = 0 \text{ and } \det(d_0(g \circ f \circ g^{-1})) < 0\};$$

finally, for every $y \in M$, denote by $\mathrm{Comp}(M, y)$ the connected component of M which contains y.

Theorems 0.1 and 0.2 are simple consequences of the following proposition.

PROPOSITION 1.1. *Let (M, N, E, π, r, n) be a quasi-natural bundle and let $v \in N$. Suppose that $r \geq 1$ is an integer or infinity. Then the following conditions are equivalent*:

(1) *π is continuous at v*;

(2) *if $w \in \mathrm{Orb}(v)$, then π is continuous at w*;

(3) *for all $u, w \in \mathrm{Orb}(v)$, the condition $\pi(u) \neq \pi(w)$ implies that $\{u, w\}$ is Hausdorff in N*;

(4) *there exists $w \in \mathrm{Orb}(v) - \pi^{-1}(\pi(v))$ such that*:

- (a) $\pi(w) \in \mathrm{Comp}(M, \pi(v))$,
- (b) *$\{v, w\}$ is Hausdorff in N, and*
- (c) *if $D^-(\pi(v)) \neq \emptyset$, then there exists $\varphi \in D^-(\pi(v))$ such that $\{E(\varphi)(v), w\}$ is Hausdorff in N*;

(5) *there exists an open* (*in N*) *neighbourhood V of v such that $\pi|V$ is continuous.*

In the proof of Proposition 1.1 we shall use the following lemmas:

LEMMA 1.1. *Let $r \geq 1$ be an integer or infinity. Let M be an n-dimensional manifold of class C^r and let $x \in M$. Let (U, g, x) be a chart of M at x such that $g(U) = \mathbb{R}^n$ and $g(x) = 0$. If $y \in \mathrm{Comp}(M, x)$, then there* *exists a C^r diffeomorphism $f : M \to M$ such that* $\mathrm{germ}_x(f) = \mathrm{germ}_x(\mathrm{id}_M)$ *and $f(y) \in U$.*

LEMMA 1.2. *Let (M, N, E, π, r, n) be a quasi-natural bundle such that $r \geq 1$ is an integer or infinity. Let $u, u', w \in N$ be such that*:

(1) $u' \in \mathrm{Orb}(u) \cap \pi^{-1}(\pi(u))$,

(2) $w \in \mathrm{Orb}(u) - \pi^{-1}(\pi(u))$,

(3) *$\{u, w\}$ is Hausdorff in N*,

(4) *if $D^-(\pi(u)) \neq \emptyset$, then there is a $\psi \in D^-(\pi(u))$ such that $\{E\psi(u), w\}$ is Hausdorff in N.*

Then $\{u', w\}$ is also Hausdorff in N.

Proof of Lemma 1.1. Let

$$A = \{z \in \mathrm{Comp}(M, x) : \text{ there exists } f \in \mathrm{Diff}(M) \text{ such that } f(z) \in U \text{ and } \mathrm{germ}_x(f) = \mathrm{germ}_x(\mathrm{id}_M)\}.$$

We have to prove that $A = \mathrm{Comp}(M, x)$. It is clear that $U \subset A$. Thus it is sufficient to show that $A - \{x\}$ and $\mathrm{Comp}(M, x) - A$ are open in $\mathrm{Comp}(M, x)$.

For each $z \in A$, let $f_z \in \mathrm{Diff}(M)$ be such that $f_z(z) \in U$ and $\mathrm{germ}_x(f_z) = \mathrm{germ}_x(\mathrm{id}_M)$. For each $t \in M - \{x\}$, let (U_t, g_t) be a chart of M at t such that $g_t(U_t) = \mathbb{R}^n$, $g_t(t) = 0$ and $x \in M - U_t$. For each $z \in U_t$ and $t \in M - \{x\}$, let $h_{t,z} \in \mathrm{Diff}(M)$ be such that $h_{t,z}(t) = z$ and $\mathrm{germ}_x(h_{t,z}) = \mathrm{germ}_x(\mathrm{id}_M)$ (see [5]).

Let $z \in A - \{x\}$. We shall show that $U_z \subset A - \{x\}$. Let $t \in U_z$. Then $f_z \circ (h_{z,t})^{-1} \in \mathrm{Diff}(M)$, $f_z \circ (h_{z,t})^{-1}(t) = f_z(z) \in U$ and $\mathrm{germ}_x(f_z \circ (h_{z,t})^{-1}) = \mathrm{germ}_x(\mathrm{id}_M)$, and thus $t \in A - \{x\}$.

Let now $z \in \mathrm{Comp}(M, x) - A$. We shall show that $U_z \cap A = \emptyset$. Assume the contrary. Let $t \in U_z \cap A$. Then $f_t \circ h_{z,t} \in \mathrm{Diff}(M)$, $f_t \circ h_{z,t}(z) = f_t(t) \in U$ and $\mathrm{germ}_x(f_t \circ h_{z,t}) = \mathrm{germ}_x(\mathrm{id}_M)$, and hence $z \in A$. ∎

Proof of Lemma 1.2. Fix $f \in \mathrm{Diff}(M)$ such that $f(\pi(u)) = \pi(u)$ and $u' = E(f)(u)$, and a chart (U, g) of M at $\pi(u)$ such that $g(U) = \mathbb{R}^n$, $g(\pi(u)) = 0$ and $\pi(w) \in M - U$. Consider two cases.

(I) $\det(d_0(g \circ f \circ g^{-1})) > 0$. Then there exists $F \in \mathrm{Diff}(M)$ such that $\mathrm{germ}_{\pi(u)}(F) = \mathrm{germ}_{\pi(u)}(f)$ and $\mathrm{germ}_{\pi(w)}(F) = \mathrm{germ}_{\pi(w)}(\mathrm{id}_M)$ (see [5]). Definition 0.1 ensures that $E(F)(w) = w$ and $E(F)(u) = E(f)(u) = u'$; since $\{u, w\}$ is Hausdorff in N and $E(F)$ is a homeomorphism, $\{u', w\}$ is also Hausdorff in N.

(II) $\det(d_0(g \circ f \circ g^{-1})) < 0$. Then $f \in D^-(\pi(u))$, i.e. $D^-(\pi(u)) \neq \emptyset$. By the assumptions $\{E(\psi)(u), w\}$ is Hausdorff in N for some $\psi \in D^-(\pi(u))$. Replacing u by $E(\psi)(u)$ and f by $f \circ \psi^{-1}$ we apply Case (I) to conclude the proof. ∎

P r o o f o f P r o p o s i t i o n 1.1. (1)$\Rightarrow$(2) is a simple consequence of Definition 0.1. (2)$\Rightarrow$(3) and (5)$\Rightarrow$(1) are obvious.

(3)$\Rightarrow$(4). Let (U,g) be a chart of M at $\pi(v)$ such that $g(U)=\mathbb{R}^n$ and $g(\pi(v))=0$. Let $y\in U-\{\pi(v)\}$. There exist $h\in \mathrm{Diff}(M)$ and a compact set $K\subset U$ such that $h(\pi(v))=y$ and $h|M-K=\mathrm{id}_M|M-K$. Let $w:=E(h)(v)$. It is clear that $w\in \mathrm{Orb}(v)-\pi^{-1}(\pi(v))$ and $\pi(w)=h(\pi(v))\in U\subset \mathrm{Comp}(M,\pi(v))$. Thus $\{v,w\}$ is Hausdorff in N. Similarly, if $\varphi\in D^-(\pi(v))$, then $\pi(E(\varphi)(v))=\pi(v)\neq\pi(w)$, and so $\{E(\varphi)(v),w\}$ is Hausdorff in N.

The proof of (4)$\Rightarrow$(1) is more complicated. Assume the contrary. Then π is not continuous at w. Let (U,g) be a chart of M at $\pi(w)$ such that $g(U)=\mathbb{R}^n$, $g(\pi(w))=0$ and $\pi(W)\not\subset U$ for every $W\in \mathrm{top}(N,w)$, where $\mathrm{top}(N,w)$ is the set of all open neighbourhoods of w. By Lemma 1.1, there is an $f\in\mathrm{Diff}(M)$ such that $f(\pi(v))\in U$ and $\mathrm{germ}_{\pi(w)}(f)=\mathrm{germ}_{\pi(w)}(\mathrm{id}_M)$. Moreover, there are $h\in \mathrm{Diff}(M)$ and a compact subset K of U such that $h|M-K=\mathrm{id}_M|M-K$ and $h(\pi(w))=f(\pi(v))$ (see [5]).

First assume that $E(f)(v)$, $E(h)(w)$ and w satisfy the assumptions of Lemma 1.2 (with $E(f)(v)$ and $E(h)(w)$ playing the roles of u and u', respectively). Then $\{E(h)(w),w\}$ is Hausdorff in N, and hence there exist $W_1\in\mathrm{top}(N,E(h)(w))$ and $W_2\in\mathrm{top}(N,w)$ such that $W_1\cap W_2=\emptyset$. Since $E(h):N\to N$ is a homeomorphism, we can find $W\in\mathrm{top}(N,w)$ such that $W\subset W_2$ and $E(h)(W)\subset W_1$. Recalling that $\pi(\widetilde{W})\not\subset U$ for every $\widetilde{W}\in\mathrm{top}(N,w)$, we deduce that $W-\pi^{-1}(U)\neq\emptyset$. On the other hand, since $\pi^{-1}(M-K)\supset N-\pi^{-1}(U)$ and $E(h)|\pi^{-1}(M-K)=\mathrm{id}_N|\pi^{-1}(M-K)$, we get $E(h)|N-\pi^{-1}(U)=\mathrm{id}_N|N-\pi^{-1}(U)$. Hence $\emptyset=W_1\cap W_2\supset E(h)(W-\pi^{-1}(U))\cap(W-\pi^{-1}(U))=W-\pi^{-1}(U)\neq\emptyset$, a contradiction.

It remains to prove that $E(f)(v),E(h)(w)$ and w satisfy the assumptions of Lemma 1.2. It is clear that $\mathrm{Orb}(E(h)(w))=\mathrm{Orb}(w)=\mathrm{Orb}(v)=\mathrm{Orb}(E(f)(v))$ as $w\in\mathrm{Orb}(v)$. On the other hand, since f is a bijection, $h(\pi(w))=f(\pi(v)),\pi(w)\neq\pi(v)$ and $\mathrm{germ}_{\pi(w)}(f)=\mathrm{germ}_{\pi(w)}(\mathrm{id}_M)$, we have $\pi(E(h)(w))=h(\pi(w))=f(\pi(v))=\pi(E(f)(v))$, $\pi(w)=f(\pi(w))\neq f(\pi(v))=\pi(E(f)(v))$ and $\{E(f)(v),w\}=E(f)(\{v,w\})$. Hence $E(h)(w)\in\mathrm{Orb}(E(f)(v))\cap\pi^{-1}(\pi(E(f)(v)))$, $w\in\mathrm{Orb}(E(f)(v))-\pi^{-1}(\pi(E(f)(v)))$ and (since $\{v,w\}$ is Hausdorff in N) $\{E(f)(v),w\}$ is Hausdorff in N.

If $q\in D^-(f(\pi(v)))$, then $f^{-1}\circ q\circ f\in D^-(\pi(v))$ (i.e. $D^-(\pi(v))\neq\emptyset$), and therefore (by the assumption on w) there exists $\varphi\in D^-(\pi(v))$ such that $\{E(\varphi)(v),w\}$ is Hausdorff in N. It is obvious that $f\circ\varphi\circ f^{-1}\in D^-(f(\pi(v)))$ and $\{E(f\circ\varphi\circ f^{-1})(E(f)(v)),w\}=E(f)(\{E(\varphi)(v),w\})$, and thus $\{E(f\circ\varphi\circ f^{-1})(E(f)(v)),w\}$ is Hausdorff in N.

(1)$\Rightarrow$(5). We can assume (2)–(4) to hold. Fix $w\in(\mathrm{Orb}(v)-\pi^{-1}(\pi(v)))\cap\pi^{-1}(\mathrm{Comp}(M,\pi(v)))$. If $D^-(\pi(v))\neq\emptyset$, we fix $\psi\in D^-(\pi(v))$. Let $f\in$

$\mathrm{Diff}(M)$ be such that $w = E(f)(v)$. Let $W \in \mathrm{top}(M, \pi(v))$ be such that $\pi(w) \in M - \mathrm{cl}(W)$ and let (U, g) be a chart of M at $\pi(v)$ such that $g(U) = \mathbb{R}^n$, $g(\pi(v)) = 0$, $U \subset W$ and $\psi(U) \subset W$. Since π is continuous at v and w, there are $V_1 \in \mathrm{top}(N, v)$ and $V_2 \in \mathrm{top}(N, w)$ such that $\pi(V_1) \subset U$ and $\pi(V_2) \subset M - \mathrm{cl}(W)$. Let $V \in \mathrm{top}(N, v)$ be such that $V \subset V_1$ and $E(f)(V) \subset V_2$.

We shall prove that $\pi|V$ is continuous. Let $\widetilde{v} \in V$ and $\widetilde{w} := E(f)(\widetilde{v})$. We shall prove that $\widetilde{w}$ satisfies (4) (with $\widetilde{v}$ and $\widetilde{w}$ playing the roles of v and w, respectively).

We see that $\pi(\widetilde{v}) \in \pi(V) \subset \pi(V_1) \subset U \subset W$ and $\pi(\widetilde{w}) \in \pi(E(f)(V)) \subset \pi(V_2) \subset M - \mathrm{cl}(W)$. Therefore $\widetilde{w} \in \mathrm{Orb}(\widetilde{v}) - \pi^{-1}(\pi(\widetilde{v}))$.

It is clear that $\pi(\widetilde{v}) \in U \subset \mathrm{Comp}(M, \pi(v))$. Then $\pi(\widetilde{w}) = f(\pi(\widetilde{v})) \in f(\mathrm{Comp}(M, \pi(v))) = \mathrm{Comp}(M, f(\pi(v))) = \mathrm{Comp}(M, \pi(w))$, and thus (as $\pi(w) \in \mathrm{Comp}(M, \pi(v))$) $\pi(\widetilde{w}) \in \mathrm{Comp}(M, \pi(\widetilde{v}))$.

Since $\widetilde{v} \in V \subset V_1$ and $\widetilde{w} \in E(f)(V) \subset V_2$ and $\pi(V_1) \cap \pi(V_2) = \emptyset$, $\{\widetilde{v}, \widetilde{w}\}$ is Hausdorff in N.

Let $h \in \mathrm{Diff}(M)$ be such that $h|M - U = \mathrm{id}_M|M - U$ and $h(\pi(v)) = \pi(\widetilde{v})$. Assume $\eta \in D^-(\pi(\widetilde{v}))$. Then $h^{-1} \circ q \circ h \in D^-(\pi(v))$, and then $D^-(\pi(v)) \neq \emptyset$. Let $\varphi := h \circ \psi \circ h^{-1}$. Of course, $\varphi \in D^-(\pi(\widetilde{v}))$. We shall prove that $\{E(\varphi)(\widetilde{v}), \widetilde{w}\}$ is Hausdorff in N. It is sufficient to prove that $E(\varphi)(V) \cap E(f)(V) = \emptyset$. We know that $E(h)|\pi^{-1}(M - \mathrm{cl}(W)) = \mathrm{id}_N|\pi^{-1}(M - \mathrm{cl}(W))$, as $M - \mathrm{cl}(W) \subset M - W \subset M - U$. On the other hand, $E(h^{-1})(V) \subset \pi^{-1}(U)$, for if $x \in V$, then $\pi(x) \in \pi(V) \subset \pi(V_1) \subset U$, and thus $\pi(E(h^{-1})(x)) = h^{-1}(\pi(x)) \in U$. Therefore (as $\psi(U) \subset W$) we have

$$\begin{aligned} E(f)(V) \cap E(\varphi)(V) &\subset V_2 \cap E(\varphi)(V) \\ &\subset \pi^{-1}(M - \mathrm{cl}(W)) \cap E(h) \circ E(\psi) \circ E(h^{-1})(V) \\ &\subset E(h)(\pi^{-1}(M - \mathrm{cl}(W))) \cap E(h) \circ E(\psi)(\pi^{-1}(U)) \\ &= E(h)(\pi^{-1}(M - \mathrm{cl}(W)) \cap E(\psi)(\pi^{-1}(U))) \\ &\subset E(h)(\pi^{-1}(M - \mathrm{cl}(W)) \cap \pi^{-1}(W)) = \emptyset. \ \blacksquare \end{aligned}$$

We have the following interesting application of Theorem 0.1. Let M be an n-dimensional C^r manifold, where $n \geq 1$ and $r \geq 1$ is an integer or infinity. Let $\mathrm{top}(M)$ be the topology on M. We say that a topology τ on M is *natural* iff any $\varphi \in \mathrm{Diff}(M)$ is a homeomorphism with respect to τ. Of course, $\mathrm{top}(M)$ is natural.

COROLLARY 1.1. *Let M be as above and let τ be a Hausdorff natural topology on M. Then* $\mathrm{top}(M) \subset \tau$.

Proof. Putting $N = (M, \tau)$, $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$, $E(\varphi) = \varphi$, and $\pi : N \to M$, $\pi = \mathrm{id}_M$, we obtain a quasi-natural bundle. Therefore π is continuous, and thus $\mathrm{top}(M) \subset \tau$. $\blacksquare$

Remark. Let $M, N, \pi, r, n, \mathrm{Diff}(M), \mathrm{Homeo}(N)$ be as in Definition 0.1. Let D be a subgroup in $\mathrm{Diff}(M)$ and let $E : D \to \mathrm{Homeo}(N)$ be a group homomorphism. The collection (M, N, E, π, r, n, D) is called *a D-quasi-natural bundle* if the conditions of Definition 0.1 are satisfied with D playing the role of $\mathrm{Diff}(M)$. We see that Proposition 1.1 (and hence Theorems 0.1 and 0.2) with D playing the role of $\mathrm{Diff}(M)$ is true for D-quasi-natural bundles provided D satisfies the following conditions:

A) for any chart (U, g) of M at x such that $g(U) = \mathbb{R}^n$, $g(x) = 0$, and $y \in U$ there exist a compact set $K \subset U$ and $h \in D$ such that $h|M - K = \mathrm{id}_M|M - K$ and $h(x) = y$,

B) for any chart (U, g) of M at x and $f \in D - D^-(x)$ with $f(x) = x$ there exist a compact set $K \subset U$ and $h \in D$ such that $h = f$ near x and $h|M - K = \mathrm{id}_M|M - K$.

For example, if either $D \subset \mathrm{Diff}(M)$ is the subgroup of all diffeomorphisms equal to the identity map outside a compact subset, or M is oriented and D is the subgroup of all orientation preserving diffeomorphisms, then D satisfies the above conditions A) and B).

2. Counterexamples. In this section we present some counterexamples.

In connection with Theorem 0.1 we present the following example of a quasi-natural bundle $(M, N, E, \pi, \omega, 1)$, where M is a connected analytic manifold, N is a second countable metrizable space, $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$ is a group homomorphism of the group of all analytic diffeomorphisms of M onto M into the group of all homeomorphisms of N onto N and $\pi : N \to M$ is not continuous.

Example 2.1. Let $M = \mathbb{R}$ and let N be the set of all global analytic mappings of $\mathbb{R}$ into $\mathbb{R}$. There is an injection $I : N \to \mathbb{R}^{\mathbb{N}}$ given by $I(f) = (f(1), f(\frac{1}{2}), f(\frac{1}{3}), \ldots)$. We equip N with the topology induced by I, i.e. $U \subset N$ is open iff $U = I^{-1}(V)$ for some V open in $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}^{\mathbb{N}}$ has the Tikhonov topology. Define $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$ by $E(g)(f) = g \circ f$. Let $\pi : N \to M$ be given by $\pi(f) = f(0)$. It is clear that $(M, N, E, \pi, \omega, 1)$ is a quasi-natural bundle. Since $f_n := (n^{-n} + (\mathrm{id}_{\mathbb{R}})^2)^{-n} \to 1$ (in N) and $\pi(f_n) = 1/n \nrightarrow \pi(1) = 1$, we see that π is not continuous at 1.

In connection with Theorem 0.2 we give the following example.

Example 2.2. Let $M = \mathbb{R}$. We equip $N := \mathbb{R} \times \mathbb{R}$ with a topology defined as follows: $U \subset N$ is open if and only if either $U = N$ or $U = V \times \{0\}$ for some V open in $\mathbb{R}$. Let $r = \infty$. Define $E : \mathrm{Diff}(M) \to \mathrm{Homeo}(N)$ by $E(f)(x, y) = (f(x), y)$. Let $\pi : N \to M$ be the projection onto the first factor. It is obvious that $(M,N,E,\pi,\infty,1)$ is a quasi-natural bundle, but the set $\{v\in N:\pi \text{ is continuous at } v\}$ is not closed in N.

Example 2.3. Let $M=\mathbb{R}\cup S^1$, where $S^1\subset\mathbb{R}^2$ is the unit circle. Let $N=\mathbb{R}$. Define $E:\mathrm{Diff}(M)\to\mathrm{Diff}(N)$ by $E(f)=f|\mathbb{R}$. Let $\pi:N\to M$ be the inclusion. Then $(M,N,E,\pi,\infty,1)$ is a regular quasi-natural bundle such that π is not surjective.

In connection with the implication (4)⇒(1) of Proposition 1.1 we give the following two examples.

Example 2.4. Let $M=\mathbb{R}\times\{0\}\cup\mathbb{R}\times\{1\}$. We equip $N:=M$ with the topology generated by N, $\emptyset$, $\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\{1\}$. Let $E:\mathrm{Diff}(M)\to\mathrm{Homeo}(N)$ be the inclusion and $\pi:N\to M$ the identity. It is clear that $(M,N,E,\pi,\infty,1)$ is a quasi-natural bundle. If $f:M\to M$ is defined by $f(t,x)=(t,x+1\pmod 2)$, $\varphi:M\to M$ is defined by $\varphi(t,x)=(-t,x)$, $v=(0,0)$ and $w=(0,1)$, then $w=f(v)\in\mathrm{Orb}(v)-\pi^{-1}(\pi(v))$, $\{v,w\}$ is Hausdorff in N, $\varphi\in D^-(\pi(v))$ and $\{E(\varphi)(v),w\}$ is also Hausdorff in N but π is not continuous at v, and $\pi(w)\notin\mathrm{Comp}(M,\pi(v))$.

Example 2.5. Let $M=\mathbb{R}$ and let N be as in Example 2.4. Define $E:\mathrm{Diff}(M)\to\mathrm{Homeo}(N)$ by $E(f)(x,t)=(f(x),t)$ if f is orientation preserving, and $=(f(x),t+1\pmod 2)$ if f is orientation reversing. Let $\pi:N\to M$ be the projection given by $\pi(x,t)=x$. It is easy to see that $(M,N,E,\pi,\infty,1)$ is a quasi-natural bundle. If $v=(1,0)$ and $w=(-1,1)$, then $w=E(-\mathrm{id})(v)\in\mathrm{Orb}(v)-\pi^{-1}(\pi(v))$, $\pi(w)\in\mathrm{Comp}(M,\pi(v))$ and $\{v,w\}$ is Hausdorff in N but π is not continuous at v. Of course $\{E(\varphi)(v),w\}$ is not Hausdorff in N for every $\varphi\in D^-(\pi(v))$.

With regard to the implication (1)⇒(3) of Proposition 1.1 the following example is interesting.

Example 2.6. Let $M=\mathbb{R}$. We equip $N:=\mathbb{R}\times\mathbb{R}$ with the topology in which $U\subset N$ is open if and only if there exists an open subset V of $\mathbb{R}$ such that $U=V\times\mathbb{R}$. Define $E:\mathrm{Diff}(M)\to\mathrm{Homeo}(N)$ by $E(f)(x,y)=(f(x),\frac{df}{dx}(x)y)$. Let $\pi:N\to M$ be the projection onto the first factor. Then $(M,N,E,\pi,\infty,1)$ is a quasi-natural bundle. It is clear that π is continuous at $v=(0,1)$ but $\mathrm{Orb}(v)$ is not Hausdorff in N.

In Example 2.7 we show that condition (3) of Proposition 1.1 cannot be replaced by the following one: If $u,w\in\mathrm{Orb}(v)$ and $\pi(u)\neq\pi(w)$, then $\{u,w\}$ is Hausdorff with respect to the induced topology.

Example 2.7. Let $M=\mathbb{R}$. Endow $N:=\mathbb{R}\times\mathbb{R}$ with the minimal topology such that for all $\alpha,\beta\in\mathbb{R}$, $\mathbb{R}\times\mathbb{R}-((-\infty,\alpha]\times\{0\}\cup[\beta,\infty)\times\{0\})$ is open. Define $E:\mathrm{Diff}(M)\to\mathrm{Homeo}(N)$ by $E(f)(x,y)=(f(x),y)$. Let $\pi:N\to M$ be the projection onto the first factor. Then $(M,N,E,\pi,\infty,1)$

is a quasi-natural bundle. Moreover, π is not continuous at $v = (0,0)$ though for all $u, w \in \mathrm{Orb}(v)$ the condition $\pi(u) \neq \pi(w)$ implies that $\{u, w\}$ is Hausdorff with respect to the induced topology.

3. Regularity of projections. In this section we prove Theorem 0.3.

We shall use the following notation. For any smooth manifold M, we denote by $A(M)$ the Lie algebra of vector fields on M. We also consider $A(M)$ as a $C^\infty(M)$ module. For any $p \in M$ we write M_p for $\{f \in C^\infty(M) : f(p) = 0\}$. For any $p \in M$ we put $A_p = M_p A(M)$. Let $A^0 = \{X \in A(M) : \mathrm{supp}(X) \text{ is compact}\}$.

In the proof of Theorem 0.3 we use the following proposition.

PROPOSITION 3.1. *Let* $(M, N, E, \pi, \infty, n)$ *be a regular quasi-natural bundle.* (*Then* $\dim(N) > 0$.) *Define* $\widetilde{E} : A^0(M) \to A(N)$ *by* $\widetilde{E}(X)(y) = [t \to E(\varphi_t)(y)]_{t=0}$, *where* $\{\varphi_t\}$ *is the flow of* X. *Then* $\widetilde{E}$ *is a Lie algebra homomorphism.*

Remark. For π a C^∞ submersion, this proposition is well known (see [12], [3]).

Proof of Proposition 3.1. If $X \in A^0(M)$, $X(\pi(x)) \neq 0$ and $\{\varphi_t\}$ is the flow of X, then $\mathbb{R} \ni t \to E(\varphi_t)(x) \in N$ is a nontrivial curve, and thus $\dim(N) > 0$. Therefore Proposition 3.1 is an immediate consequence of the following four lemmas.

LEMMA 3.1. *The mapping* $\widetilde{E} : A^0(M) \to A(N)$ *is regular in the following sense*: *If* $\widetilde{X} : U \times M \to TM$ *is a* C^∞ *mapping* (U *is a* C^∞ *manifold*) *such that for all* $\tau \in U$ *the mapping* $\widetilde{X}_\tau : M \to TM$, $\widetilde{X}_\tau(x) = \widetilde{X}(\tau, x)$, *is an element of* $A^0(M)$, *then*

$$U \times N \ni (\tau, y) \to \widetilde{E}(\widetilde{X}_\tau)(y) \in TN$$

is also of class C^∞.

LEMMA 3.2. $\widetilde{E} : A^0(M) \to A(N)$ *is* $\mathbb{R}$-*linear.*

LEMMA 3.3. *For any* $X \in A^0(M)$ *and any* $\varphi \in \mathrm{Diff}(M)$ *we have* $\widetilde{E}(\varphi_* X) = (E(\varphi))_* \widetilde{E}(X)$.

LEMMA 3.4. *If* $X, Y \in A^0(M)$, *then* $[\widetilde{E}(X), \widetilde{E}(Y)] = \widetilde{E}([X, Y])$.

Proof of Lemma 3.1. It is clear that $\widetilde{E}$ is local. Therefore since smoothness is also a local property, we may assume that there exists a compact set $K \subset U \times M$ such that $\widetilde{X}(\tau, x) = 0$ for all $(\tau, x) \in U \times M - K$. Then we can consider $\widetilde{X}$ as an element of $A^0(U \times M)$. Let $\{\Phi_t\}$ be the global flow of $\widetilde{X}$. Clearly, $\{\Phi_t\}$ is of the form $\Phi_t(\tau, x) = (\tau, \varphi(t, \tau, x))$, where $\varphi : \mathbb{R} \times U \times M \to M$ is of class C^∞. It is easy to verify that for every $\tau \in U$,

$\{\varphi(t,\tau,\cdot)\}$ is the flow of $\widetilde{X}_\tau$. From the regularity of (M,N,E,π,∞,n) it follows that

$$U\times N\ni(\tau,y)\to[t\to E(\varphi(t,\tau,\cdot))(y)]_{t=0}\in TN$$

is of class C^∞. ∎

Proof of Lemma 3.2. We fix $y\in N$ and $X,Y\in A^0(M)$. Define $E_y:\mathbb{R}^2\to T_yN$ by $E_y(\alpha,\beta)=\widetilde{E}(\alpha X+\beta Y)(y)$. If $\{\varphi_t^{\alpha,\beta}\}$ is the flow of $\alpha X+\beta Y$, then $\{\varphi_{\tau t}^{\alpha,\beta}\}$ is the flow of $\tau(\alpha X+\beta Y)$, and thus since $\{E(\varphi_{\tau t}^{\alpha,\beta})\}$ is the flow of $\widetilde{E}(\tau(\alpha X+\beta Y))$ and $\{E(\varphi_t^{\alpha,\beta})\}$ is the flow of $\widetilde{E}(\alpha X+\beta Y)$, we find that $\tau\widetilde{E}(\alpha X+\beta Y)=\widetilde{E}(\tau(\alpha X+\beta Y))$, where $\alpha,\beta,\tau\in\mathbb{R}$. Therefore for every $\alpha,\beta,t\in\mathbb{R}$ we have $E_y(t(\alpha,\beta))=tE_y(\alpha,\beta)$. It follows from Lemma 3.1 that E_y is of class C^∞. Hence E_y is $\mathbb{R}$-linear. ∎

Proof of Lemma 3.3. Let $X\in A^0(M)$ and $\varphi\in\mathrm{Diff}(M)$. If $\{\varphi_t\}$ is the flow of X, then $\{\varphi\circ\varphi_t\circ\varphi^{-1}\}$ is the flow of φ_*X, and so $\{E(\varphi)\circ E(\varphi_t)\circ(E(\varphi))^{-1}\}$ is the flow of both $\widetilde{E}(\varphi_*X)$ and $(E(\varphi))_*\widetilde{E}(X)$. Thus $\widetilde{E}(\varphi_*X)=(E(\varphi))_*\widetilde{E}(X)$. ∎

Proof of Lemma 3.4. Let $X,Y\in A^0(M)$ and let $\{\varphi_t\}$ be the flow of X. If $F:\mathbb{R}\times M\to TM$ is given by $F(t,x)=((\varphi_t)_*X)(x)-X(x)$, then (since $F(0,\cdot)=0$) there exists $\widetilde{F}:\mathbb{R}\times M\to TM$ of class C^∞ such that $F(t,x)=t\widetilde{F}(t,x)$ for every $(t,x)\in\mathbb{R}\times M$. Therefore $\widetilde{X}:\mathbb{R}\times M\to TM$ given by

$$\widetilde{X}(t,x)=\begin{cases}-\dfrac{((\varphi_t)_*X)(x)-X(x)}{t}, & t\neq 0,\\[2mm] -\lim\limits_{t\to0}\dfrac{((\varphi_t)_*X)(x)-X(x)}{t}, & t=0,\end{cases}$$

is of class C^∞. It follows from Lemmas 3.1–3.3 that

$$\begin{aligned}\widetilde{E}([X,Y])(y)&=\widetilde{E}(\lim_{t\to0}\widetilde{X}_t)(y)=\lim_{t\to0}\widetilde{E}(\widetilde{X}_t)(y)\\&=\lim_{t\to0}\widetilde{E}\Big(-\frac{(\varphi_t)_*X-X}{t}\Big)(y)\\&=-\lim_{t\to0}\frac{((E(\varphi_t))_*\widetilde{E}(X))(y)-\widetilde{E}(X)(y)}{t}\\&=[\widetilde{E}(X),\widetilde{E}(Y)](y).\ \blacksquare\end{aligned}$$

Proof of Theorem 0.3. Let $\widetilde{E}:A^0(M)\to A(N)$ be the Lie algebra homomorphism defined in Proposition 3.1. Define a Lie algebra homomorphism $\varphi:A(M)\to A(N)$ by $\varphi(X)(y):=\widetilde{E}(\widetilde{X})(y)$, where $X\in A(M)$, $y\in N$ and $\widetilde{X}\in A^0(M)$ is such that $\mathrm{germ}_{\pi(y)}(\widetilde{X})=\mathrm{germ}_{\pi(y)}(X)$.

First we show that $N^+:=\{q\in N:\varphi^{-1}(A_q(N))\neq A(M)\}=N$. Let $q\in N$ and let $X\in A^0(M)$ be such that $X(\pi(q))\neq 0$. Then $\varphi(X)(q)\neq 0$, for if $\{\varphi_t\}$ is the flow of X, then there exists $\varepsilon > 0$ such that $\varphi_t(\pi(q)) \neq \pi(q)$ for all $t \in (0, \varepsilon)$; hence $E(\varphi_t)(q) \neq q$ for all $t \in (0, \varepsilon)$, and thus $\varphi(X)(q) \neq 0$. We have proved that $N^+ = N$.

By the results of K. Masuda [7, pp. 509–511] for every $q \in N^+$ there exists a unique non-empty finite subset $\psi(q) = \{p_1, \ldots, p_l\}$ of M such that $d/n \geq l$ and

$$\bigcap_{i=1}^{l} M_{p_i} A(M) \supset \varphi^{-1}(A_q(N)) \supset \bigcap_{i=1}^{l} M_{p_i}^{h+1} A(M),$$

where $h = 2((d - nl)^2 + d - nl) + 1$, $d = \dim N$ and $n = \dim M$. We prove that $\psi(q) = \{\pi(q)\}$ for every $q \in N$.

Let $q \in N = N^+$. It is sufficient to show that $p \notin \psi(q)$ for every $p \in M - \{\pi(q)\}$. Assume the contrary. Let $p \in M - \{\pi(q)\}$ with $p \in \psi(q)$. Then (in particular) $\varphi^{-1}(A_q(N)) \subset M_p A(M)$. On the other hand, if $X \in A^0(M)$ is such that $X(p) \neq 0$ and $\mathrm{germ}_{\pi(q)}(X) = \mathrm{germ}_{\pi(q)}(0)$, then $\varphi(X)(q) = 0$ (i.e. $X \in \varphi^{-1}(A_q(N))$) and $X \notin M_p A(M)$, a contradiction.

Therefore $\mathrm{int}\{q \in N^+ : \mathrm{card}\, \psi(q) = 1\} = N$. Hence by the results of K. Masuda [7, pp. 509–511], for every $q \in N$ there exist an open neighbourhood U of q, an open dense subset $V \subset U$, an open neighbourhood U_1 of $\pi(q)$ and a continuous mapping $\widetilde{\psi} : U \to U_1$ such that $\widetilde{\psi}|V$ is a C^∞ submersion and for any $q \in U$ the equality $\widetilde{\psi}(q) = p$ implies $\psi(q) = \{p\}$. ∎

Problem. Let $(M, N, E, \pi, \infty, n)$ be a regular quasi-natural bundle. Is π of class C^∞?

4. Prolongation functors with compact fibres. In this section we prove Theorem 0.4. In the proof we use the following proposition which is similar to Proposition 14 of [13].

Proposition 4.1. (I) *Let F be a quasi-prolongation functor over C^r manifolds such that conditions* (1)–(3) *of Theorem* 0.4(I) *are satisfied. Suppose that $f : P \times M \to N$ is of class C^r, where M, N, P are C^r manifolds. Then*

$$P \times FM \ni (t, x) \to F(f(t, \cdot))(x) \in FN$$

is continuous.

(II) *Let F be a regular quasi-prolongation functor over C^∞ manifolds such that condition* (1) *of Theorem* 0.4(II) *is satisfied. Suppose that $f : P \times M \to N$ is of class C^∞, where M, N, P are C^∞ manifolds. Then*

$$P \times FM \ni (t, x) \to F(f(t, \cdot))(x) \in FN$$

is of class C^∞.

P r o o f o f P r o p o s i t i o n 4.1. Let $(t_0, x_0) \in P \times M$. Let (U, g) be a chart of P at t_0 and let (V, h) be a chart of M at x_0 such that $g(U) = \mathbb{R}^p$ and $h(V) = \mathbb{R}^m$, where $p = \dim(P)$ and $m = \dim(M)$. It is clear that $f(t, \cdot) \circ h^{-1}(y) = f \circ (g^{-1} \times h^{-1}) \circ \tau_{(g(t),0)} \circ i(y)$ for every $(t, y) \in U \times \mathbb{R}^m$, where $i : \mathbb{R}^m \to \mathbb{R}^p \times \mathbb{R}^m$ is given by $i(y) = (0, y)$ and $\tau_{(g(t),0)} : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p \times \mathbb{R}^m$ is the translation by $(g(t), 0)$. Therefore

$$F(f(t, \cdot))(y) = F(f \circ (g^{-1} \times h^{-1})) \circ F(\tau_{(g(t),0)}) \circ F(i) \circ (F(h^{-1}))^{-1}(y)$$

for every $(t, y) \in U \times \pi_M^{-1}(V)$. On the other hand, for every C^r manifold $\widetilde{N}$, $(\widetilde{N}, F\widetilde{N}, F|\operatorname{Diff}(\widetilde{N}), \pi_{\widetilde{N}}, r, \dim(\widetilde{N}))$ is a quasi-natural bundle and $F\widetilde{N}$ is a Hausdorff space, and thus (by Theorem 0.1) $\pi_{\widetilde{N}} : F\widetilde{N} \to \widetilde{N}$ is continuous. Hence for every n the restriction of F to the category of n-dimensional C^r manifolds and embeddings is a locally determined associated space such that $F\mathbb{R}^n$ is a second countable Hausdorff space. Therefore for every n the mapping

$$\mathbb{R}^n \times F\mathbb{R}^n \ni (x, y) \to F(\tau_x)(y) \in F\mathbb{R}^n ,$$

where $\tau_x : \mathbb{R}^n \to \mathbb{R}^n$ is the translation by x, is continuous (see Proposition 1.1 of [8]). It follows that $U \times \pi_M^{-1}(V) \ni (t, y) \to F(f(t, \cdot))(y) \in FN$ is continuous. Since continuity is a local property, Proposition 4.1(I) is proved. The proof of Proposition 4.1(II) is similar. (We need the well-known theorem of Montgomery and Zippin [9] on continuous Lie group actions.) ■

The main difficulty in proving Theorem 0.4 is to show the following propositions.

PROPOSITION 4.2. *Suppose the assumptions of Theorem* 0.4(I) *are satisfied. Then F is of order* 0, *i.e. if $f, g : M \to N$ are C^r mappings of C^r manifolds such that $f(x) = g(x)$ for some $x \in M$, then $F(f)|\pi_M^{-1}(x) = F(g)|\pi_M^{-1}(x)$.*

PROPOSITION 4.3. *Let F be a regular quasi-prolongation functor over C^∞ manifolds. Assume that F is of order* 0 *and $F\mathbb{R}^n$ has a countable basis for every n. Then for every C^∞ manifold M, $\pi_M : FM \to M$ is a C^∞ submersion.*

R e m a r k. If F is a regular quasi-prolongation functor over C^∞ manifolds such that $F\mathbb{R}^n$ has a countable basis for every n, then from Proposition 4.1(II) and Theorem 0.3 it follows that if $F\mathbb{R}^n$ is connected, then for every C^∞ manifold M, $\pi_M : FM \to M$ is of class C^∞ on some open dense subset of FM. The answer to the question "Is π_M of class C^∞?" is unknown. Proposition 4.3 ensures that for F of order 0 the answer is affirmative.

P r o o f o f P r o p o s i t i o n 4.2. Let (U, h) be a chart of M at x such that $h(U) = \mathbb{R}^m$ and $h(x) = 0$. It is sufficient to show that $F(f \circ h^{-1})(z) = F(g \circ h^{-1})(z)$ for every $z \in \pi^{-1}(0)$, where $\pi = \pi_{\mathbb{R}^m}$.

Let $z \in \pi^{-1}(0)$ and let $\varphi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be given by $\varphi(t, x) = tx$. Since $\varphi(t, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism for every $t \in \mathbb{R} - \{0\}$, we have $F(\varphi(t, \cdot))(\pi^{-1}(0)) = \pi^{-1}(0)$ for every $t \in \mathbb{R} - \{0\}$. In particular, there exists a sequence $y_n \in \pi^{-1}(0)$ such that $F(\varphi(1/n, \cdot))(y_n) = z$ for every n. Since $\pi^{-1}(0)$ is compact and second countable, it is a metrizable compact space, and thus there exist $y \in \pi^{-1}(0)$ and a subsequence $n(k)$ $(k = 1, 2, \ldots)$ of $(1, 2, \ldots)$ such that $y_{n(k)} \to y$ as $k \to \infty$. Then $F(\varphi(1/n(k), \cdot))(y_{n(k)}) \to F(0)(y)$ as $k \to \infty$ because of Proposition 4.1(I). Since $\pi^{-1}(0)$ is Hausdorff, $F(0)(y) = z$. Therefore $F(f \circ h^{-1})(z) = F(f \circ h^{-1}) \circ F(0)(y) = F(f \circ h^{-1} \circ 0)(y) = F(g \circ h^{-1} \circ 0)(y) = F(g \circ h^{-1})(z)$. ∎

P r o o f o f P r o p o s i t i o n 4.3. First we prove that $\pi_M : FM \to M$ is of class C^∞. It suffices to show that $f \circ \pi_M \in C^\infty(FM)$ for every $f \in C^\infty(M)$.

Let $Y \in A^0(M)$. Since F satisfies the assumptions of Proposition 4.1(II), $(M, FM, F|\operatorname{Diff}(M), \pi_M, \infty, \dim(M))$ is a regular quasi-natural bundle. Let $\widetilde{F} : A^0(M) \to A(FM)$ be the Lie algebra homomorphism described in Proposition 3.1 (with $F|\operatorname{Diff}(M)$ playing the role of E). In the proof of Theorem 0.3 we have shown that $\widetilde{F}(Y)(y) \neq 0$ for all $y \in \pi_M^{-1}(\{x \in M : Y(x) \neq 0\})$. On the other hand, if $X \in A^0(M)$ is such that $X(x_0) = 0$, then $\widetilde{F}(X)(z) = 0$ for all $z \in \pi_M^{-1}(x_0)$, for if $\{\varphi_t\}$ is the flow of X and $z \in \pi_M^{-1}(x_0)$, then $\varphi_t(x_0) = x_0$ for all $t \in \mathbb{R}$, and thus (since F is of order 0) $F(\varphi_t)(z) = F(\mathrm{id}_M)(z) = z$ for all $t \in \mathbb{R}$, hence $\widetilde{F}(X)(z) = 0$. Moreover, for every $y \in FM$, $(f - f(\pi_M(y)))Y|\pi_M(y) = 0$. Owing to these facts, we have $0 = \widetilde{F}((f - f(\pi_M(y)))Y)(y) = \widetilde{F}(fY)(y) - f(\pi_M(y))\widetilde{F}(Y)(y)$ for every $y \in FM$, and hence $f \circ \pi_M|\pi_M^{-1}(\{x \in M : Y(x) \neq 0\})$ is of class C^∞. Therefore $f \circ \pi_M$ is of class C^∞.

To show that π_M is a submersion, let $y \in FM$ and let (U, g) be a chart of M at $\pi_M(y)$ such that $g(U) = \mathbb{R}^m$ and $g(\pi_M(y)) = 0$. Since $\pi_M|\pi_M^{-1}(U) = g^{-1} \circ \pi \circ (F(g^{-1}))^{-1}$ where $\pi = \pi_{\mathbb{R}^m}$, it suffices to show that $\operatorname{rank}(d_v\pi) = m$ for every $v \in \pi^{-1}(0)$. Indeed, it follows from Proposition 4.1(II) that the mapping $\sigma : \mathbb{R}^m \to F\mathbb{R}^m$, $\sigma(x) = F(\tau_x)(v)$, is of class C^∞. We see that $\pi \circ \sigma = \mathrm{id}$ and $\sigma(0) = v$. Therefore $\operatorname{rank}(d_v\pi) = m$. ∎

We are now in a position to prove Theorem 0.4.

P r o o f o f T h e o r e m 0.4. Let $G := \pi_{\mathbb{R}}^{-1}(0)$. Then G is a compact second countable space. If F is regular it follows from Proposition 4.3 that G is a C^∞ manifold. For every C^r manifold M we define a mapping $I_M : M \times G \to FM$ by $I_M(m, f) = F(m)(f)$, where $m : \mathbb{R} \to M$ is given by $m(x) = m$. We are going to prove that $\{I_M\}$ gives an isomorphism between $(\ ) \times G$ and F.

By Proposition 4.2, F is of order 0. Therefore the mapping $T_M : FM \to M \times G$ given by $T_M(v) = (\pi_M(v), F(0)(v))$ is equal to I_M^{-1}. From Proposition 4.1 we know that I_M is continuous (and of class C^∞ if F is regular). Similarly, it follows from Theorem 0.1 and Proposition 4.3 that T_M is continuous (and of class C^∞ if F is regular). Hence I_M is a homeomorphism (and a C^∞ diffeomorphism if F is regular).

We see that $p_M(m, f) = m = \pi_M \circ F(m)(f) = \pi_M \circ I_M(m, f)$ and $I_N \circ ((\ ) \times G)(g)(m, f) = I_N(g(m), f) = F(g(m))(f) = F(g \circ m)(f) = F(g) \circ F(m)(f) = F(g) \circ I_M(m, f)$ for every $(m, f) \in M \times G$. ■

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