On Cauchy–Riemann submanifolds whose local geodesic symmetries preserve the fundamental form

by Sorin Dragomir (Stony Brook, N.Y.) and Mauro Capursi (Bari)

Abstract. We classify generic Cauchy–Riemann submanifolds (of a Kaehlerian manifold) whose fundamental form is preserved by any local geodesic symmetry.

Introduction. Let $(M^{2m}, g, J)$ be a Hermitian manifold of complex dimension $m$, where $g$ denotes the Hermitian metric, while $J$ stands for the complex structure. Let $\Psi : M^n \to M^{2m}$ be an isometric immersion of a real $n$-dimensional Riemannian manifold $(M^n, g)$ in $M^{2m}$. Let $E \to M^n$ be the normal bundle of $\Psi$. Then $M^n$ is a Cauchy–Riemann (C.R.) submanifold of $M^{2m}$ if it carries a pair of complementary distributions $(D, D^\perp)$ such that $D$ is holomorphic (i.e. $J_x(D_x) = D_x$, $x \in M^n$) and $D^\perp$ is totally real (i.e. $J_x(D^\perp_x) \subseteq E_x$, $x \in M^n$). Let $\tan_x, \nor_x$ be the natural projections associated with the direct sum decomposition $T_x(M^{2m}) = T_x(M^n) \oplus E_x$, $x \in M^n$.

Each C.R. submanifold $M^n$ of an (almost) Hermitian manifold $M^{2m}$ is known to possess a natural $f$-structure $P$ (in the sense of K. Yano [7]) given by $PX = \tan(JX)$. The fundamental 2-form $\Omega$ of $M^n$ is given by $\Omega(X,Y) = g(X,PY)$, for any tangent vector fields $X, Y$ on $M^n$. In the present note we are concerned with the following:

Problem. Let $M^{2m}$ be an (almost) Hermitian manifold. Classify its C.R. submanifolds $M^n$ all of whose local geodesic symmetries preserve the fundamental form.

Let $M^n$ be such a C.R. submanifold. Set $q = \dim \ker D^\perp_x$, $x \in M^n$. By a result of K. Sekigawa–L. Vanhecke [5], if $M^n$ is invariant (i.e. $q = 0$) then...
$M^n$ is a locally symmetric Kaehler manifold. Our contribution regards the generic case (i.e. $q = \text{codim} M^n$) and consists in the following:

**Theorem.** Let $M^n$ be an $n$-dimensional generic C.R. submanifold of the Kaehlerian manifold $M^{2m}$. If all local geodesic symmetries of $M^n$ preserve the fundamental form, then $M^n$ is locally a Riemannian product $M^{2(n-m)} \times M^{2m-n}$, where $M^{2(n-m)}$ is a totally geodesic complex submanifold of $M^{2m}$, while $M^{2m-n}$ is a totally real submanifold of $M^{2m}$.

The key ingredient in the proof is a result by A. Gray [4], which provides power series expansions for analytic covariant tensor fields in normal coordinates.

**2. Proof of the Theorem.** Let $(M^n, D, D^\perp)$ be a C.R. submanifold of the Kaehlerian manifold $(M^{2m}, g, J)$. Let $x \in M^n$ and $(U_x, x^i)$ a local system of normal coordinates at $x$, $x_i(x) = 0$, $1 \leq i \leq n$. Let $R_{ijkl}$ denote the Riemann–Christoffel tensor field of $(M^n, g)$ and $\nabla_i$ covariant differentiation. Let $W_{\alpha_1...\alpha_s}$ be an analytic covariant tensor field of type $(0,s)$ on $M^n$. By a result of A. Gray [4], if $p \in U_x$ then

(2.1) $W_{\alpha_1...\alpha_s}(p) = W_{\alpha_1...\alpha_s}(x) + \sum_{i=1}^n (\nabla_i W_{\alpha_1...\alpha_s})(x)x^i$

$$+ \frac{1}{2} \sum_{i,j=1}^n \left\{ \nabla_{ij}^2 W_{\alpha_1...\alpha_s} - \frac{1}{3} \sum_{t=1}^n \sum_{h=1}^s R_{ijhkt} W_{\alpha_1...\alpha_{h-1}t\alpha_{h+1}...\alpha_s} \right\}(x)x^i x^j$$

$$+ \frac{1}{6} \sum_{i,j,k=1}^n \left\{ \nabla_{ijk}^3 W_{\alpha_1...\alpha_s} - \sum_{t=1}^n \sum_{h=1}^s R_{ijhkt}(\nabla_k W_{\alpha_1...\alpha_{h-1}t\alpha_{h+1}...\alpha_s}) \right\}(x)x^i x^j x^k + \theta(x^4)$$

where $x^i = x^i(p)$ (cf. also B. Y. Chen–L. Vanhecke [1], p. 31).

Let $\gamma: r \rightarrow \exp_x(rX) \in U_x$ be a geodesic (parametrized by arc length) of $(M^n, g)$, where $X \in T_x(M^n)$, $\|X\| = 1$. Let $\Omega$ be the fundamental 2-form of $M^n$. We work under the basic assumption that each local geodesic symmetry $f : \exp_x(rX) \rightarrow \exp_x(\rho rX)$ preserves $\Omega$, i.e. $f^* \Omega = \Omega$, or

(2.2) $\Omega_{ij}(\exp_x(rX)) = \Omega_{ij}(\exp_x(\rho rX))$.

The local parametric equations (in normal coordinates) of $\gamma$ are $x^i(r) = rX^i$, $1 \leq i \leq n$. Thus the power series expansion formula (2.1) leads
to the expansion

\[(2.3)\quad \Omega_{\alpha_1\alpha_2}(\gamma(r)) = \Omega_{\alpha_1\alpha_2}(x) + \sum_{i=1}^{n} X^i(\nabla_i \Omega_{\alpha_1\alpha_2})(x)r^i + \frac{1}{2} \sum_{i,j=1}^{n} X^i X^j \{ \nabla_{\alpha_1}^2 \Omega_{\alpha_1\alpha_2} - \frac{1}{3} \sum_{i=1}^{n} (R_{\alpha_1\alpha_2} \Omega_{\alpha_1\alpha_2} + R_{\alpha_2\alpha_1} \Omega_{\alpha_1\alpha_2}) \} (x)r^i r^j \]

\[+ \frac{1}{6} \sum_{i,j,k=1}^{n} X^i X^j X^k \{ \nabla_{\alpha_1}^3 \Omega_{\alpha_1\alpha_2} - \sum_{i=1}^{n} (R_{\alpha_1\alpha_2} \nabla_k \Omega_{\alpha_1\alpha_2} + R_{\alpha_2\alpha_1} \nabla_k \Omega_{\alpha_1\alpha_2}) \} (x)r^i r^j r^k \]

\[\quad - \frac{1}{2} \sum_{i=1}^{n} (\nabla_{\alpha_1} \nabla_i R_{j\alpha_2 \beta \kappa} + \nabla_{\alpha_2} \nabla_i R_{j\alpha_1 \beta \kappa} ) (x)r^i r^j \Omega_{\alpha_1\alpha_2} = 0, \text{ i.e. the canonical}\]

\[\text{f-structure of } M^n \text{ is parallel and} \]

\[\nabla X R(PY, X)X + P(\nabla X R)(Y, X)X = 0 \]

for any \( Y \in T_x(M^n) \). Set \( FZ = \text{nor}(JZ) \) for any tangent vector field \( Z \) on \( M^n \). Then \( F \) is a normal bundle valued 1-form on \( M^n \) vanishing on the holomorphic distribution. Set \( t\xi = \tan(J\xi) \), \( f\xi = \text{nor}(J\xi) \), for any cross-section \( \xi \) in \( E \rightarrow M^n \). Clearly, if \( M^n \) is generic (\( q = 2m - n \)) then \( f = 0 \). Let \( \sigma \) be the second fundamentanl form of \( \Psi \) and \( \alpha \xi \) the Weingarten operator (associated with the normal section \( \xi \)). Let \( \nabla_{\xi} \) be the Levi-Civita connection of \((M^{2m}, \tilde{g})\). Yet \( \tilde{g} \) is Kaehlerian, i.e. \( \nabla J = 0 \); by the Gauss and Weingarten formulae (see e.g. eqs. (1.1)–(1.2) of [9], p. 19) and identification of tangential, respectively normal, components, one obtains

\[(2.6)\quad (\nabla X P)Y = a_{PY}X + t\sigma(X, Y), \]

\[(2.7)\quad (\nabla X F)Y = -\sigma(X, PY) \]

for any tangent vector fields \( X, Y \) on \( M^n \). As \( P \) is parallel, \( FP = 0 \) and
\( t = J \), formula (2.6) gives

\[
(2.8) \quad \sigma(X, PY) = 0
\]

and by (2.7), \( F \) is parallel, too. As a consequence both the holomorphic and totally real distributions are parallel and thus \( M^n \) is locally a Riemannian product. It is easily seen that (2.8) also yields that \( M^{2(n-m)} \) is totally geodesic.

**Remarks.** (i) Let \( M^{2m} \) be a complete simply connected complex space-form (of constant holomorphic curvature \( c \)). Combining our Theorem with a result by K. Yano–M. Kon [8], one shows that if \( M^n \) is a complete generic submanifold obeying (2.2) then either \( M^n \) is an \( m \)-dimensional totally real (i.e. \( P = 0 \)) submanifold of \( M^{2m} \), or \( c = 0 \) and\( M^n \) is congruent to \( \mathbb{C}^{n-m} \times M^{2m-n} \), where \( M^{2m-n} \) is a totally real submanifold of \( \mathbb{C}^m \).

(ii) The curvature identity (2.5) does not contribute further to the classification of generic submanifolds (subject to (2.2)) of complex space-forms. Indeed, by remark (i), either \( P = 0 \) and thus (2.5) is identically satisfied, or \( c = 0 \) and then (by the Gauss eq. (1.10) of [9], p. 78),

\[
R(Y, Z)W = a_\sigma(Z, W)Y - a_\sigma(Y, W)Z
\]

for any tangent vector fields \( Y, Z, W \) on \( M^n \). As \( M^n \) is generic, any normal field \( \xi \) may be written as \( \xi = FY \) for some \( Y \) tangent to \( M^n \). Thus (2.6) and (2.8) yield \( a_\xi PY = 0 \) for any \( Y, \xi \). Consequently, \( R(PY, Z)W = R(Y, PZ)W = R(Y, Z)PW = 0 \) and (2.5) turns into \( P\nabla_X(R(Y, X)i_\perp X) = 0 \), where \( i_\perp = -tF \); now this is identically satisfied, as \( D_\perp \) is parallel.

(iii) Let \( M^{2m} \) be a locally conformal Kaehler manifold (see e.g. I. Vaisman [6]). Let \( M^n \) be a generic C.R. submanifold subject to (2.2). Repeating the arguments in the proof of our Theorem, we obtain \( \nabla P = 0 \). Then, by a result of [2; II, Th. 1, p. 2], if the 1-form \( \omega \) induced by the Lee form of \( M^{2m} \) has no singular points (i.e. \( \omega_x \neq 0 \), for any \( x \in M^n \)) then \( M^n \) is a totally real submanifold of \( M^{2m} \) (see also [3]).

**References**


\textit{Cauchy–Riemann submanifolds}