Nonnegative solutions of a class of second order nonlinear differential equations

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Abstract. A differential equation of the form

\[(q(t)k(u)u')' = \lambda f(t)h(u)u'\]

depending on the positive parameter \(\lambda\) is considered and nonnegative solutions \(u\) such that \(u(0) = 0\), \(u(t) > 0\) for \(t > 0\) are studied. Some theorems about the existence, uniqueness and boundedness of solutions are given.

1. Introduction. In [6] the equation

\[(k(u)u')' = f(t)u'\]

was considered and the author has given sufficient conditions for the existence and uniqueness of nonnegative solutions \(u\) such that \(u(0) = 0\), \(u(t) > 0\) for \(t > 0\). This problem is connected with the description of the mathematical model of the infiltration of water. For more details see e.g. [3]–[5].

In [4] and [5] the existence and uniqueness of nonnegative solutions was proved for the differential equations

\[(uu')' = (1 - t)u' \quad (t \in [0, 1])\]

and

\[(uu')' = A^{-t}u' \quad (A > 1).\]

The methods are based on the special form of the equations and on the Banach fixed point theorem. In [1] and [2], the following equation was considered:

\[(k(u)u')' = (1 - t)u'.\]

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In this paper we consider the equation
\[(2) \quad (q(t)k(u)u')' = f(t)h(u)u'\]
which is a generalization of (1), and give sufficient conditions for the existence and uniqueness of solutions \(u\) of (2) satisfying \(u(0) = 0, u(t) > 0\) for \(t > 0\), as well as for their boundedness and unboundedness. In the last section we discuss the dependence of solutions of the equation \((q(t)k(u)u')' = \lambda f(t)h(u)u'\) on the positive parameter \(\lambda\) and we consider the boundary value problem \((q(t)k(u)u')' = \lambda f(t)h(u)u'\), \(\lim_{t \to \infty} u(t; \lambda) = a \in (0, \infty)\). In accordance with [6] the proof of the existence theorem is based on an iterative method and a monotone behaviour of some operator. The proof of the uniqueness is different from the one in [6]. For the special case of (2), namely (1), we obtain the same results as in [6] (where \(\int_0^\infty (k(s)/s) \, ds = \infty\) should be required).

2. Notations, lemmas. We will consider the differential equation (2) in which \(q, k, f, h\) satisfy the following assumptions:

(H1) \(q \in C^0([0, \infty)), q(t) > 0\) for all \(t > 0\) and \(\int_0^t \frac{dt}{q(t)} < \infty\);

(H2) \(k \in C^0([0, \infty)), k(0) = 0, k(u) > 0\) for all \(u > 0\);

(H3) \(\int_0 k(s) \, ds < \infty\) and \(\int_0^\infty \frac{k(s)}{s} \, ds = \infty\);

(H4) \(f \in C^1([0, \infty)), f(t) > 0, f'(t) \leq 0\) for all \(t \geq 0\);

(H5) \(h \in C^0([0, \infty)), h(u) \geq 0\) and the function \(H(u) := \int_0^u h(s) \, ds\) is strictly increasing for all \(u \geq 0\);

(H6) \(\int_0^\infty \frac{k(u)}{H(u)} \, du < \infty\) and \(\int_0^\infty \frac{k(u)}{H(u)} \, du = \infty\).

By a solution of (2) we mean a function \(u \in C^0([0, \infty)) \cap C^1((0, \infty))\) such that \(u(0) = 0, u(t) > 0\) for all \(t > 0\), \(\lim_{t \to 0^+} q(t)k(u(t))u'(t) = 0, q(t)k(u(t))u'(t)\) is continuously differentiable for all \(t > 0\) and (2) is satisfied on \((0, \infty)\).

For \(u \in [0, \infty)\) we define the strictly increasing functions \(K\) and \(V\) by
\[K(u) = \int_0^u k(s) \, ds, \quad V(u) = \int_0^u \frac{k(s)}{H(s)} \, ds.\]
Clearly $K \in C^1([0, \infty)), V \in C^0([0, \infty)) \cap C^1((0, \infty)), \lim_{u \to \infty} K(u) = \infty = \lim_{u \to \infty} V(u)$.

Set $M = \{ u; u \in C^0([0, \infty)), u(0) = 0, u(t) > 0 \text{ for } t > 0 \}$.

**Lemma 1.** If $u$ is a solution of (2), then $u$ is a solution of the integral equation

$$K(u(t)) = \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds$$

and conversely, if $u \in M$ is a solution of (3), then $u$ is a solution of (2).

**Proof.** Let $u$ be a solution of (2). Integrating (2) from $a \ (> 0)$ to $t$, we obtain

$$q(t)k(u(t))u'(t) - q(a)k(u(a))u'(a) = \int_a^t f(s)h(u(s))u'(s) \, ds$$

$$= f(t)H(u(t)) - f(a)H(u(a)) - \int_a^t f'(s)H(u(s)) \, ds.$$  \(\text{Let } a \to 0^+. \) We get

$$K(u(t))' = \frac{1}{q(t)} \left[ f(t)H(u(t)) - \int_0^t f'(s)H(u(s)) \, ds \right]$$

for $t > 0$, and integrating (4) from 0 to $t$, we have

$$K(u(t)) = \int_0^t \frac{1}{q(s)} \left[ f(s)H(u(s)) - \int_0^s f'(z)H(u(z)) \, dz \right] \, ds$$

$$= \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds,$$

and consequently, $u$ is a solution of (3).

Now, let $u \in M$ be a solution of (3). Then

$$u(t) = K^{-1} \left[ \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds \right]$$

for $t \geq 0$, where $K^{-1}$ denotes the inverse function to $K$ on $[0, \infty)$. From (4) it follows that $u' \in C^0((0, \infty))$ and

$$u'(t) = \frac{1}{q(t)k(u(t))} \left[ f(t)H(u(t)) - \int_0^t f'(s)H(u(s)) \, ds \right].$$
therefore
\[(6) \quad q(t)k(u(t))u'(t) = f(t)H(u(t)) - \int_0^t f'(s)H(u(s)) \, ds.\]

Hence
\[
\lim_{t \to 0^+} q(t)k(u(t))u'(t) = 0, \quad q(t)k(u(t))u'(t) \in C^1((0, \infty)),
\]
\[
(q(t)k(u(t))u'(t))' = f(t)h(u(t))u'(t) \quad \text{for } t > 0,
\]
consequently, \(u\) is a solution of (2).

**Remark 1.** It follows from Lemma 1 that solving (2) is equivalent to solving the integral equation (3) in the set \(M\).

**Lemma 2.** If \(u \in M\) is a solution of (3), then
\[
(7) \quad V^{-1}\left(\int_0^t \frac{f(s)}{q(s)} \, ds\right) \leq u(t) \leq V^{-1}\left(f(0) \int_0^t \frac{ds}{q(s)}\right) \quad \text{for } t \geq 0.
\]

**Proof.** Let \(u \in M\) be a solution of (3). Then \(u'(t) > 0\) for \(t > 0\) and (cf. (6))
\[
f(t)H(u(t)) \leq q(t)k(u(t))u'(t) \leq \left[f(t) - \int_0^t f'(s) \, ds\right]H(u(t))
\]
\[
= f(0)H(u(t)),
\]

hence
\[
(8) \quad \frac{f(t)}{q(t)} \leq \frac{k(u(t))u'(t)}{H(u(t))} = (V(u(t)))' \leq \frac{f(0)}{q(t)} \quad \text{for } t > 0.
\]

Integrating (8) from 0 to \(t\), we obtain
\[
(9) \quad \int_0^t \frac{f(s)}{q(s)} \, ds \leq V(u(t)) \leq f(0) \int_0^t \frac{ds}{q(s)} \quad \text{for } t \geq 0
\]
and (7) follows.

Define the operator \(T : M \to M\) by
\[
(Tu)(t) = K^{-1}\left[\int_0^t \left(\frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{ds}{q(z)}\right)H(u(s)) \, ds\right] \quad \text{for } t \geq 0
\]
and set
\[
\varphi(t) = V^{-1}\left(\int_0^t \frac{f(s)}{q(s)} \, ds\right), \quad \overline{\varphi}(t) = V^{-1}\left(f(0) \int_0^t \frac{ds}{q(s)}\right) \quad \text{for } t \geq 0.
\]
Lemma 3. For $t \in [0, \infty)$,

\begin{align*}
(T \varphi)(t) & \geq \varphi(t), \quad (T \varphi)(t) \leq \varphi(t) .
\end{align*}

Proof. Setting

$$
\alpha(t) = \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(\varphi(s))\,ds - K(\varphi(t)) ,
$$

$$
\beta(t) = \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(\varphi(s))\,ds - K(\varphi(t))
$$

for $t \geq 0$ we see that to prove Lemma 3 it is enough to show $\alpha(t) \geq 0$ and $\beta(t) \leq 0$ on $[0, \infty)$. Since

$$
\alpha'(t) = \frac{f(t)}{q(t)} H(\varphi(t)) - \frac{1}{q(t)} \int_0^t f'(s) H(\varphi(s))\,ds - K'(\varphi(t)) \varphi'(t)
$$

$$
= - \frac{1}{q(t)} \int_0^t f'(s) H(\varphi(s))\,ds \geq 0 ,
$$

$$
\beta'(t) = \frac{f(t)}{q(t)} H(\varphi(t)) - \frac{1}{q(t)} \int_0^t f'(s) H(\varphi(s))\,ds - K'(\varphi(t)) \varphi'(t)
$$

$$
\leq \frac{f(t) - f(0)}{q(t)} H(\varphi(t)) - \frac{H(\varphi(t))}{q(t)} \int_0^t f'(s)\,ds = 0
$$

for $t > 0$ and $\alpha(0) = 0 = \beta(0)$, we see $\alpha(t) \geq 0$, $\beta(t) \leq 0$ on $[0, \infty)$ and inequalities (10) are true.

3. Existence theorem. We define sequences $\{u_n\} \subset M$, $\{v_n\} \subset M$ by the recurrence formulas

$$
u_0 = \varphi, \quad u_{n+1} = T(u_n),
$$

$$
v_0 = \varphi, \quad v_{n+1} = T(v_n)
$$

for $n = 0, 1, 2, \ldots$

Theorem 1. Let assumptions $(H_1)$–$(H_6)$ be fulfilled. Then the limits

$$
\lim_{n \to \infty} u_n(t) =: u(t), \quad \lim_{n \to \infty} v_n(t) =: \varphi(t)
$$

exist for all $t \geq 0$. The functions $u$, $\varphi$ are solutions of (2), and if $u$ is any solution of (2) then

$$
\varphi(t) \leq u(t) \leq \varphi(t) \quad \text{for} \ t \geq 0 .
$$
Proof. By Lemma 3 we have

\[ u_0(t) \leq u_1(t), \quad v_1(t) \leq v_0(t) \quad \text{for } t \geq 0. \]

Since \( \alpha, \beta \in M \) and \( \alpha(t) \leq \beta(t) \) for \( t \geq 0 \) implies \( (Ta)(t) \leq (T\beta)(t) \) for \( t \geq 0 \), we deduce

\[ \varphi(t) = u_0(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq v_n(t) \leq \ldots \leq v_1(t) \leq v_0(t) = \overline{\varphi}(t) \]

for \( t \geq 0 \) and \( n \in \mathbb{N} \). Therefore the limits \( \lim_{n \to \infty} u_n(t) =: \underline{u}(t), \lim_{n \to \infty} v_n(t) =: \overline{v}(t) \) exist for all \( t \geq 0 \), \( \varphi(t) \leq v(t) \leq \overline{\varphi}(t) \leq \overline{\varphi}(t) \) on \( [0, \infty) \) and using the Lebesgue theorem we see that \( \underline{u}, \overline{v} \) are solutions of (3) and \( \underline{u}, \overline{v} \in M \).

If \( u \in M \) is a solution of (3), by Lemma 2 we have

\[ \varphi(t) \leq u(t) \leq \overline{\varphi}(t) \quad \text{for } t \geq 0 \]

and (11) follows by the monotonicity of \( T \).

Lemma 3. If (2) admits two different solutions \( u \) and \( v \), then \( u(t) \neq v(t) \) for all \( t > 0 \).

Proof. Let \( u, v \) be two different solutions of (2). First, suppose there exists a \( t_1 > 0 \) such that \( u(t) < v(t) \) for \( t \in (0, t_1) \) and \( u(t_1) = v(t_1) \). Since \( H(u(t)) - H(v(t)) < 0 \) on \( (0, t_1) \), we have

\[
K(u(t_1)) - K(v(t_1)) = \int_0^{t_1} \left( \frac{f(s)}{q(s)} - f'(s) \frac{1}{q(s)} \right) \left( H(u(s)) - H(v(s)) \right) ds < 0,
\]

contradicting \( K(u(t_1)) = K(v(t_1)) \).

Secondly, suppose there exists \( 0 < t_1 < t_2 \) such that \( u(t_n) = v(t_n) \) (\( n = 1, 2 \)) and \( u(t) \neq v(t) \) on \( (t_1, t_2) \). Suppose

\[ u(t) < v(t) \quad \text{for } t \in (t_1, t_2). \]

Then \( u'(t_1) - v'(t_1) \leq 0, u'(t_2) - v'(t_2) \geq 0, H(u(t)) - H(v(t)) < 0 \) on \( (t_1, t_2) \), therefore

\[
0 \leq q(t_2)k(u(t_2))(u'(t_2) - v'(t_2)) - q(t_1)k(u(t_1))(u'(t_1) - v'(t_1))
\]

\[ = - \int_{t_1}^{t_2} f'(s)(H(u(s)) - H(v(s))) ds \leq 0 \]

and consequently, \( f'(t) = 0 \) on \( [t_1, t_2] \). Hence \( u'(t_1) = v'(t_1), f(t) = \text{const} \) (\( =: k \)) for \( t \in [t_1, t_2] \) and

\[
K(u(t)) - K(v(t)) = \int_{t_1}^{t} \frac{k}{q(s)}(H(u(s)) - H(v(s))) ds \quad \text{for } t \in [t_1, t_2].
\]
Then we have
\[ 0 = K(u(t_2)) - K(v(t_2)) = \int_{t_1}^{t_2} \frac{k}{q(s)} (H(u(s)) - H(v(s))) \, ds, \]
which contradicts \( H(u(t)) - H(v(t)) \neq 0 \) for \( t \in (t_1, t_2) \).

4. Bounded and unbounded solutions

**Theorem 2.** Let assumptions \((H_1)\)–\((H_6)\) be fulfilled. Then

(i) some (and then any) solution of \((2)\) is bounded if and only if
\[ \int_0^\infty \frac{ds}{q(s)} < \infty, \]

(ii) some (and then any) solution of \((2)\) is unbounded if and only if
\[ \int_0^\infty \frac{ds}{q(s)} = \infty. \]

**Proof.** First observe that either \( \int_0^\infty ds/q(s) < \infty \) or \( \int_0^\infty ds/q(s) = \infty \).

Suppose \( \int_0^\infty ds/q(s) < \infty \). Then according to Lemma 2 any solution of \((3)\) (and by Lemma 1 also any solution of \((2)\)) is bounded.

Suppose \( \int_0^\infty ds/q(s) = \infty \) and let \( u \) be a solution of \((2)\). Then
\[ K(u(t)) = \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds \quad \text{for } t \geq 0, \]
and for \( t \geq t_1 \), where \( t_1 \) is a positive number, we have
\[ K(u(t)) = \int_0^{t_1} \left( \frac{f(s)}{q(s)} - f'(s) \int_s^{t_1} \frac{dz}{q(z)} \right) H(u(s)) \, ds \]
\[ + \int_{t_1}^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds \]
\[ \geq H(u(t_1)) \int_{t_1}^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) \, ds \]
\[ = H(u(t_1)) f(t_1) \int_{t_1}^t \frac{dz}{q(z)}. \]

Therefore \( \lim_{t \to \infty} K(u(t)) = \infty \) and \( u \) is necessarily unbounded.
5. Uniqueness theorem

Theorem 3. Let assumptions (H₁)–(H₆) be fulfilled. Assume that there exists \( \varepsilon > 0 \) such that the modulus of continuity \( \gamma(t) := \sup\{|q(t_1) - q(t_2)|; t_1, t_2 \in [0, \varepsilon], |t_1 - t_2| \leq \varepsilon\} \) of \( q \) on \( [0, \varepsilon] \) satisfies
\[
\limsup_{t \to 0^+} \frac{\gamma(t)}{t} < \infty.
\]

Then (2) admits a unique solution.

Proof. According to Lemma 1 and Theorem 1, it is sufficient to show that (3) admits a unique solution, that is, \( \mathbf{u} = \mathbf{v} \), where \( \mathbf{u}, \mathbf{v} \) are defined in Theorem 1. Since \( 0 < q(t) \leq \overline{q}(t) \) on \( (0, \infty) \), we see that \( \overline{q}'(t) > 0 \), \( \overline{q}'(t) > 0 \) for \( t > 0 \). Set \( u_1 = \mathbf{u}, u_2 = \mathbf{v}, A_i = \lim_{t \to \infty} u_i(t) \) and \( w_i = u_i^{-1} \), where \( u_i^{-1} \) denotes the inverse function to \( u_i \) \( (i = 1, 2) \). Then
\[
w'_i(x) = q(w_i(x))k(x)\left[ \int_0^x f(w_i(s))h(s) \, ds \right]^{-1} \quad \text{for} \quad x \in (0, A_i), \ i = 1, 2
\]
and
\[
w_i(x) = \int_0^x q(w_i(s))k(s)\left[ \int_0^s f(w_i(z))h(z) \, dz \right]^{-1} \, ds \quad \text{for} \quad x \in [0, A_i), \ i = 1, 2.
\]

Therefore, for \( x \in [0, A_1) \) we have
\[
(12) \quad (0 \leq) \ w_1(x) - w_2(x)
\]
\[
= \int_0^x \left( q(w_1(s)) - q(w_2(s)) \right)k(s)\left[ \int_0^s f(w_2(z))h(z) \, dz \right]^{-1} \, ds
\]
\[
+ \int_0^x \left\{ q(w_1(s))k(s)\left[ \int_0^s f(w_1(z))h(z) \, dz \right]^{-1} \right. \int_0^s f(w_2(z))h(z) \, dz - f(w_2(z))h(z) \, dz \bigg\} \, ds.
\]

Define \( a = u_1(\varepsilon), X(x) = \max\{w_1(t) - w_2(t); 0 \leq t \leq x\} \) for \( x \in [0, a] \). Suppose \( X(x) > 0 \) on \( [0, a] \). Then
\[
|q(w_1(x)) - q(w_2(x))| \leq \gamma(X(x)) \quad \text{for} \quad x \in [0, a]
\]
and using (12) we have
\[
w_1(x) - w_2(x) \leq (LX(x) + T\gamma(X(x)))V(x) \quad \text{for} \quad 0 \leq x \leq a,
\]
where
\[
T = \frac{1}{f(\varepsilon)} , \quad L = T^2 \max_{t \in [0, \varepsilon]} f'(t) \max_{t \in [0, \varepsilon]} q(t).
\]

Hence
\[
X(x) \leq LX(x) + T\gamma(X(x))V(x)
\]
and 
\[
\frac{\gamma(X(x))}{X(x)} V(x) \geq (1 - LV(x)) T^{-1} \quad \text{for } x \in (0, a].
\]
By the assumption of Theorem 2, \(\limsup_{x \to 0^+} \frac{\gamma(X(x))}{X(x)} V(x) < \infty\), therefore \(\lim_{x \to 0^+} \frac{\gamma(X(x))}{X(x)} V(x) = 0\), which contradicts the fact that \(\lim_{x \to 0^+} (1 - LV(x)) T^{-1} = T^{-1}\). This proves that there exists an interval \([0, b]\) \((0 < b < \infty)\) such that \(u_1 = u_2\) on \([0, b]\).

Assume \(u_1 \neq u_2\) on \([0, \infty)\) and let \([0, c]\) be the maximal interval where \(u_1(t) = u_2(t)\). Define
\[
Y(t) = \max\{u_2(s) - u_1(s); c \leq s \leq t\} \quad \text{for } t \geq c.
\]
Then \(Y(c) = 0\) and \(Y(t) > 0\) for all \(t > c\). Since
\[
K(u_2(t)) - K(u_1(t)) = \int_c^t \left( f(s) - f'(s) \int_s^t \frac{dz}{q(z)} \right) (H(u_2(s)) - H(u_1(s))) \, ds
\]
for \(t \geq c\), we have
\[
u_2(t) - u_1(t) \leq L_1 Y(t) \int_c^t \left( f(s) - f'(s) \int_s^t \frac{dz}{q(z)} \right) \, ds \quad \text{for } t \in [c, c+1],
\]
where
\[L_1 = \max\{h(u); u \in [u_1(c), u_2(c+1)]\} [\min\{k(u); u \in [u_1(c), u_2(c+1)]\}]^{-1}.
\]
Hence
\[
Y(t) = L_1 Y(t) \int_c^t \left( f(s) - f'(s) \int_s^t \frac{dz}{q(z)} \right) \, ds
\]
and
\[
1 \leq L_1 \int_c^t \left( f(s) - f'(s) \int_s^t \frac{dz}{q(z)} \right) \, ds \quad \text{for } t \in (c, c+1],
\]
which is a contradiction. This completes the proof.

6. Dependence of solutions on the parameter. Consider the differential equation
\[
(q(t)k(u)u')' = \lambda f(t)h(u)u', \quad \lambda > 0,
\]
depending on the positive parameter \(\lambda\). Assume that assumptions (H1)–(H6) are satisfied. Set
\[
\varphi(t; \lambda) = V^{-1} \left( \lambda \int_0^t \frac{f(s)}{q(s)} \, ds \right), \quad \psi(t; \lambda) = V^{-1} \left( \lambda f(0) \int_0^t \frac{dz}{q(z)} \right)
\]
and define
\[
(T_\lambda u)(t) = K^{-1} \left( \lambda \int_0^t \left( \frac{f(s)}{q(s)} - f'(s) \int_s^t \frac{dz}{q(z)} \right) H(u(s)) \, ds \right),
\]
\[
u_0(t; \lambda) = \varphi(t; \lambda), \quad u_{n+1}(t; \lambda) = (T_\lambda u_n)(t),
\]
\[
u_0(t; \lambda) = \varphi(t; \lambda), \quad v_{n+1}(t; \lambda) = (T_\lambda v_n)(t)
\]
for \( t \in [0, \infty), \lambda \in (0, \infty) \) and \( n \in \mathbb{N} \).

**Theorem 4.** Let assumptions \((H_1)-(H_6)\) be fulfilled. Then the limits
\[
\lim_{n \to \infty} u_n(t; \lambda) =: \underline{u}(t; \lambda), \quad \lim_{n \to \infty} v_n(t; \lambda) =: \overline{u}(t; \lambda)
\]
exist for \( t \in [0, \infty) \) and \( \lambda > 0 \). The functions \( \underline{u}(t; \lambda) \) and \( \overline{u}(t; \lambda) \) are solutions of (13), and if \( u(t; \lambda) \) is any solution of (13) then
\[
\underline{u}(t; \lambda) \leq u(t; \lambda) \leq \overline{u}(t; \lambda) \quad \text{for } t \geq 0.
\]
Moreover, for all \( 0 < \lambda_1 < \lambda_2 \) we have
\[
\underline{u}(t; \lambda_1) < \underline{u}(t; \lambda_2), \quad \overline{u}(t; \lambda_1) < \overline{u}(t; \lambda_2) \quad \text{for } t > 0.
\]

**Proof.** The proof of the existence of the limits \( \lim_{n \to \infty} u_n(t; \lambda) \) and \( \lim_{n \to \infty} v_n(t; \lambda) \) and of (15) is similar to the proof of Theorem 1 and therefore it is omitted here.

Let \( 0 < \lambda_1 < \lambda_2 \). Then \( \varphi(t; \lambda_1) < \varphi(t; \lambda_2), \varphi(t; \lambda_1) < \overline{\varphi}(t; \lambda_2) \) and \( (T_\lambda u)(t) < (T_\lambda u)(t) \) for each \( u \in M \) and \( t > 0 \). Since \( H \) is strictly increasing on \( [0, \infty) \), we have
\[
u_n(t; \lambda_1) < \nu_n(t; \lambda_2), \quad v_n(t; \lambda_1) < v_n(t; \lambda_2) \quad \text{for } t > 0 \text{ and } n \in \mathbb{N},
\]
and consequently,
\[
\underline{u}(t; \lambda_1) < \underline{u}(t; \lambda_2), \quad \overline{u}(t; \lambda_1) < \overline{u}(t; \lambda_2) \quad \text{for } t > 0.
\]
If \( v(t_0; \lambda_1) = v(t_0; \lambda_2) \) for a \( t_0 > 0 \), where \( v \) is either \( \underline{u} \) or \( \overline{u} \), then in view of Lemma 1 we get
\[
\lambda_1 \int_0^{t_0} \left( \frac{f(s)}{q(s)} - f'(s) \int_s^{t_0} \frac{dz}{q(z)} \right) H(v(s; \lambda_1)) \, ds
\]
\[
= \lambda_2 \int_0^{t_0} \left( \frac{f(s)}{q(s)} - f'(s) \int_s^{t_0} \frac{dz}{q(z)} \right) H(v(s; \lambda_2)) \, ds,
\]
contradicting \( \lambda_1 < \lambda_2 \) and
\[
\left( \frac{f(t)}{q(t)} - f'(t) \int_t^{t_0} \frac{ds}{q(s)} \right) (H(v(t; \lambda_1)) - H(v(t; \lambda_2))) \leq 0 \quad \text{for } t \in (0, t_0].
\]
Hence (16) is proved.
Theorem 5. Let the assumptions of Theorem 3 be fulfilled and \( \int_{0}^{\infty} ds/q(s) < \infty \). Then for each \( a \in (0, \infty) \) there exists a unique \( \lambda_0 > 0 \) such that equation (13) with \( \lambda = \lambda_0 \) has a (necessarily unique) solution \( u(t; \lambda_0) \) with

\[
\lim_{t \to \infty} u(t; \lambda_0) = a.
\]

Proof. According to Theorem 3 equation (13) has for each \( \lambda > 0 \) a unique solution \( u(t; \lambda) \), and by Theorem 1 this solution is bounded. Since \( u(t; \lambda) \) is strictly increasing in \( t \) on \([0, \infty)\), we can define \( g: (0, \infty) \to (0, \infty) \) by

\[
g(\lambda) = \lim_{t \to \infty} u(t; \lambda).
\]

According to Theorem 4, \( g \) is nondecreasing on \((0, \infty)\). If \( g(\lambda_1) = g(\lambda_2) \) for some \( 0 < \lambda_1 < \lambda_2 \), then

\[
\int_{0}^{\infty} \left( \frac{f(s)}{q(s)} - f'(s) \int_{s}^{\infty} \frac{dz}{q(z)} \right) \left( H(u(s; \lambda_2)) - H(u(s; \lambda_1)) \right) ds = 0,
\]

contradicting \( H(u(t; \lambda_1)) - H(u(t; \lambda_2)) < 0 \) for \( t \in (0, \infty) \). Hence \( g \) is strictly increasing on \((0, \infty)\). To prove Theorem 5 it is enough to show that \( g \) maps \((0, \infty)\) onto itself. First, we see from \( \varphi(t; \lambda) \leq u(t; \lambda) \leq \overline{\varphi}(t; \lambda) \) that

\[
\lim_{\lambda \to 0^+} g(\lambda) = 0 \text{ and } \lim_{\lambda \to \infty} g(\lambda) = \infty.
\]

Secondly, assume to the contrary that \( \lim_{\lambda \to \lambda_0^-} g(\lambda) < \lim_{\lambda \to \lambda_0^+} g(\lambda) \) for a \( \lambda_0 > 0 \). Setting

\[
v_1(t) = \lim_{\lambda \to \lambda_0^-} u(t; \lambda), \quad v_2(t) = \lim_{\lambda \to \lambda_0^+} u(t; \lambda) \text{ for } t \geq 0,
\]

we get \( v_1 \neq v_2 \). On the other hand, using the Lebesgue dominated convergence theorem as \( \lambda \to \lambda_0^- \) and \( \lambda \to \lambda_0^+ \) in the equality

\[
u(t; \lambda) = K^{-1} \left[ \lambda \int_{0}^{t} \left( \frac{f(s)}{q(s)} - f'(s) \int_{s}^{t} \frac{dz}{q(z)} \right) H(u(s; \lambda)) ds \right]
\]

we see that

\[
v_i(t) = K^{-1} \left[ \lambda_0 \int_{0}^{t} \left( \frac{f(s)}{q(s)} - f'(s) \int_{s}^{t} \frac{dz}{q(z)} \right) H(v_i(s)) ds \right]
\]

for \( t \geq 0 \) and \( i = 1, 2 \).

Therefore \( v_1, v_2 \) are solutions of (13) with \( \lambda = \lambda_0 \), contradicting the fact that equation (13) with \( \lambda = \lambda_0 \) has a unique solution.
References


