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Univalent harmonic mappings

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Abstract. Let $a < 0$, $\Omega = \mathbb{C} - (-\infty, a]$ and $U = \{z : |z| < 1\}$. We consider the class $S_H(U,\Omega)$ of functions f which are univalent, harmonic and sense preserving with $f(U) = \Omega$ and satisfy $f(0) = 0, f_z(0) > 0$ and $f_{\bar z}(0) = 0$. We describe the closure $\overline{S_H(U,\Omega)}$ of $S_H(U,\Omega)$ and determine the extreme points of $\overline{S_H(U,\Omega)}$.

1. Introduction. Let S_H be the class of functions f which are univalent sense preserving harmonic mappings of the unit disk $U = \{z : |z| < 1\}$ and satisfy $f(0) = 0$ and $f_z(0) > 0$. Let F and G be analytic in U with $F(0) = G(0) = 0$ and $\operatorname{Re} f(z) = \operatorname{Re} F(z)$ and $\operatorname{Im} f(z) = \operatorname{Re} G(z)$ for z in U. Then $h = (F + iG)/2$ and $g = (F - iG)/2$ are analytic in U and $f = h + \overline{g}$. f is locally one-to-one and sense preserving if and only if $|g'(z)| < |h'(z)|$ for z in U [3]. If $h(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 > 0$, and $g(z) = b_1 z + b_2 z^2 + \ldots$ for z in U, it follows that $|b_1| < a_1$ and hence $a_1 f - \overline{b_1} \overline{f}$ also belongs to S_H. Thus, consideration is often restricted to the subclass S_H^0 of S_H consisting of those functions in S_H with $f_{\bar z}(0) = 0$.

Since harmonic mappings are not essentially determined by their image domains, various authors have studied subclasses of S_H, consisting of functions mapping U onto a specific simply connected domain D. In particular, Hengartner and Schober [5] considered the case of D being a strip, Abu-Muhanna and Schober [1] considered the case of D being a wedge or a half plane. Cima and the author [2] also considered the case of D being a strip.

In this paper we will study the case of D being the plane $\mathbb{C}$ slit along a ray pointing at the origin. This type of domain is often extremal for certain problems over classes of functions mapping U onto domains that are starlike or convex in one direction. Also, Hengartner and Schober [6] considered the case of D being the plane $\mathbb{C}$ slit along the interval $(-\infty, 0]$. They studied these mappings as they related to minimal surfaces. Our purpose is to study

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extreme points and to use the knowledge of extreme points to solve some extremal problems.

Let $a < 0$ and $\Omega = \mathbb{C} - (-\infty, a]$. $S_H(U, \Omega)$ is the class of functions f which are univalent sense preserving harmonic maps with $f(U) = \Omega$ and satisfy $f(0) = 0$, $f_z(0) > 0$ and $f_{\bar{z}}(0) = 0$.

In the sequel F and G will be functions analytic in U with $F(0) = G(0) = 0$ and $\operatorname{Re} f(z) = \operatorname{Re} F(z)$ and $\operatorname{Re} G(z) = \operatorname{Im} f(z)$ for z in U. If $h = (F + iG)/2$ and $g = (F - iG)/2$ then $f = h + \overline{g}$ and $|g'(z)| < |h'(z)|$ for z in U.

2. The class $S_H(U, \Omega)$. Let $\mathcal{P}$ be the class of functions $P(z)$ which are analytic in U with $P(0) = 1$ and $\operatorname{Re} P(z) > 0$ for z in U.

Lemma 1. *If $P(z)$ is in $\mathcal{P}$, then*

$$-\frac{1}{2} \le \operatorname{Re} \int_0^{-1} \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta \le -\frac{1}{6}.$$

Proof.

$$\operatorname{Re} \int_0^{-1} \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta = \int_0^{1} \frac{-(1-t)}{(1+t)^3} \operatorname{Re} P(-t)\, dt.$$

However, it is well known that

$$\frac{1-t}{1+t} \le \operatorname{Re} P(-t) \le \frac{1+t}{1-t}.$$

Thus

$$\int_0^1 \frac{-1}{(1+t)^2}\, dt \le \int_0^1 \frac{-(1-t)}{(1+t)^3} \operatorname{Re} P(-t)\, dt \le \int_0^1 \frac{-(1-t)^2}{(1+t)^4}\, dt,$$

and the lemma follows.

We now let $\mathcal{F}$ be the class of functions f which have the form

$$f(z) = \frac{a}{\operatorname{Re} \int_0^{-1} \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta} \left[\operatorname{Re} \int_0^{z} \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta + i \operatorname{Im} \frac{z}{(1-z)^2} \right] \tag{1.1}$$

where P is in $\mathcal{P}$.

THEOREM 1. *If $f \in \mathcal{F}$, then f is harmonic, sense preserving and univalent in U and $f(U)$ is convex in the direction of the real axis with $f(U) \subset \Omega$.*

Proof. Let $f = h + \overline{g} = \operatorname{Re} F + i \operatorname{Re} G$. Then with

$$A = a \Big/ \operatorname{Re} \int_0^{-1} \left[(1+\zeta)/(1-\zeta)^3\right] P(\zeta)\, d\zeta$$

we have

$$F(z) = A \int_0^z \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta \quad \text{and} \quad G(z) = -\frac{iAz}{(1-z)^2}.$$

Since

$$\frac{g'(z)}{h'(z)} = \frac{F'(z) - iG'(z)}{F'(z) + iG'(z)} = \frac{P(z)-1}{P(z)+1},$$

it follows that $|g'(z)| < |h'(z)|$ for z in U. Thus f is locally one-to-one and sense preserving in U. Also

$$h(z) - g(z) = iG(z) = \frac{Az}{(1-z)^2}$$

is convex in the direction of the real axis. By a theorem of Clunie and Sheil-Small [3, Theorem 5.3], f is univalent and $f(U)$ is convex in the direction of the real axis.

Moreover, $f(z)$ is real if and only if z is real. Since $A > 0$ and $\operatorname{Re} P(z) > 0$ it follows that $f(r) = \operatorname{Re} F(r)$ is increasing on $(-1, 1)$ and bounded on $(-1, 0)$. Thus $\lim_{r \to -1^+} f(r)$ exists and equals a. Moreover, $\lim_{r \to 1^-} f(r) = +\infty$. Thus $f(U)$ omits the interval $(-\infty, a]$. Therefore $f(U) \subset \Omega$.

The next theorem up to translation is contained in [6]. However, for the sake of completeness and since our point of view is somewhat different, we include a proof here.

THEOREM 2. *$S_H(U, \Omega) \subset \mathcal{F}$.*

Proof. Let f be in $S_H(U, \Omega)$. Since Ω is convex in the direction of the real axis, by a result of Clunie and Sheil-Small, $h - g = iG$ is univalent and convex in the direction of the real axis.

Let $h(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 > 0$, and $g(z) = b_2 z^2 + \ldots$ Then $G(z) = -i(h(z) - g(z)) = -a_1 iz + \ldots$ Since $f(U) = \Omega$, $\operatorname{Re} G(z) = \operatorname{Im} f(z)$ is 0 on the boundary of U. Since G is convex in the direction of the imaginary axis, it follows that $G(U)$ is $\mathbb{C}$ slit along one or two infinite rays along the imaginary axis. Thus $G(z)/(-a_1 i)$ maps U onto $\mathbb{C}$ slit along one or two infinite rays along the real axis. However, $G(z)/(-a_1 i)$ is a member of the class S of functions f analytic and univalent in U and normalized by $f(0) = f'(0) - 1 = 0$. Making use of subordination arguments, it follows that $G(z)/(-a_1 i)$ has the form

$$\frac{G(z)}{-a_1 i} = \frac{z}{1+cz+z^2}, \qquad -2 \le c \le 2,$$

and hence $\operatorname{Im} f(r) = \operatorname{Re} G(r) = 0$ for $-1 < r < 1$. Since $f_z(0) > 0$, the function f is increasing on $(-1,1)$, so that $\lim_{r\to -1^+} f(r) = a$ and $\lim_{r\to 1^-} f(r) = +\infty$.

Now if $f = h + \overline{g}$, then $h' - g' = iG'$ and

$$\frac{h'+g'}{h'-g'} = \frac{1+g'/h'}{1-g'/h'}.$$

Since $|g'(z)/h'(z)| < 1$ for z in U, it follows that

$$\frac{h'+g'}{h'-g'} = P,$$

where P is in $\mathcal{P}$. Thus $h' + g' = (h'-g')P = iG'P$, and

$$F(z) = h(z) + g(z) = \int_0^z iG'(\zeta)P(\zeta)\,d\zeta.$$

Now suppose $G(z) = -a_1 iz/(1+cz+z^2)$, $-2 < c \le 2$. Then

$$F(z) = a_1 \int_0^z \frac{1-\zeta^2}{(1+c\zeta+\zeta^2)^2} P(\zeta)\,d\zeta.$$

If $0 < r < 1$, then

$$\begin{aligned} f(r) = \operatorname{Re} f(r) = \operatorname{Re} F(r) &= a_1 \int_0^r \frac{1-t^2}{(1+ct+t^2)^2} \operatorname{Re} P(t)\,dt \\ &\le a_1 \int_0^r \frac{(1-t^2)(1+t)}{(1+ct+t^2)^2(1-t)}\,dt \\ &= a_1 \int_0^r \frac{(1+t)^2}{(1+ct+t^2)^2}\,dt \le M \end{aligned}$$

for some M, since $-2 < c \le 2$. However, this is impossible since $\lim_{r\to 1^-} f(r) = +\infty$.

Thus the only possibility is that $G(z) = -a_1 iz/(1-z)^2$, and

$$F(z) = a_1 \int_0^z \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\,d\zeta.$$

Thus,

$$a = \lim_{r\to -1^+} f(r) = a_1 \operatorname{Re} \int_0^{-1} \frac{1+\zeta}{(1-\zeta)^3} P(\zeta)\, d\zeta,$$

and the theorem follows.

THEOREM 3. $\overline{S_H(U,\Omega)} = \mathcal{F}$.

Proof. Let $f(z)$ have the form (1.1), and let r_n be a sequence with $0 < r_n < 1$ and $\lim r_n = 1$. Let $P_n(z) = P(r_n z)$, and denote by $f_n(z)$ the function obtained from (1.1) by replacing $P(z)$ by $P_n(z)$. We claim that f_n is in $S_H(U,\Omega)$. To see this, let

$$A = a \Big/ \operatorname{Re} \int_0^{-1} \left[(1+\zeta)/(1-\zeta)^3\right] P_n(\zeta)\, d\zeta\,,$$

$$F_n(z) = A \int_0^{z} [(1+\zeta)/(1-\zeta)^3] P_n(\zeta)\, d\zeta\,.$$

There exists $\delta > 0$ so that we may write for $|z-1| < \delta$,

$$P_n(z) = P_n(1) + P_n'(1)(z-1) + \frac{P_n''(1)}{2}(z-1)^2 + \ldots$$

Then, for $|z-1| < \delta$,

$$\begin{aligned} F_n'(z) &= A\frac{1+z}{(1-z)^3} P_n(z) \\ &= A\left[\frac{-2P_n(1)}{(z-1)^3} - \frac{2P_n'(1)+P_n(1)}{(z-1)^2} - \frac{P_n''(1)+P_n'(1)}{z-1} + \ldots\right]. \end{aligned}$$

Let $D = \{z : |z-1| < \delta\} - \{z : 1 \le z \le 1+\delta\}$. If $1-\delta < c < 1$, then for z in D

$$F_n(z) - F_n(c) = \int_c^z F_n'(\zeta)\, d\zeta,$$

where the path of integration is in D. This gives, for z in D,

$$F_n(z) = A\Big[\frac{2P_n(1)}{(z-1)^2} + \frac{2P_n'(1)+P_n(1)}{z-1} - (P_n''(1)+P_n'(1))\log(z-1) + \sum_{j=0}^{\infty} c_j (z-1)^j\Big]$$

where $\sum_{j=0}^{\infty} c_j(z-1)^j$ converges for $|z-1| < \delta$ and $\log(z-1) = \ln|z-1| + i\arg(z-1)$, $0 < \arg(z-1) < 2\pi$. That is, for z in D, F_n has the form

$$F_n(z) = A\left[\frac{c}{(z-1)^2} + \frac{d}{(z-1)} + e\log(z-1) + q(z)\right],$$

where $\operatorname{Re} c = 2 \operatorname{Re} P_n(1) > 0$ and $q(z)$ is analytic at $z = 1$. Thus, for z in D,

$$\begin{aligned}\operatorname{Re} f_n(z) &= \operatorname{Re} F_n(z) \\ &= A\Bigg[\operatorname{Re}\left(\frac{c}{(z-1)^2} + \frac{d}{(z-1)}\right) + \operatorname{Re}(e)\ln|z-1| \\ &\qquad\qquad - \operatorname{Im}(e)\arg(z-1) + \operatorname{Re} q(z)\Bigg].\end{aligned}$$

We want to prove that f_n cannot have a nonreal finite cluster point at $z = 1$. To see this, suppose $z_j = 1 + t_j e^{i\theta_j}$ is in U with $t_j > 0$ and $\lim t_j = 0$ and is such that

$$\lim_{j\to\infty} \operatorname{Im}\left[\frac{z_j}{(1-z_j)^2}\right] = l,$$

where l is finite and $l \neq 0$. Then

$$\lim_{j\to\infty} \frac{-(\sin 2\theta_j + t_j \sin\theta_j)}{t_j^2} = l \neq 0.$$

This implies that $(\sin 2\theta_j + t_j \sin\theta_j)$ approaches 0, which in turn implies that $\sin 2\theta_j$ approaches 0. Thus $e^{-2i\theta_j}$ approaches ± 1. Therefore, $\operatorname{Re}(ce^{-i2\theta_j})$ approaches $\pm \operatorname{Re} c \neq 0$. It now follows that

$$\begin{aligned}|\operatorname{Re} f_n(z_j)| = \Bigg| &\frac{\operatorname{Re}(ce^{-i2\theta_j}) + t_j \operatorname{Re}(de^{-i\theta_j}) + \operatorname{Re}(e)t_j^2 \ln(t_j)}{t_j^2} \\ &\qquad\qquad - (\operatorname{Im} e)\arg(z_j - 1) + \operatorname{Re} q(z_j)\Bigg|\end{aligned}$$

approaches $+\infty$ as n approaches $+\infty$. Thus f_n has no finite nonreal cluster points at $z = 1$. At all other points of $|z| = 1$, the finite cluster points of f_n are real. Since $f_n(U) \subset \Omega$ and $\lim_{r\to -1} f_n(r) = a$, it follows that $f_n(U) = \Omega$.

Thus f_n is in $S_H(U, \Omega)$ and hence f is in $\overline{S_H(U, \Omega)}$. Since $\mathcal{F}$ is closed under uniform limits on compact subsets of U, it follows that $\mathcal{F} = \overline{S_H(U, \Omega)}$.

3. Extreme points of $\mathcal{F}$. If $P \in \mathcal{P}$, then it is known [4] that

$$P(z) = \int\limits_{|\eta|=1} \frac{1+\eta z}{1-\eta z}\, d\mu(\eta) \tag{3.1}$$

where μ is a probability measure on $X = \{\eta : |\eta| = 1\}$. Thus if f is in $\mathcal{F}$, there is a probability measure μ on X so that

$$(3.2)\qquad f(z)=\frac{a}{\operatorname{Re}\int_{|\eta|=1}k(-1,\eta)d\mu(\eta)}\times\Big[\operatorname{Re}\int\limits_{|\eta|=1}k(z,\eta)\,d\mu(\eta)+i\operatorname{Im}\frac{z}{(1-z)^2}\Big]$$

where

$$(3.3)\qquad k(z,\eta)=\int\limits_0^z\frac{(1+\zeta)(1+\eta\zeta)}{(1-\zeta)^3(1-\eta\zeta)}\,d\zeta$$
$$=\begin{cases}\dfrac{2\eta(1+\eta)}{(1-\eta)^3}\log\left(\dfrac{1-\eta z}{1-z}\right)-\dfrac{(1+4\eta-\eta^2)z}{(1-\eta)^2(1-z)}\\ \qquad\qquad+\dfrac{(1+\eta)(2z-z^2)}{(1-\eta)(1-z)^2}, & \eta\neq1,\\ \dfrac{z+\frac13z^3}{(1-z)^3}, & \eta=1.\end{cases}$$

The extreme points of $\mathcal{F}$ are easily obtained by making use of a property of a nonlinear homeomorphism observed by Szapiel [7].

LEMMA 2 [7]. *Suppose X is a convex linear Hausdorff space, $\phi: X\to\mathbb{C}$ is homogeneous, $c\in\mathbb{C}\setminus\{0\}$ and A is a compact convex subset of $\phi^{-1}(c)$. Let $\psi: A\to\mathbb{R}$ be affine continuous with $0\notin\psi(A)$ and let $B=\{a/\psi(a): a\in A\}$. Then*

1) *B is compact convex,*
2) *the map $a\to a/\psi(a)$ is a homeomorphism of A onto B,*
3) *$EB=\{a/\psi(a): a\in EA\}$, where ED means the set of all extreme points of D.*

Proof. For the sake of completeness we include the proof as communicated to me by the referee. We observe that if a_1 and a_2 are in A and $a_1/\psi(a_1)=a_2/\psi(a_2)$, then $c/\psi(a_1)=\phi(a_1/\psi(a_1))=\phi(a_2/\psi(a_2))=c/\psi(a_2)$. Thus $\psi(a_1)=\psi(a_2)$ and hence $a_1=a_2$. Next we note that $(\lambda_1a_1+\lambda_2a_2)/\psi(\lambda_1a_1+\lambda_2a_2)=\mu_1a_1/\psi(a_1)+\mu_2a_2/\psi(a_2)$, where

$$(3.4)\qquad \mu_j=\lambda_j\psi(a_j)/\psi(\lambda_1a_1+\lambda_2a_2),\qquad j=1,2\,.$$

Thus if $\lambda_j>0$ with $j=1,2$, and $\lambda_1+\lambda_2=1$ then $\mu_j>0$ with $\mu_1+\mu_2=1$. If $\mu_j>0$, $j=1,2$, with $\mu_1+\mu_2=1$, we seek $\lambda_j>0$, $j=1,2$, with $\lambda_1+\lambda_2=1$ such that equation (3.4) is satisfied. It is easily verified that $\lambda_1=\mu_1\psi(a_2)/[\mu_1\psi(a_2)+\mu_2\psi(a_1)]$ and $\lambda_2=\mu_2\psi(a_1)/[\mu_1\psi(a_2)+\mu_2\psi(a_1)]$ satisfy the requirements. Thus B is convex.

Next suppose $a \notin EA$. Then $a = \lambda_1 a_1 + \lambda_2 a_2$ with $\lambda_j > 0$, $j = 1,2$, $\lambda_1 + \lambda_2 = 1$ and $a_1 \neq a_2$. Thus

$$\frac{a}{\psi(a)} = \frac{\lambda_1 a_1 + \lambda_2 a_2}{\psi(\lambda_1 a_1 + \lambda_2 a_2)} = \mu_1 \frac{a_1}{\psi(a_1)} + \mu_2 \frac{a_2}{\psi(a_2)}.$$

So $a/\psi(a) \notin EB$.

Conversely, suppose $a/\psi(a) \notin EB$. Then there exist $a_1 \neq a_2$ in A and $\mu_j > 0$, $j = 1,2$, with $\mu_1 + \mu_2 = 1$ so that

$$\frac{a}{\psi(a)} = \mu_1 \frac{a_1}{\psi(a_1)} + \mu_2 \frac{a_2}{\psi(a_2)} = \frac{\lambda_1 a_1 + \lambda_2 a_2}{\psi(\lambda_1 a_1 + \lambda_2 a_2)}.$$

Since the map $a \to a/\psi(a)$ is a homeomorphism, it follows that $a = \lambda_1 a_1 + \lambda_2 a_2$ and hence $a \notin EA$. Therefore $EB = \{a/\psi(a) : a \in EA\}$.

We apply Lemma 2 with

$$Q_P(z) = \operatorname{Re}\Big[\int_0^z (1+\zeta)(1-\zeta)^{-3} P(\zeta)\,d\zeta\Big] + i\operatorname{Im}[z(1-z)^{-2}],$$

$$A = \{Q_P : P \in \mathcal{P}), \phi(f) = f_z(0), c = 1, \psi(Q_P) = Q_P(-1)/a.$$

Then $\mathcal{F} = B$ is convex and $EB = \{f(z)/f(-1) : f \in EA\}$. However, the map $Q_P \to P$ is a linear homeomorphism between A and $\mathcal{P}$. Since $E\mathcal{P} = \{(1+\mu z)/(1-\mu z) : |\mu| = 1\}$ [4], we obtain the following theorem.

Theorem 4. *The extreme points of* $\mathcal{F}$ *are*

$$f_\eta(z) = \frac{a}{\operatorname{Re} k(-1,\eta)}\Big[\operatorname{Re} k(z,\eta) + i\operatorname{Im}\frac{z}{(1-z)^2}\Big], \qquad |\eta| = 1.$$

4. The mapping properties of extreme points. If $\eta = e^{i\beta}$, then

$$\begin{aligned}\operatorname{Re} k(z,\eta) = {} & \frac{\cot(\beta/2)}{2\sin^2(\beta/2)} \arg\Big(\frac{1-e^{i\beta}z}{1-z}\Big) \\ & - \cot(\beta/2)\operatorname{Im}\frac{z}{(1-z)^2} + \frac{1}{\sin^2(\beta/2)}\operatorname{Re}\frac{z}{1-z}\end{aligned}$$

and

$$f_\eta(z) = \frac{a}{\frac{(\beta/2)\cot(\beta/2)}{2\sin^2(\beta/2)} - \frac{1}{2\sin^2(\beta/2)}}\Big[\operatorname{Re} k(z,\eta) + i\operatorname{Im}\frac{z}{(1-z)^2}\Big].$$

Suppose $\eta = e^{i\beta}$, $0 < \beta < \pi$. If ζ is on the open arc of the unit circle going from 1 to η to -1 to $\overline{\eta}$ in the counterclockwise direction, then $\arg(1-\eta\zeta)/(1-\zeta) = \beta/2$. For these ζ,

$$\lim_{z\to\zeta} f_\eta(z) = a.$$

If ζ is on the open arc from $\overline{\eta}$ to 1, then $\arg(1-\eta\zeta)/(1-\zeta) = \beta/2 - \pi$ and we obtain

$$\lim_{z\to\zeta} f_\eta(z) = a - \frac{\pi a \cot(\beta/2)}{(\beta/2)\cot(\beta/2) - 1} = b < a.$$

The cluster set of $\arg(1-\eta z)/(1-z)$ at $\overline{\eta}$ is the interval $[\beta/2-\pi, \beta/2]$. Thus, the cluster set of $f_\eta(z)$ at $\overline{\eta}$ is the interval $[b, a]$.

We now use a technique similar to that used in Example 5.4 of [3]. Let $(1+z)/(1-z) = u + iv$ and note

$$\begin{aligned}
z/(1-z) &= \tfrac{1}{2}[(1+z)/(1-z) - 1]\,,\\
((1+z)/(1-z))^2 &= 4z/(1-z)^2 + 1\,,\\
(1-\eta z)/(1-z) &= [(1-\eta)/2](1+z)/(1-z) + (1+\eta)/2\,,\\
\operatorname{Im}[z/(1-z)^2] &= \tfrac{1}{4}\operatorname{Im}(u+iv)^2\,.
\end{aligned}$$

Using these observations, we obtain

$$\begin{aligned}
f_\eta(z) = {} & \frac{a}{\operatorname{Re} k(-1,\eta)}\\
& \times \left[\frac{\cot(\beta/2)}{2\sin^2(\beta/2)} \arg[(1+e^{i\beta})(1+v\tan(\beta/2) - iu\tan(\beta/2))]\right.\\
& \left. -\frac{uv}{2}\cot(\beta/2) + \frac{u-1}{2\sin^2(\beta/2)} + i\frac{uv}{2}\right] = x + iy.
\end{aligned}$$

If $uv = 2c, u > 0$, then $y = ac/\operatorname{Re} k(-1,\eta)$ and

$$\begin{aligned}
x = {} & \frac{a}{\operatorname{Re} k(-1,\eta)}\\
& \times \left[\frac{\cot(\beta/2)}{2\sin^2(\beta/2)} \arg\left[(1+e^{i\beta})\left(1 + \frac{2c\tan(\beta/2)}{u} - iu\tan(\beta/2)\right)\right]\right.\\
& \left. -c\cot(\beta/2) + \frac{u-1}{2\sin^2(\beta/2)}\right].
\end{aligned}$$

Now suppose that $c > 0$. Then $\lim_{u\to+\infty} x = +\infty$, and

$$\begin{aligned}
\lim_{u\to 0} x = {} & \frac{a}{\operatorname{Re} k(-1,\eta)}\\
& \times \left[\frac{\cot(\beta/2)}{2\sin^2(\beta/2)} \arg(1+e^{i\beta}) - c\cot(\beta/2) - \frac{1}{2\sin^2(\beta/2)}\right]\\
= {} & a - \frac{ca\cot(\beta/2)}{\operatorname{Re} k(-1,\eta)}.
\end{aligned}$$

A calculation gives $dx/du > 0$ for any c. Thus x is increasing. Thus if $c > 0$, x takes on all values in the interval $(a - ca\cot(\beta/2)/\operatorname{Re} k(-1,\eta), +\infty)$.

It follows that $f_\eta(z)$ covers the horizontal line segment $y = ac/\operatorname{Re} k(-1,\eta)$, with left end point on the line $y = -[\tan(\beta/2)](x-a)$.

By the same reasoning, if $c<0$, $f_\eta(z)$ covers the horizontal line segment $y = ac/\operatorname{Re} k(-1,\eta)$ with left end point on the line $y = -[\tan(\beta/2)](x-b)$.

If $c=0$, $uv=0$. Since $u>0$, it follows that $v=0$ and

$$x = \frac{a}{\operatorname{Re} k(-1,\eta)} \times \left[\frac{\cot(\beta/2)}{2\sin^2(\beta/2)} \arg[(1+e^{i\beta})(1-iu\tan(\beta/2))] + \frac{u-1}{2\sin^2(\beta/2)}\right].$$

Thus,

$$\lim_{u\to 0} x = a \quad \text{and} \quad \lim_{u\to+\infty} x = +\infty.$$

Since x is increasing, f_η covers the interval $(a,+\infty)$ on the real axis and no other part of the real axis. It follows that $f_\eta(z)$ maps U onto the domain given by the shaded region in Figure 4.1.

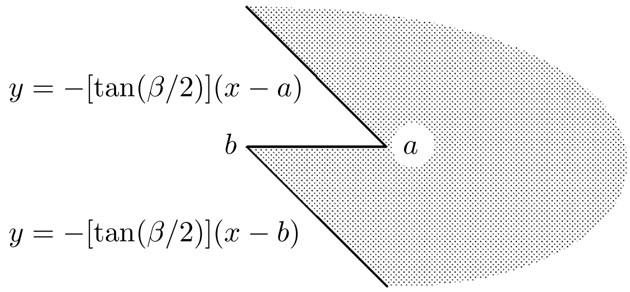

Fig. 4.1

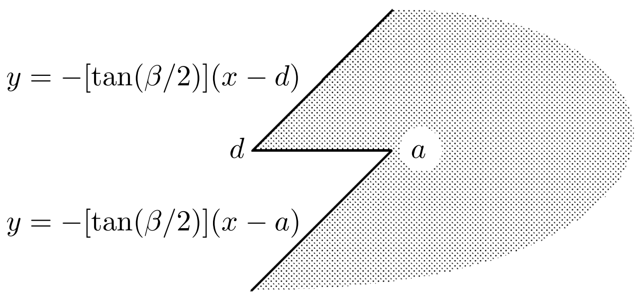

Fig. 4.2

Similarly if $-\pi<\beta<0$, $f_\eta(z)$ maps U onto the domain pictured in Figure 4.2, where

$$d = a + \frac{a\pi\cot(\beta/2)}{(\beta/2)\cot(\beta/2)-1} < a.$$

5. Applications. In this section we will use our knowledge of extreme points to solve some extremal problems on $\overline{S_H(U,\Omega)}$.

LEMMA 3. *If* $|\eta| = 1$, *then*

$$|\operatorname{Re} k(-1,\eta)| \ge |\operatorname{Re} k(-1,1)| = 1/6\,.$$

Proof.

$$k(-1,\eta) = \begin{cases} \dfrac{2\eta(1+\eta)}{(1-\eta)^3}\log\left(\dfrac{1+\eta}{2}\right) + \dfrac{1+4\eta-\eta^2}{2(1-\eta)^2} - \dfrac{3}{4}\cdot\dfrac{1+\eta}{1-\eta}\,, & \eta \ne 1,\\ -1/6, & \eta = 1. \end{cases}$$

If $\eta = e^{i\theta}$, then

$$\begin{aligned} k(-1,e^{i\theta}) &= \frac{2e^{i\theta}}{(1-e^{i\theta})^2}\cdot\frac{1+e^{i\theta}}{1-e^{i\theta}}\log\left(\frac{1+e^{i\theta}}{2}\right)\\ &\quad+\frac{1+4e^{i\theta}-e^{i2\theta}}{2(1-e^{i\theta})^2} - \frac{3}{4}\cdot\frac{1+e^{i\theta}}{1-e^{i\theta}}\\ &= \frac{2}{(e^{-i\theta/2}-e^{i\theta/2})^2}\cdot\frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}\log\left(\frac{1+e^{i\theta}}{2}\right)\\ &\quad+\frac{e^{-i\theta}+4-e^{i\theta}}{2(e^{-i\theta/2}-e^{i\theta/2})^2} - \frac{3}{4}\cdot\frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}\\ &= \frac{-i\cos(\theta/2)}{2\sin^3(\theta/2)}\log\left(\frac{1+e^{i\theta}}{2}\right) - \frac{2-i\sin\theta}{4\sin^2(\theta/2)} - \frac{3i\cos(\theta/2)}{4\sin(\theta/2)}\,. \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{Re} k(-1,e^{i\theta}) &= \frac{\cos(\theta/2)}{2\sin^3(\theta/2)}\arg\left(\frac{1+e^{i\theta}}{2}\right) - \frac{1}{2\sin^2(\theta/2)}\\ &= \frac{(\theta/2)\cos(\theta/2)}{2\sin^3(\theta/2)} - \frac{1}{2\sin^2(\theta/2)}\\ &= \frac{(\theta/2)\cos(\theta/2)-\sin(\theta/2)}{2\sin^3(\theta/2)}\,. \end{aligned}$$

Let $q(\theta) = [(\theta/2)\cos(\theta/2)-\sin(\theta/2)]/[2\sin^3(\theta/2)]$ if $\theta \ne 0$ and $q(0) = -1/6$. We want to find the maximum of $q(\theta)$ in $[-\pi,\pi]$, or equivalently the maximum of $h(\phi)$ in $[-\pi/2,\pi/2]$, where $h(\phi) = (\phi\cos\phi - \sin\phi)/(2\sin^3\phi)$ for $\phi \ne 0$ and $h(0) = -1/6$. Since $h(-\phi) = h(\phi)$, we need to consider $h(\phi)$ in $[0,\pi/2]$. Let $g(\phi) = 6\phi\cos\phi - 6\sin\phi + 2\sin^3\phi$ for $0 \le \phi \le \pi/2$. Then

$$\begin{aligned} g'(\phi) &= -6\phi\sin\phi + 6\sin^2\phi\cos\phi\\ &= 3\sin\phi(\sin 2\phi - 2\phi) \le 0. \end{aligned}$$

Thus $g(\phi) \le g(0) = 0$. It follows that $h(\phi) \le -1/6$ in $[0,\pi/2]$ and thus

$q(\theta) \le -1/6$ on $[-\pi,\pi]$. Therefore

$$|\operatorname{Re} k(-1,\eta)| = |q(\theta)| \ge 1/6 = |\operatorname{Re} k(-1,1)|\,.$$

THEOREM 5. *Let* $f(z) = h(z) + \overline{g(z)}$ *be in* $\overline{S_H(U,\Omega)}$. *If* $h(z) = \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=2}^{\infty} b_n z^n$, *then*

$$|a_n| \le (n+1)(2n+1)|a|, \tag{5.1}$$

$$|b_n| \le (n-1)(2n-1)|a|, \tag{5.2}$$

$$||a_n| - |b_n|| \le |a_n - b_n| \le 6|a|n. \tag{5.3}$$

Equality is attained in (5.1), (5.2) *and* (5.3) *for all* n *by*

$$f(z) = -6a\left[\operatorname{Re}\frac{z+\frac{1}{3}z^3}{(1-z)^3} + i\operatorname{Im}\frac{z}{(1-z)^2}\right].$$

Proof. We need only prove (5.1)–(5.3) for the extreme points of $\overline{S_H(U,\Omega)}$. Let $f_\eta(z) = A[\operatorname{Re} k(z,\eta) + i\operatorname{Im} z/(1-z)^2]$, $A = a/\operatorname{Re} k(-1,\eta)$. In our notation $F(z) = Ak(z,\eta)$ and $G(z) = -Aiz/(1-z)^2$. Thus

$$h(z) = \frac{F(z)+iG(z)}{2} = \frac{A}{2}\left[k(z,\eta) + \frac{z}{(1-z)^2}\right] = \sum_{n=1}^{\infty} a_n z^n\,,$$

$$g(z) = \frac{F(z)-iG(z)}{2} = \frac{A}{2}\left[k(z,\eta) - \frac{z}{(1-z)^2}\right] = \sum_{n=2}^{\infty} b_n z^n\,.$$

Thus if $\eta \ne 1$, then

$$h(z) = \frac{A}{2}\bigg[\sum_{n=1}^{\infty}\frac{2\eta(1+\eta)(1-\eta^n)}{(1-\eta)^3 n}z^n - \sum_{n=1}^{\infty}\frac{1+4\eta-\eta^2}{(1-\eta)^2}z^n + \sum_{n=1}^{\infty}\frac{(n+1)(1+\eta)}{1-\eta}z^n + \sum_{n=1}^{\infty} nz^n\bigg]\,.$$

Therefore

$$\begin{aligned} a_n &= \frac{A}{2}\left[\frac{2\eta(1+\eta)(1-\eta^n)}{n(1-\eta)^3} - \frac{1+4\eta-\eta^2}{(1-\eta)^2} + \frac{(n+1)(1+\eta)}{1-\eta} + n\right] \\ &= \frac{A}{2}\bigg[\frac{2\eta(1+\eta)(1+\eta+\eta^2+\ldots+\eta^{n-1})}{n(1-\eta)^2} - \frac{1+4\eta-\eta^2}{(1-\eta)^2} \\ &\qquad + \frac{(n+1)(1+\eta)}{1-\eta} + n\bigg] \\ &= \frac{A}{2}\left[\frac{n^2 + (2-4n)\eta + (4-n^2)\eta^2 + 4\eta^3 + \ldots + 4\eta^n + 2\eta^{n+1}}{n(1-\eta)^2} + n\right] \end{aligned} \tag{5.4}$$

$$= \frac{A}{2}\left[\frac{(1-\eta)^2(n^2+\sum_{k=1}^{n-1}2(n-k)^2\eta^k)}{n(1-\eta)^2}+n\right]$$
$$= A\left[\frac{n^2+\sum_{k=1}^{n-1}(n-k)^2\eta^k}{n}\right].$$

Similarly, for $n \ge 2$

$$b_n = \frac{A}{2}\left[\frac{n^2+\sum_{k=1}^{n-1}2(n-k)^2\eta^k}{n}-n\right] = A\left[\frac{\sum_{k=1}^{n-1}(n-k)^2\eta^k}{n}\right]. \tag{5.5}$$

We note that (5.4) and (5.5) also hold for $\eta = 1$.

Now by Lemma 3

$$|A| = \frac{|a|}{|\mathrm{Re}\, k(-1,\eta)|} \le \frac{|a|}{|\mathrm{Re}\, k(-1,1)|} = 6|a|.$$

Thus from (5.4) and (5.5) we obtain

$$|a_n| \le 6|a|\frac{n^2+\sum_{k=1}^{n-1}(n-k)^2}{n} = \frac{6|a|}{n}\cdot\frac{n(n+1)(2n+1)}{6}$$
$$= (n+1)(2n+1)|a|$$

with equality when $\eta = 1$, and

$$|b_n| \le \frac{6|a|}{n}\sum_{k=1}^{n-1}(n-k)^2 = \frac{6|a|}{n}\cdot\frac{(n-1)n(2n-1)}{6}$$
$$= (n-1)(2n-1)|a|$$

with equality for $\eta = 1$.

To obtain (5.3) we note that for an extreme point $f_\eta(z)$ we have $a_n - b_n = An$.

THEOREM 6. *If $f = h + \bar{g}$ is in $S_H(U,\Omega)$, then*

$$|f_z(z)| = |h'(z)| \le \frac{6|a|}{1-|z|}\left|\frac{1+z}{(1-z)^3}\right| \le \frac{6(1+|z|)|a|}{(1-|z|)^4}.$$

Equality occurs for z real and positive and

$$f(z) = -6a\left[\mathrm{Re}\,\frac{z+\frac{1}{3}z^3}{(1-z)^3} + i\,\mathrm{Im}\,\frac{z}{(1-z)^2}\right].$$

P r o o f. We need only consider extreme points $f_\eta(z)$. In this case

$$h(z) = \frac{A}{2}\left[k(z,\eta)+\frac{z}{(1-z)^2}\right]$$
$$= \frac{A}{2}\left[\frac{2\eta(1+\eta)}{(1-\eta)^3}\log\left(\frac{1-\eta z}{1-z}\right) - \frac{4\eta}{(1-\eta)^2}\cdot\frac{z}{1-z} + \frac{2}{1-\eta}\cdot\frac{z}{(1-z)^2}\right].$$

After straightforward computations we get

$$h'(z) = \frac{a}{\operatorname{Re} k(-1,\eta)} \cdot \frac{(1+z)}{(1-z)^3(1-\eta z)},$$

$$|h'(z)| = \frac{|a|}{|\operatorname{Re} k(-1,\eta)||1-\eta z|} \left| \frac{1+z}{(1-z)^3} \right|$$

$$\leq \frac{6|a|}{1-|z|} \left| \frac{1+z}{(1-z)^3} \right| \leq \frac{6|a|(1+|z|)}{(1-|z|)^4}.$$

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