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On Cauchy–Riemann submanifolds whose local geodesic symmetries preserve the fundamental form

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Abstract. We classify generic Cauchy–Riemann submanifolds (of a Kaehlerian manifold) whose fundamental form is preserved by any local geodesic symmetry.

Introduction. Let $(M^{2m}, \overline{g}, J)$ be a Hermitian manifold of complex dimension m, where $\overline{g}$ denotes the Hermitian metric, while J stands for the complex structure. Let $\Psi : M^n \to M^{2m}$ be an isometric immersion of a real n-dimensional Riemannian manifold (M^n, g) in M^{2m}. Let $E \to M^n$ be the normal bundle of Ψ. Then M^n is a *Cauchy–Riemann* (C.R.) *submanifold* of M^{2m} if it carries a pair of complementary distributions $(D, D^\perp)$ such that D is holomorphic (i.e. $J_x(D_x) = D_x$, $x \in M^n$) and $D^\perp$ is totally real (i.e. $J_x(D_x^\perp) \subseteq E_x$, $x \in M^n$). Let $\tan_x$, nor_x be the natural projections associated with the direct sum decomposition

$$T_x(M^{2m}) = T_x(M^n) \oplus E_x\,, \quad x \in M^n\,.$$

Each C.R. submanifold M^n of an (almost) Hermitian manifold M^{2m} is known to possess a natural f-structure P (in the sense of K. Yano [7]) given by $PX = \tan(JX)$. The *fundamental* 2-*form* Ω of M^n is given by

$$\Omega(X, Y) = g(X, PY)\,,$$

for any tangent vector fields X, Y on M^n. In the present note we are concerned with the following:

Problem. *Let M^{2m} be an* (*almost*) *Hermitian manifold. Classify its C.R. submanifolds M^n all of whose local geodesic symmetries preserve the fundamental form.*

Let M^n be such a C.R. submanifold. Set $q = \dim_{\mathbb{R}} D_x^\perp$, $x \in M^n$. By a result of K. Sekigawa–L. Vanhecke [5], if M^n is invariant (i.e. $q = 0$) then

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M^n is a locally symmetric Kaehler manifold. Our contribution regards the generic case (i.e. $q = \operatorname{codim} M^n$) and consists in the following:

THEOREM. *Let M^n be an n-dimensional generic C.R. submanifold of the Kaehlerian manifold M^{2m}. If all local geodesic symmetries of M^n preserve the fundamental form, then M^n is locally a Riemannian product $M^{2(n-m)} \times M^{2m-n}$, where $M^{2(n-m)}$ is a totally geodesic complex submanifold of M^{2m}, while M^{2m-n} is a totally real submanifold of M^{2m}.*

The key ingredient in the proof is a result by A. Gray [4], which provides power series expansions for analytic covariant tensor fields in normal coordinates.

2. Proof of the Theorem. Let $(M^n, D, D^\perp)$ be a C.R. submanifold of the Kaehlerian manifold $(M^{2m}, \overline{g}, J)$. Let $x \in M^n$ and (U_x, x^i) a local system of normal coordinates at x, $x^i(x) = 0$, $1 \le i \le n$. Let R_{ijkl} denote the Riemann–Christoffel tensor field of (M^n, g) and ∇_i covariant differentiation. Let $W_{\alpha_1 \ldots \alpha_s}$ be an analytic covariant tensor field of type $(0, s)$ on M^n. By a result of A. Gray [4], if $p \in U$ then

$$
\begin{aligned}
(2.1)\qquad W_{\alpha_1\ldots\alpha_s}(p) &= W_{\alpha_1\ldots\alpha_s}(x) + \sum_{i=1}^{n} (\nabla_i W_{\alpha_1\ldots\alpha_s})(x) x^i \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} \Big\{ \nabla^2_{ij} W_{\alpha_1\ldots\alpha_s} - \frac{1}{3}\sum_{t=1}^{n}\sum_{h=1}^{s} R_{i\alpha_h jt} W_{\alpha_1\ldots\alpha_{h-1} t \alpha_{h+1}\ldots\alpha_s} \Big\}(x) x^i x^j \\
&+ \frac{1}{6}\sum_{i,j,k=1}^{n} \Big\{ \nabla^3_{ijk} W_{\alpha_1\ldots\alpha_s} - \sum_{t=1}^{n}\sum_{h=1}^{s} R_{i\alpha_h jt} (\nabla_k W_{\alpha_1\ldots\alpha_{h-1} t \alpha_{h+1}\ldots\alpha_s}) \\
&- \frac{1}{2}\sum_{t=1}^{n}\sum_{h=1}^{s} (\nabla_i R_{j\alpha_h kt}) W_{\alpha_1\ldots\alpha_{h-1} t \alpha_{h+1}\ldots\alpha_s} \Big\}(x) x^i x^j x^k + \theta(x^4)
\end{aligned}
$$

where $x^i = x^i(p)$ (cf. also B. Y. Chen–L. Vanhecke [1], p. 31).

Let $\gamma : r \to \exp_x(rX) \in U_x$ be a geodesic (parametrized by arc length) of (M^n, g), where $X \in T_x(M^n)$, $||X|| = 1$. Let Ω be the fundamental 2-form of M^n. We work under the basic assumption that each local geodesic symmetry $f : \exp_x(rX) \to \exp_x(-rX)$ preserves Ω, i.e. $f^*\Omega = \Omega$, or

$$(2.2)\qquad \Omega_{ij}(\exp_x(rX)) = \Omega_{ij}(\exp_x(-rX))\,.$$

The local parametric equations (in normal coordinates) of γ are $x^i(r) = rX^i$, $1 \le i \le n$. Thus the power series expansion formula (2.1) leads to the expansion

$$\Omega_{\alpha_1\alpha_2}(\gamma(r)) = \Omega_{\alpha_1\alpha_2}(x) + \sum_{i=1}^{n} X^i(\nabla_i \Omega_{\alpha_1\alpha_2})(x)r \tag{2.3}$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n} X^i X^j \Big\{\nabla^2_{ij}\Omega_{\alpha_1\alpha_2} - \frac{1}{3}\sum_{t=1}^{n}(R_{i\alpha_1 jt}\Omega_{t\alpha_2} + R_{i\alpha_2 jt}\Omega_{\alpha_1 t})\Big\}(x)r^2$$
$$+ \frac{1}{6}\sum_{i,j=1}^{n} X^i X^j X^k \Big\{\nabla^3_{ijk}\Omega_{\alpha_1\alpha_2} - \sum_{t=1}^{n}(R_{i\alpha_1 jt}\nabla_k\Omega_{t\alpha_2} + R_{i\alpha_2 jt}\nabla_k\Omega_{\alpha_1 t})$$
$$- \frac{1}{2}\sum_{t=1}^{n}(\Omega_{t\alpha_2}\nabla_i R_{j\alpha_1 kt} + \Omega_{\alpha_1 t}\nabla_i R_{j\alpha_2 kt})\Big\}(x)r^3 + \theta(r^4)\,.$$

Let $\{e_i\}_{1\le i\le n}$ be an orthonormal basis of $T_x(M^n)$ such that $e_1 = X$. We suppose the normal coordinates at x have been chosen such that $\partial/\partial x^i\,|_x = e_i$, $1 \le i \le n$. By straightforward computation our (2.3) turns into

$$\begin{aligned}\Omega_{ij}(\gamma(r)) = {} & \langle e_i, P_x e_j\rangle + \langle e_i, (\nabla P)_x(X, e_j)\rangle r \\ & + \tfrac{1}{2}\langle e_i, (\nabla^2 P)_x(X,X,e_j) - \tfrac{1}{3}(RP + PR)_x e_j\rangle r^2 \\ & + \tfrac{1}{6}\langle e_i, (\nabla^3 P)_x(X,X,X,e_j) - R_x(\nabla P)_x(X,e_j) \\ & - (\nabla P)_x(X, R_x e_j) - \tfrac{1}{2}\{(\nabla_X R)(P_x e_j, X)X \\ & + P_x(\nabla_X R)(e_j, X)X\}\rangle r^3 + \theta(r^4)\end{aligned} \tag{2.4}$$

where $\langle\ ,\ \rangle = g_x$ and R_x denotes the transformation $R_x v = R(v,X)X$, $v \in T_x(M^n)$. Next (2.2) and (2.4) furnish $\nabla P = 0$, i.e. the canonical f-structure of M^n is parallel and

$$(\nabla_X R)(PY, X)X + P(\nabla_X R)(Y,X)X = 0 \tag{2.5}$$

for any $Y \in T_x(M^n)$. Set $FZ = \mathrm{nor}(JZ)$ for any tangent vector field Z on M^n. Then F is a normal bundle valued 1-form on M^n vanishing on the holomorphic distribution. Set $t\xi = \tan(J\xi)$, $f\xi = \mathrm{nor}(J\xi)$, for any cross-section ξ in $E \to M^n$. Clearly, if M^n is generic ($q = 2m - n$) then $f = 0$. Let σ be the second fundamental form of Ψ and a_ξ the Weingarten operator (associated with the normal section ξ). Let $\overline{\nabla}$ be the Levi-Civita connection of $(M^{2m}, \overline{g})$. Yet $\overline{g}$ is Kaehlerian, i.e. $\overline{\nabla}J = 0$; by the Gauss and Weingarten formulae (see e.g. eqs. (1.1)–(1.2) of [9], p. 19) and identification of tangential, respectively normal, components, one obtains

$$(\nabla_X P)Y = a_{FY}X + t\sigma(X,Y)\,, \tag{2.6}$$
$$(\nabla_X F)Y = -\sigma(X, PY) \tag{2.7}$$

for any tangent vector fields X, Y on M^n. As P is parallel, $FP = 0$ and

$t = J$, formula (2.6) gives

$$\sigma(X, PY) = 0 \tag{2.8}$$

and by (2.7), F is parallel, too. As a consequence both the holomorphic and totally real distributions are parallel and thus M^n is locally a Riemannian product. It is easily seen that (2.8) also yields that $M^{2(n-m)}$ is totally geodesic. ■

Remarks. (i) Let M^{2m} be a complete simply connected complex space-form (of constant holomorphic curvature c). Combining our Theorem with a result by K. Yano–M. Kon [8], one shows that if M^n is a complete generic submanifold obeying (2.2) then *either* M^n *is an* m*-dimensional totally real* (i.e. $P = 0$) *submanifold of* M^{2m}, *or* $c = 0$ *and* M^n *is congruent to* $\mathbb{C}^{n-m} \times M^{2m-n}$, *where* M^{2m-n} *is a totally real submanifold of* $\mathbb{C}^m$.

(ii) The curvature identity (2.5) does not contribute further to the classification of generic submanifolds (subject to (2.2)) of complex space-forms. Indeed, by remark (i), either $P = 0$ and thus (2.5) is identically satisfied, or $c = 0$ and then (by the Gauss eq. (1.10) of [9], p. 78), $R(Y, Z)W = a_{\sigma(Z,W)}Y - a_{\sigma(Y,W)}Z$ for any tangent vector fields Y, Z, W on M^n. As M^n is generic, any normal field ξ may be written as $\xi = FY$ for some Y tangent to M^n. Thus (2.6) and (2.8) yield $a_\xi PY = 0$ for any Y, ξ. Consequently, $R(PY, Z)W = R(Y, PZ)W = R(Y, Z)PW = 0$ and (2.5) turns into $P\nabla_X(R(Y, X)i^\perp X) = 0$, where $i^\perp = -tF$; now this is identically satisfied, as $D^\perp$ is parallel.

(iii) Let M^{2m} be a locally conformal Kaehler manifold (see e.g. I. Vaisman [6]). Let M^n be a generic C.R. submanifold subject to (2.2). Repeating the arguments in the proof of our Theorem, we obtain $\nabla P = 0$. Then, by a result of [2; II, Th. 1, p. 2], if the 1-form ω induced by the Lee form of M^{2m} has no singular points (i.e. $\omega_x \neq 0$, for any $x \in M^n$) then M^n is a totally real submanifold of M^{2m} (see also [3]).

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