Generalized Schwarzian derivatives
for generalized fractional linear transformations

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Abstract. Generalizations of the classical Schwarzian derivative of complex analysis
have been proposed by Osgood and Stowe [12, 13], Carne [5], and Ahlfors [3]. We present
another generalization of the Schwarzian derivative over vector spaces.

Introduction. Our approach is to define an analogue of the Schwarzian
derivatives in \( \mathbb{R} \cup \{ \infty \} \) using the Clifford algebra generated from \( \mathbb{R}^n \). More
precisely, we use Vahlen’s group of Clifford matrices to construct a “derivative” which in appearance bears an extremely close resemblance to the classical Schwarzian derivative. As conformal transformations in dimensions greater than two correspond to Möbius transformations we are forced to introduce a family of Schwarzians in higher dimensions. We show that a \( C^3 \) diffeomorphism annihilated by this family of Schwarzian derivatives is, up to a linear isomorphism, a Möbius transformation. We also show that these generalized Schwarzian derivatives possess a conformal invariance under Möbius transformations, and contain the generalized Schwarzian derivatives described by Ahlfors [3]. Unfortunately, this work also tells us that the method used for obtaining the chain rule for the classical Schwarzian derivative (see [10]) breaks down in higher dimensions.

Motivated by the fact that the analogue of Vahlen’s group of Clifford
matrices over Minkowski space is \( U(2, 2) \) we show that the fractional linear
transformations associated with \( U(2, 2) \), \( \text{Sp}(n, \mathbb{R}) \), the real symplectic group,
and \( H(n, n) \), the quaternionic unitary group, all have Schwarzian derivatives
associated with them. These transformations have previously been described
in [7, 9], and elsewhere. We also show that the conformal group over \( \mathbb{R}^{p,q} \)
has a generalized Schwarzian derivative.

Preliminaries. From \( \mathbb{R}^n \) we may construct a Clifford algebra \( A_n \). This
can be done [4, 14] by taking an orthonormal basis \( \{ e_j \}_{j=1}^n \) of \( \mathbb{R}^n \) and

\[ e_{j,k} = e_j e_k e_j = e_k e_j e_k \]

for \( j, k = 1, \ldots, n \).

introducing the basis

\begin{align}
1, e_1, \ldots, e_n, e_{j_1}, \ldots, e_{j_r}, \ldots, e_1 \ldots e_n
\end{align}

of \( A_n \), where 1 is the identity and \( j_1 < \ldots < j_r \) with \( 1 \leq r \leq n \). Moreover, the elements \( e_1,\ldots,e_n \) satisfy the identity

\begin{align}
e_i e_j + e_j e_i = -2 \delta_{ij} 1
\end{align}

within \( A_n \), where \( \delta_{ij} \) is the Kronecker delta. We now have \( \mathbb{R}^n \subseteq A_n \) and each non-zero vector \( x \in \mathbb{R}^n \setminus \{0\} \) has a multiplicative inverse \( x^{-1} = -x/|x|^2 \in \mathbb{R}^n \), which corresponds to the Kelvin inverse of a vector.

Writing \( x \) as \( x_1 e_1 + \ldots + x_n e_n \) we may obtain

\begin{align}
e_1(x_1 e_1 + \ldots + x_n e_n) e_1 &= -x_1 e_1 + x_2 e_2 + \ldots + x_n e_n,
\end{align}

which describes a reflection along the line spanned by \( e_1 \). In greater generality, for each \( y \in S^{n-1} \) the element \( y x y \) is a vector, and this action describes a reflection along the line spanned by \( y \). By induction, for \( y_1, \ldots, y_k \in S^{n-1} \) the element \( y_1 \ldots y_k x y_k \ldots y_1 \) is a vector and this action describes an orthogonal transformation of \( \mathbb{R}^n \). The element \( y_1 \ldots y_k \) is an element lying in \( A_n \).

This group is called \( \text{Pin}(n) \) (see [4]). More formally, we have

\begin{align}
\text{Pin}(n) = \{ a \in A_n : a = y_1 \ldots y_k \text{ where } k \in \mathbb{N} \text{ and } y_j \in S^{n-1} \text{ for } 1 \leq j \leq k \}.
\end{align}

In [4] it is shown that \( \text{Pin}(n) \) is a double covering of \( O(n) \), the orthogonal group (i.e. there is a surjective group homomorphism \( \Theta : \text{Pin}(n) \rightarrow O(n) \) such that \( \ker \Theta \cong \mathbb{Z}_2 \)).

We also need the antiautomorphism \( \sim : A_n \rightarrow A_n, e_{j_1} \ldots e_{j_r} \mapsto e_{\overline{j}_r} \ldots e_{\overline{j}_1} \). It is usual to write \( \overline{X} \) for \( \sim(X) \), where \( X \in A_n \) (see [14]). If \( a = y_1 \ldots y_k \in \text{Pin}(n) \) then \( y_k \ldots y_1 = \overline{a} \).

Besides \( \sim \) we need the antiautomorphism \( - : A_n \rightarrow A_n, e_{j_1} \ldots e_{j_r} \mapsto (-1)^r e_{\overline{j}_r} \ldots e_{\overline{j}_1} \). Again, it is usual [14] to write \( \overline{X} \) for \( -X \). If we write \( X \) as \( x_0 + \ldots + x_{1 \ldots n} e_1 \ldots e_n \) then we can easily deduce that the identity part of \( X \overline{X} \) is \( x_0^2 + \ldots + x_{1 \ldots n}^2 \). So \( A_n \) is a trace algebra.

Following Vahlen [15] and Mass [11], Ahlfors [1, 2] has used Clifford algebras to describe properties of Möbius transformations in \( \mathbb{R}^n \cup \{\infty\} \).

We shall now briefly redescribe these transformations.

The transformations

\begin{enumerate}
    \item \( T : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}, T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an orthogonal transformation and \( T(\infty) = \infty \),
    \item \( R : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}, x \mapsto x + v \text{ for } x \in \mathbb{R}^n \text{ and } v \in \mathbb{R}^n, \infty \mapsto \infty, \)
\end{enumerate}
(c) \( D : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}, \ x \mapsto \lambda x \text{ for } x \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}, \ \infty \mapsto \infty, \)

(d) \( \text{In} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}, \ x \mapsto x^{-1} \text{ for } x \in \mathbb{R}^n \setminus \{0\}, \ \infty \mapsto 0, \ 0 \mapsto \infty, \)

are all special examples of Möbius transformations.

**Definition 1.** The group of diffeomorphisms of \( \mathbb{R}^n \cup \{\infty\} \) generated by the transformations (a)–(d) is called the Möbius group, and is denoted by \( \text{Möb}(n) \). An element of \( \text{Möb}(n) \) is called a Möbius transformation.

When \( n = 1 \) the Clifford algebra is the complex field, and in this case it is extremely well known that a sense preserving Möbius transformation in two real dimensions can be written as \( (az + b)(cz + d)^{-1} \) where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) and \( z \in \mathbb{C} \cup \{\infty\} \).

In higher dimensions we have:

**Definition 2.** A matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( a, b, c, d \in A_n \) and

(i) \( a = a_1 \ldots a_{n_1}, \ b = b_1 \ldots b_{n_2}, \ c = c_1 \ldots c_{n_3}, \ d = d_1 \ldots d_{n_4} \), with \( n_1, n_2, n_3, n_4 \in \mathbb{N} \) and \( a_i, b_j, c_k, d_l \in \mathbb{R}^n \) for \( 1 \leq i \leq n_1, 1 \leq j \leq n_2, 1 \leq k \leq n_3, 1 \leq l \leq n_4 \),

(ii) \( \tilde{a}c, \tilde{c}d, \tilde{d}b, \tilde{b}a \in \mathbb{R}^n \),

(iii) \( \tilde{a}d - \tilde{b}c \in \mathbb{R} \setminus \{0\} \),

is called a Vahlen matrix.

From (2) and (i) we see that if \( \tilde{a}c \) is in \( \mathbb{R}^n \) then so is \( \tilde{c}(\tilde{a}c)c = \tilde{c}a(\tilde{c}c) \). But \( \tilde{c}c \in \mathbb{R} \), and so \( \tilde{c}a \in \mathbb{R}^n \). Consequently, (ii) is equivalent to saying \( \tilde{a}c, \tilde{d}b, \tilde{b}a \in \mathbb{R}^n \).

As \( \tilde{c}d \in \mathbb{R}^n \) we have \( \tilde{c}cx + \tilde{d}d \in \mathbb{R}^n \) for each \( x \in \mathbb{R}^n \), so if \( c \neq 0 \) then \( cx + d \) is invertible in \( A_n \) for all but one value of \( x \in \mathbb{R}^n \cup \{0\} \). If \( c = 0 \) then it follows from Definition 2 that \( d \) is invertible in \( A_n \). Consequently, \( (ax + b)(cx + d)^{-1} \) is a well defined element of \( A_n \) for all but one value of \( x \in \mathbb{R}^n \cup \{0\} \).

When \( c \neq 0 \) we have

\[
(3) \quad (ax + b)(cx + d)^{-1} = ac^{-1} + \lambda(cx + d)^{-1}
\]

where \( \lambda \in \mathbb{R} \setminus \{0\} \), and when \( c = 0 \),

\[
(4) \quad (ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1}.
\]

Both (3) and (4) are Möbius transformations.

From (3) and (4) we have
Lemma 1 [1]. Each Vahlen matrix can be expressed as a finite product of the special Vahlen matrices
\[
\begin{pmatrix}
a & 0 \\
0 & \frac{1}{a-1}
\end{pmatrix}, \quad \begin{pmatrix}
\lambda^{1/2} & 0 \\
0 & \lambda^{-1/2}
\end{pmatrix}, \quad \begin{pmatrix}1 & v \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}0 & 1 \\
1 & 0
\end{pmatrix}
\]
where \(a \in \text{Pin}(n)\), \(\lambda \in \mathbb{R}^+\), and \(v \in \mathbb{R}^n\).

These special Vahlen matrices transform into special M"obius transformations (a)–(d). Using this fact, the identities (3) and (4), and Lemma 1 it is straightforward to deduce

Proposition 1 [1]. The set \(V(n)\) of Vahlen matrices over \(\mathbb{R}^n\) forms a group under matrix multiplication, and the projection
\[
p : V(n) \to \text{M"ob}(n), \quad \begin{pmatrix}a & b \\
c & d
\end{pmatrix} \mapsto (ax + b)(cx + d)^{-1},
\]
is a surjective group homomorphism.

By trying to determine the Vahlen matrices for which the equation
\[
x = (ax + b)(cx + d)^{-1}
\]
holds for all \(x \in \mathbb{R}^n\) we may use (3) and (4) to obtain

Proposition 2.

\[
\text{Ker}(p) = \left\{ \begin{pmatrix} \lambda & 0 \\
0 & \lambda
\end{pmatrix}, \begin{pmatrix} e_1 \ldots e_n & 0 \\
0 & -\lambda(e_1 \ldots e_n)^{-1}
\end{pmatrix} : \lambda \in \mathbb{R} \setminus \{0\} \right\}.
\]

Consequently, the group \(V(n) \setminus \mathbb{R}^+\) is a four-fold covering group of \(\text{M"ob}(n)\). Now,
\[
V(n) \setminus \mathbb{R}^+ \cong \left\{ \begin{pmatrix} a & b \\
c & d
\end{pmatrix} \in V(n) : ad - bc = \pm 1 \right\}.
\]
The subgroup
\[
V_+(n) = \left\{ \begin{pmatrix} a & b \\
c & d
\end{pmatrix} \in V(n) : ad - bc = 1 \right\}
\]
of \(V(n) \setminus \mathbb{R}^+\) is a natural generalization of \(\text{SL}(2, \mathbb{R})\).

The Vahlen matrices introduced here are not quite the same as those described in [1]. We now introduce those matrices:

Definition 3. A matrix \(\begin{pmatrix} a & b \\
c & d \end{pmatrix}\) with \(a, b, c, d \in A_n\) and
\[
\begin{align*}
&\text{(i)} \quad a = a_1 \ldots a_n, \quad b = b_1 \ldots b_n, \quad c = c_1 \ldots c_n, \quad d = d_1 \ldots d_n, \\
&\text{with } a_i, b_i, c_i, d_i \in \mathbb{R}^+ + \mathbb{R}^n, \\
&\text{(ii)} \quad a_1 e_1, c_1 e_1, d_1 e_1, b_1 e_1 \in \mathbb{R}^+ + \mathbb{R}^n, \\
&\text{(iii)} \quad ad - bc \in \mathbb{R} \setminus \{0\},
\end{align*}
\]
where \(\mathbb{R} + \mathbb{R}^n\) is spanned by \(1, e_1, \ldots, e_n\), is called a refined Vahlen matrix.
We denote the set of refined Vahlen matrices over \( \mathbb{R} \times \mathbb{R}^n \) by \( V_0(n) \). By similar arguments to those given above we find [1] that \( V_0(n) \) is a group. The subgroup
\[
V_0^+(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_0(n) : ad - bc = 1 \right\}
\]
is a generalization of \( \text{SL}(2, \mathbb{C}) \). Indeed, \( V_0^+(1) = \text{SL}(2, \mathbb{C}) \).

Other properties of these types of matrices can be found in [6].

1. Now suppose that \( A \) is a real normed algebra with an identity, and \( U(A) \) is the open set of invertible elements in \( A \). Suppose that \( V \) is a domain in \( \mathbb{R}^n \) and \( f : V \to U(A) \) is a \( C^1 \) function. For \( y \in S^{n-1} \) we shall let \( f(x)_y \) denote the partial derivative of \( f \) at \( x \) in the direction of \( y \).

The following simple result is crucial to all that follows:

**Proposition 3.** Suppose that \( f(x)^{-1} \) denotes the algebraic inverse of \( f(x) \). Then \( (f(x)^{-1})_y = -f(x)^{-1}f(x)_yf(x)^{-1} \).

**Proof.**
\[
\frac{1}{h}(f(x + hy)^{-1} - f(x)^{-1}) = \frac{1}{h}f(x + hy)^{-1}(f(x) - f(x + hy))f(x)^{-1}
\]
\[
= -f(x + hy)^{-1}\left(\frac{f(x + hy) - f(x)}{h}\right)f(x)^{-1}.
\]
So
\[
\lim_{h \to 0} \frac{1}{h}(f(x + hy)^{-1} - f(x)^{-1}) = -f(x)^{-1}f(x)_yf(x)^{-1}.
\]

This result is an elementary generalization of the basic result that for \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\} \), \( f(x) = 1/x \), we have \( (df/dx)(x) = -1/x^2 \).

2. From Proposition 3 and (3) and (4) we have

**Lemma 2.** Suppose that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(n) \setminus \mathbb{R}^+ \) and \( \Phi(z) = (az + b)(cx + d)^{-1} \). Then for each \( y \in S^{n-1} \) we have
\[
\Phi(x)_y = \begin{cases} 
-\chi(c^{-1}d)y(c + c^{-1}d)^{-1}c^{-1} & \text{if } c \neq 0, \\
ayd^{-1} & \text{otherwise}.
\end{cases}
\]

From Lemma 2 and Proposition 3 it is now easy to deduce the following formula:
\[
(5) \quad \Phi(x)_y y \Phi(x)_y^{-1} - \frac{3}{2} \{\Phi(x)_y y \Phi(x)_y^{-1}\}^2 = 0.
\]

Here \( \Phi(x)_y y \Phi(x)_y^{-1} \) and \( \Phi(x)_y y \Phi(x)_y^{-1} \) mean respectively the third and second partial derivatives of \( \Phi \) at \( x \) in the direction of \( y \). Moreover, \( \Phi(x)_y^{-1} \) denotes the Kelvin inverse of the vector \( \Phi(x)_y \). (From the expressions appearing in Lemma 2 it is straightforward to see that \( \Phi(x)_y \) is a non-zero vector.)
Expression (5) is very similar in appearance to the classical Schwarzian derivative of a Möbius transformation in $\mathbb{C} \cup \{\infty\}$ (see for example [10]).

**Lemma 3.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^1$ diffeomorphism. Then $w(x)_y$ is a non-zero vector for each $x \in V$. ■

Using Lemma 3 we can now make the following definition:

**Definition 4.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism. Then we define $\{S, w\}_y$ to be $w_{yy}w_y^{-1} - \frac{3}{2}(w_{yy}w_y^{-1})^2$, and we call $\{S, w\}_y$ the Schwarzian derivative of $w$ in the direction of $y \in S^{n-1}$.

$\{S, w\}_y$ takes its values in the Lie subalgebra of $A_n$ spanned by $\{1, e_i, e_j, e_i e_j e_k e_l : 1 \leq i < k < l \leq n\}$.

From Proposition 3 we have

**Lemma 4.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism. Then

$$(w(x)_{yy}w(x)_y^{-1})_y = w(x)_{yy}w(x)_y^{-1} - (w_{yy}(x)w(x)_y^{-1})^2,$$

where $(w(x)_{yy}w(x)_y^{-1})_y$ denotes the partial derivative of $w(x)_{yy}w(x)_y^{-1}$ at $x$ in the direction of $y$. ■

As a consequence of Lemma 4 we have

**Proposition 4.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism. Then

$$(6) \quad \{S, w\}_y = (w_{yy}w_y^{-1})_y - \frac{1}{2}(w_{yy}w_y^{-1})^2.$$ ■

Expression (6) is completely analogous to the other well known form of the classical Schwarzian (see [10]).

We shall now try to determine solutions to the equation $\{S, w\}_y = 0$.

First we note

**Lemma 5.** Suppose that $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism. Then $\{S, L\}_y = 0$ for all $y \in S^{n-1}$. ■

The fact that $L$ is a solution to our generalized Schwarzian represents a departure from the results in complex analysis, and is a consequence of the fact that the Schwarzian presented here is dependent on our choice of $y$.

Bearing this in mind we are led to the following result:

**Proposition 5.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism and $\{S, w\}_{e_1} = 0$. Suppose also that $w_{e_1 e_1} \neq 0$. Then there exist $C^3$ maps $a(x_2, \ldots, x_n)$, $b(x_2, \ldots, x_n)$, $c(x_2, \ldots, x_n)$ and $d(x_2, \ldots, x_n)$ such that

$$(7) \quad w(x) = (a(x_2, \ldots, x_n) + x_1)^{-1}b(x_2, \ldots, x_n) + c(x_2, \ldots, x_n).$$
Proof. First we set \( w(x)_{e_1} w(x)^{-1} = v(x) \). So the equation \( \{S, w\}_{e_1} = 0 \) becomes
\[
\frac{\partial v}{\partial x_1} = \frac{1}{2} v^2.
\]
(8)

As \( w(x)_{e_1} \neq 0 \) we find that \( v \) is invertible in the Clifford algebra. So (8) is equivalent to
\[
v^{-1} \frac{\partial v}{\partial x_1} v^{-1} = \frac{1}{2},
\]
or
\[
-v^{-1} \frac{\partial v}{\partial x_1} v^{-1} = \frac{1}{2}.
\]
But from Proposition 3 we have
\[
v^{-1} \frac{\partial v}{\partial x_1} v^{-1} = \frac{\partial}{\partial x_1} (v^{-1}).
\]
So \( (\partial / \partial x_1)(v^{-1}) = -1/2 \). Consequently,
\[
v(x)^{-1} = -\frac{1}{2}(x_1 + a(x_2, \ldots, x_n)).
\]
As \( v(x) \) is invertible in \( A_n \), \( x_1 + a(x_2, \ldots, x_n) \) must be invertible in \( A_n \). So
\[
-2(x_1 + a(x_2, \ldots, x_n))^{-1} = v(x).
\]
We now set \( \partial w / \partial x_1 = u(x) \). So we have
\[
\frac{\partial u}{\partial x_1}(x) = -2(x_1 + a(x_2, \ldots, x_n))^{-1} u(x).
\]
Equation (9) tells us that \( u(x) \) is a \( C^\infty \) function in the variable \( x_1 \). It also enables us to deduce that \( u(x) \) is a real-analytic function in \( x_1 \).

Explicitly working out the Taylor expansion of \( u(x) \) about one fixed value \( x_1 = x_1' \) we have
\[
u(x) = -2(a(x_2, \ldots, x_n) + x_1)^{-2} b(x_1', x_2, \ldots, x_n).
\]
So
\[
w(x) = (a(x_2, \ldots, x_n) + x_1)^{-1} b(x_1', x_2, \ldots, x_n) + c(x_2, \ldots, x_n),
\]
where \( a, b \) and \( c \) are \( A_n \)-valued functions. \( \blacksquare \)

We may also easily deduce

**Proposition 6.** Suppose that \( w : V \hookrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism and \( (\partial^3 w / \partial x_1^3)(x) = 0 \) on some neighbourhood of \( x_0 \in V \). Then on that neighbourhood we have
\[
w(x) = x_1 a'(x_2, \ldots, x_n) + b'(x_2, \ldots, x_n),
\]
where \( a' \) and \( b' \) are \( A_n \)-valued functions. \( \blacksquare \)
Now using elementary continuity arguments we have, from Propositions 5 and 6,

**Proposition 7.** Suppose that \( w : V \hookrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism satisfying \( \{ S, w \}_{e_i} = 0 \) for all \( x \in V \). If \( (\partial^2 w/\partial x_1^2)(x_0) \neq 0 \) for some \( x_0 \in V \), then \( (\partial^2 w/\partial x_i^2)(x) \neq 0 \) for any \( x \in V \). ■

We now deduce

**Lemma 6.** The function \( c(x_2, \ldots, x_n) \) appearing in (7) is a vector-valued function.

**Outline proof.** The result follows immediately from allowing the term \( x_1 \), on the right hand side of (7), to vary. ■

We now see that

\[
w(x) - c(x_2, \ldots, x_n) = (a(x_2, \ldots, x_n) + x_1)^{-1} b(x_2, \ldots, x_n)
\]

is a vector. As we can take the Kelvin inverse of the left hand side of (11), we see that \( b(x_2, \ldots, x_n) \) is invertible in \( A_n \). By now allowing \( x_1 \) to vary we have, from (11),

**Lemma 7.** \( b(x_2, \ldots, x_n)^{-1} a(x_2, \ldots, x_n) \) is a vector, and so is \( b(x_2, \ldots, x_n) \).

As a consequence of Lemma 7 we have

**Lemma 8.** The function \( a(x_2, \ldots, x_n) \) lies in the subspace of \( A_n \) spanned by the set \( \{ 1, e_i e_j : 1 \leq i < j \leq n \} \).

As a consequence of all this we can rewrite (7) as

\[
(12) \quad w(x) = (\lambda_1(x_2, \ldots, x_n) + x_1 \lambda_1(x_2, \ldots, x_n))^{-1} + \gamma_1(x_2, \ldots, x_n)
\]

where \( \lambda_1, \mu_1, \) and \( \gamma_1 \) are all vectors.

Similar calculations tell us that the functions \( a'(x_2, \ldots, x_n) \) and \( b'(x_2, \ldots, x_n) \) appearing in (10) are vectors.

(10) and (12) give us

**Theorem 1.** Suppose that \( w : V \hookrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism satisfying \( \{ S, w \}_{y} = 0 \) for each \( y \in S^{n-1} \). Then for any line \( l \subseteq \mathbb{R}^n \) with \( l \cap V \neq \emptyset \), on each connected line segment of \( V \cap l \) the diffeomorphism \( w \) is the restriction of a Möbius transformation on \( \mathbb{R}^n \cup \{ \infty \} \).

In fact, elementary geometry and continuity arguments give us

**Theorem 2.** Suppose that \( w : V \hookrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism satisfying \( \{ S, w \}_{y} = 0 \) for each \( y \in S^{n-1} \). Then for any line \( l \subseteq \mathbb{R}^n \) with \( l \cap V \neq \emptyset \), \( w|_{V \cap l} \) is the restriction of a Möbius transformation on \( \mathbb{R}^n \cup \{ \infty \} \). ■

It might initially be suspected that if \( w : V \hookrightarrow \mathbb{R}^n \) is \( C^3 \) diffeomorphism and \( \{ S, w \}_{e_j} = 0 \) for \( j = 1, \ldots, n \) then

\[
w(x) = (a(Lx) + b)(c(Lx) + d)^{-1}
\]
where \((a, b, c, d)\) is a Vahlen matrix and \(L : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is an isomorphism. Unfortunately, this is not true.

Consider \(w(x_1e_1 + x_2e_2) = (1/x_1)e_1 + (1/x_2)e_2\). Then \(\{S, w\}_{x_1} = \{S, w\}_{x_2} = 0\), but \(w(x_1e_1 + x_2e_2)\) is not a Möbius transformation. Bearing the example in mind we shall continue to look at \(C^3\) diffeomorphisms whose generalized Schwarzian vanishes at all points in \(V\) and in all directions. First we prove:

**Proposition 8.** Suppose that \(w : V \hookrightarrow \mathbb{R}^n\) is a \(C^3\) diffeomorphism and \(\{S, w\}_y = 0\) for \(y \in S^{n-1}\). Suppose also that on each line \(l\) with \(V \cap l \neq \emptyset\) we have

\[
\begin{align*}
(13) \quad w(x) &= (\lambda_l(x_2^+) + x_1\mu_l(x_1^+))^{-1} + \gamma_l(x_1^+),
\end{align*}
\]

where \(x_1^+\) is a variable independent of \(x_1\), and \(x_l\) is a parametrization of \(l\). Then \(\gamma_l(x_1^+)\) is a constant.

**Proof.** Choose a point \(x_0 \in V\), and a ball \(B(x_0, r)\). For each ray \(r_{x_0}\) passing through \(x_0\) we have

\[
(14) \quad w(x) = (\lambda(x_0)(\theta_1, \ldots, \theta_{n-1}) + |r_{x_0}|\mu(x_0)(\theta_1, \ldots, \theta_{n-1}))^{-1} + \gamma_{x_0}(\theta_1, \ldots, \theta_{n-1}),
\]

where \(\theta_1, \ldots, \theta_{n-1}\) is a parametrization of \(S^{n-1}\). So on each ray \(w(x)\) has a unique continuation.

From (14) we have \(\lim_{|r_{x_0}| \to \infty} w(x) = \gamma_{x_0}(\theta_1', \ldots, \theta_{n-1}'), \) where \((\theta_1', \ldots, \theta_{n-1}') \in \gamma_{x_0} \cap S^{n-1}\). Similarly, for \(x_1 \in B(x_0, r) \setminus \{x_0\}\) we have

\[
\begin{align*}
 w(x) &= (\lambda_{x_1}(\theta_1, \ldots, \theta_{n-1}) + |r_{x_1}|\mu_{x_1}(\theta_1, \ldots, \theta_{n-1}))^{-1} + \gamma_{x_1}(\theta_1, \ldots, \theta_{n-1})
\end{align*}
\]

and therefore \(\lim_{|r_{x_1}| \to \infty} w(x) = \gamma_{x_1}(\theta_1', \ldots, \theta_{n-1}').\)

Now choose a continuous function \(z : (0, \infty) \rightarrow \mathbb{R}^n\) so that \(z(0) = x_0\) and \(z(t)\) is asymptotic to the ray \(r_{x_1}\). As \(\lambda, \mu\) and \(\gamma\) are continuous we obtain \(\lim_{t \to \infty} w(z(t)) = \gamma_{x_0}(\theta_1', \ldots, \theta_{n-1}').\) Consequently, \(\gamma_{x_1}(\theta_1', \ldots, \theta_{n-1}') = \gamma_{x_0}(\theta_1', \ldots, \theta_{n-1}').\) As this is true for each \(x_1 \in B(x_0, r)\), \(\gamma_l(x_1^+)\) is a constant.

We shall denote this constant vector by \(\gamma\). Trivially we have:

**Lemma 9.** Suppose that \(w(x)\) is as in Proposition 8. Then the \(C^3\) diffeomorphism \(w(x) - \gamma\) also has the generalized Schwarzian zero for all \(y \in S^{n-1}\). Moreover, on each line \(l\) we have

\[
 \begin{align*}
 w(x) - \gamma &= (\lambda_l(x_1^+) + x_1\mu_l(x_1^+))^{-1}. 
\end{align*}
\]

Via direct computation we may deduce

**Proposition 9.** Suppose that \(w : V \hookrightarrow \mathbb{R}^n\) is a \(C^3\) diffeomorphism and \(\{S, w(x)\}_y = 0\) for all \(x \in V\) and all \(y \in S^{n-1}\). Then \(\{S, w(x)^{-1}\}_y = 0\) for all \(x \in V\) and all \(y \in S\).
On taking the Kelvin inverse of $w(x) - \gamma$ it follows from Proposition 6 that on any two-dimensional hyperspace of $\mathbb{R}^n$ spanned by $e_i$ and $e_j$ and intersecting $V$ we have
\[
(w(x) - \gamma)^{-1} = v_1(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) + x_i v_1(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) + x_j v_j(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) + x_i x_j v_{ij}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n),
\]
where $v_1, v_i, v_j$ and $v_{ij}$ are vectors. On setting $x_i = u_i - u_j$ and $x_j = u_i + u_j$ it now follows from Propositions 6 and 9 that $v_{ij} = 0$. Consequently, we have

**Theorem 3.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism satisfying $\{S, w\}_y = 0$ for each $y \in S^{n-1}$. Then there is an isomorphism $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a Vahlen matrix $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ such that $w(x) = (a(Lx) + b)(c(Lx) + d)^{-1}$. ■

We now turn to look at other properties of this generalized Schwarzian. We begin with

**Theorem 4.** Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism, and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in V(n) \setminus \mathbb{R}^n_\ast$. Then
\[
\{S, (aw + b)(cw + d)^{-1}\}_y = (w\tilde{c} + \tilde{d})^{-1}\{S, w\}_y (w\tilde{c} + \tilde{d}).
\]

**Outline proof.** When $c = 0$, the result follows from (4). When $c \neq 0$ we have $(aw + b)(cw + d)^{-1} = ac^{-1} + \lambda(cw\tilde{c} + d\tilde{d})^{-1}$ where $\lambda \neq 1$. The result now follows from Proposition 3.

As $cw\tilde{c} + d\tilde{d}$ is a vector in $\mathbb{R}^n$, $cw + d$ can be expressed as a product of vectors in $\mathbb{R}^n$. Consequently, (15) can be rewritten as
\[
\{S, (aw + b)(cw + d)^{-1}\}_y = \text{sgn}(cw + d)\frac{(cw + d)\{S, w\}_y (cw + d)}{|cw + d|^2}
\]
where $\text{sgn}(cw + d)$ is the sign of $(cw + d)(cw + d)$.

If we dictate that the basis (1) is an orthonormal basis for $A_n$ then (16) yields

**Proposition 10.** If $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in V(n) \setminus \mathbb{R}^n_\ast$ then for each $y_1, y_2 \in S^{n-1}$ we have
\[
\langle \{S, w\}_{y_1}, \{S, w\}_{y_2} \rangle = \langle \{S, (aw + b)(cw + d)^{-1}\}_{y_1}, \{S, (aw + b)(cw + d)^{-1}\}_{y_2} \rangle. ■
\]

If $w : V \hookrightarrow \mathbb{R}^n$ is a $C^3$ diffeomorphism we shall let $\{S, w\}_{y,ij}$ denote the identity component of $\{S, w\}_y$, while $\{S, w\}_{y,ij}$ denotes the bivector component of $\{S, w\}_y$, that is, the component spanned by $\{e_i e_j : 1 \leq i < j \leq n\}$. 

Moreover, \( \{S,w\}_{ij} \) denotes the four-vector component of \( \{S,w\} \), spanned by \( \{e_i e_j e_k e_l : 1 \leq i < j < k < l \leq n\} \). As

\[
(cw + d)e_i e_j (c\bar{w} + d) = \frac{(cw + d)e_i (c\bar{w} + d)(cw + d)e_j (c\bar{w} + d)}{(cw + d)(c\bar{w} + d)},
\]

we have from (16)

PROPOSITION 11. Suppose \( w : V \leftrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism and \( (\frac{a}{b} \frac{c}{d}) \in V(n) \setminus \mathbb{R}^+ \). Then

\[
\{S,(aw + b)(cw + d)^{-1}\}_{ij} = \text{sgn}(cw + d)\frac{\{S,w\}_{ij}(c\bar{w} + d)}{|cw + d|^2},
\]

\[
\{S,(aw + b)(cw + d)^{-1}\}_{ijkl} = \text{sgn}(cw + d)\frac{\{S,w\}_{ijkl}(c\bar{w} + d)}{|cw + d|^2}.
\]

We also have

PROPOSITION 12. Suppose \( w : V \leftrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism and \( (\frac{a}{b} \frac{c}{d}) \in V(n) \setminus \mathbb{R}^+ \). Then

\[
\{S,(aw + b)(cw + d)^{-1}\}_{y,0} = \{S,w\}_{y,0}.
\]

Propositions 11 and 12 give us

\[
\langle \{S,(aw + b)(cw + d)^{-1}\}_{yi,ij}, \{S,(aw + b)(cw + d)^{-1}\}_{y2,ij} \rangle = \langle \{S,w\}_{yi,ij}, \{S,w\}_{y2,ij} \rangle,
\]

and

\[
\langle \{S,(aw + b)(cw + d)^{-1}\}_{yi,ijkl}, \{S,(aw + b)(cw + d)^{-1}\}_{y2,ijkl} \rangle = \langle \{S,w\}_{yi,ijkl}, \{S,w\}_{y2,ijkl} \rangle.
\]

Explicitly computing \( \{S,w\}_{y,0} \) we get

\[
\langle w_{yyy}, w_y \rangle |w_y|^2 - \frac{3}{2} \langle w_{yy}, w_y \rangle^2 |w_y|^4 + \frac{9}{4} |w_{yy}|^2 |w_y|^{-2}.
\]

This expression corresponds to one of the generalizations of the Schwarzian derivative given in [3].

Using differential forms we find that \( \{S,w\}_{y,ij} \) is equivalent to

\[
w_y \wedge w_{yyy} - 3 \langle w_y, w_{yy} \rangle (w_y \wedge w_{yy}) |w_y|^{-1},
\]

where \( w_y, w_{yy} \) are all regarded as 1-forms. This expression is identical to the second generalized Schwarzian derivative appearing in [3].

We now show that the usual method of obtaining a chain rule for the Schwarzian in one complex variable breaks down.

Suppose now \( g(w) : V \leftrightarrow \mathbb{R}^n \) is a \( C^3 \) diffeomorphism. Ideally we would like to obtain an expression for \( \{S,g(w)\}_{y} \) in terms of \( \{S,g\}_{w_y} \) and \( \{S,w\}_{y} \). First we note that \( g(w)_{yyy} \) contains the term \( Dg(x) w_{yyy} \), while \( g(w)_{yy} \)
contains the term \( Dg_{w(x)}w_{yy} \), and \( g(w)_y \) is equal to \( Dg_{w(x)}w_y \). We could re-express \( Dg_{w(x)}w_{yy} \), \( Dg_{w(x)}w_{yy} \) and \( Dg_{w(x)}w_y \) as \( a_1(x, y)w_{yy}a_1(x, y) \), \( a_2(x, y)w_{yy}a_2(x, y) \) and \( a_3(x, y)w_{yy}a_3(x, y) \), respectively, where \( a_j(x, y) = b_{j, 1}(x, y) \ldots b_{j, n_j}(x, y) \) with \( b_{j, i}(x, y) \in \mathbb{R}^n \setminus \{0\} \) for \( j = 1, 2, 3 \) and \( 1 \leq i \leq n_j \).

In general \( a_j(x, y) = a_k(x, y) \) only for \( j = k \) so we are unable to use this approach to extend the chain rule given in Theorem 4 to obtain a generalization of the Schwarzian chain rule described in [10].

3. Besides \( A_n \) we can also construct [14] the Clifford algebra \( A_{p,q} \) from the vector space \( \mathbb{R}^{p,q} \). The space \( \mathbb{R}^{p,q} \) is spanned by the elements \( f_1, \ldots, f_p, e_{p+1}, \ldots, e_{p+q} \), and it is endowed with the quadratic form \( \langle \cdot, \cdot \rangle \), where
\[
\langle x, x \rangle = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2
\]
for \( x = x_1f_1 + \ldots + x_pf_p + x_{p+1}e_{p+1} + \ldots + x_{p+q}e_{p+q} \). To construct \( A_{p,q} \) we define the relations
\[
e_if_j = -f_je_i, \quad e_ie_j + e_je_i = -2\delta_{ij}, \quad f_i f_j + f_j f_i = 2\delta_{ij}.
\]
It may now be deduced that \( A_{p,q} \) has dimension \( 2^{p+q} \). When \( p = 0 \) and \( q = n \) we have \( A_{0,n} = A_n \). It is straightforward to extend the antiautomorphisms \( \sim \) and \( - \) to \( A_{p,q} \) (see [14]). Also, we have the following extension of the Pin group:
\[
\text{Pin}(p, q) = \{ a \in A_{p,q} : a = a_1 \ldots a_k, \ k \in \mathbb{N} \text{ and } a_j \in \mathbb{R}^{p,q} \ \\
\quad \text{ where } a_j^2 = \pm 1 \text{ for } 1 \leq j \leq k \}. 
\]
Moreover [14], \( \langle axa, axa \rangle = \langle x, x \rangle \) for each \( a \in \text{Pin}(p, q) \). It may easily be verified that \( \text{Pin}(p, q) \) is a covering group of
\[
O(p, q) = \{ T : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q} : T \text{ is linear and } \langle Tx, Tx \rangle = \langle x, x \rangle \text{ for all } x \in \mathbb{R}^{p,q} \}. 
\]
If we take the closure, within the algebra \( A_{p,q}(2) \) (of \( 2 \times 2 \) matrices with coefficients in \( A_{p,q} \)), of the group generated by
\[
\left\{ \left( \begin{array}{cc} a & 0 \\ 0 & \tilde{a}^{-1} \end{array} \right), \left( \begin{array}{cc} 1 & v \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & \pm 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) : \\
\quad a \in \text{Pin}(p, q), \ v \in \mathbb{R}^{p,q}, \ \lambda \in \mathbb{R}^+ \right\}
\]
we obtain a new group which we denote by \( V(p, q) \). Again, when \( p = 0 \) and \( q = n \) we obtain \( V(n) \setminus \mathbb{R}^+ \).

We could also take the closure, within \( A_{p,q}(2) \), of the group generated
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by

\[
\left\{ \begin{pmatrix} a & 0 \\ \frac{1}{a^{-1}} & 1 \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : a = a_1 \ldots a_r, r \in \mathbb{N}, \right. \\
\left. a_j \in \mathbb{R} + \mathbb{R}^{p,q} \text{ with } a_j^2 = \pm 1 \text{ for } 1 \leq j \leq r, v \in \mathbb{R} + \mathbb{R}^{p,q}, \lambda \in \mathbb{R}^+ \right\}
\]

where \( \mathbb{R} + \mathbb{R}^{p,q} \) is spanned by \( 1, f_1, \ldots, f_p, e_{p+1}, \ldots, e_{p+q} \). We denote this group by \( V_0(p,q) \). When \( p = 0 \) and \( q = n \) we have \( V_0(p,q) = V_0(n) / \mathbb{R}^+ \).

For \( x = x_0 + x_1 f_1 + \ldots + x_p f_p \in \mathbb{R} + \mathbb{R}^{p,0} \) we have \( x\mathbb{T} = x_0^2 - x_1^2 - \ldots - x_p^2 \), so \( \mathbb{R} + \mathbb{R}^{3,0} \) inherits the same structure as the four-dimensional Minkowski space. On making the identifications

\[
\begin{align*}
1 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
f_1 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
f_2 & \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
f_3 & \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\end{align*}
(17)
\]

we see \([8]\) that \( \mathbb{R} + \mathbb{R}^{3,0} \) is identified with \( H_2 \), the space of \( 2 \times 2 \) Hermitean matrices. Also, for

\[
A = \begin{pmatrix} x_0 + x_1 & x_2 + i x_3 \\ x_2 - i x_3 & x_0 - x_1 \end{pmatrix} \in H_2
\]

we have \( \text{det} A = x_0^2 - x_1^2 - x_2^2 - x_3^2 \). Using the identifications (17) it is straightforward calculation to see that \( A_{3,0} \) is isomorphic to \( \mathbb{C}(2) \), the algebra of \( 2 \times 2 \) complex matrices.

Via this isomorphism it may now be deduced from the description of \( V_0(p,q) \) that

\[
V_0(3,0) \cong \mathbb{U}(2,2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{C}(2) \text{ and} \right. \\
\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} A^T & \overline{C}^T \\ B^T & \overline{D}^T \end{pmatrix} = \pm \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\},
\]

where \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

In greater generality, we have the group

\[
\mathbb{U}(n,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{C}(n) \text{ and} \right. \\
\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A^T & \overline{C}^T \\ B^T & \overline{D}^T \end{pmatrix} = \pm \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},
\]

where \( I_n \) is the \( n \times n \) identity matrix.

We shall let \( H_n \) denote the space of \( n \times n \) Hermitean matrices.
As \( U(n, n) \) is the closure of the subgroup of \( \mathbb{C}(2n) \) generated by the set
\[
\left\{ \begin{pmatrix} A & 0 \\ 0 & (AT)^{-1} \end{pmatrix}, \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_n & \pm I_n \\ 0 & 0 \end{pmatrix} : A \in \mathbb{C}(n), B \in H(n) \right\}
\]
we can deduce that for each \( (A B) \in U(n, n) \) the function
\[
\det_{C, D} : H_n \to \mathbb{C}, \quad X \mapsto \det(CX + D)
\]
is non-zero on an open, dense subset of \( H_n \). Hence \( (AX + B)(CX + D)^{-1} \) is well defined on this open, dense set. Moreover, using (18) we see that \( (AX + B)(CX + D)^{-1} \in H_n \) whenever \( (CX + D)^{-1} \) is defined.

The fractional linear transformation \( (AX + B)(CX + D)^{-1} \) has previously been described in [7, 9], and elsewhere.

4. From the previous section we may deduce:

**Proposition 13.** Suppose that \( (A B) \in U(n, n) \), and \( z \in H_n \setminus \{0\} \). Let \( \Phi(X) = (AX + B)(CX + D)^{-1} \). Then
\[
\Phi(X)_{zzz}\Phi(X)_{z}^{-1} - \frac{3}{2}(\Phi(X)_{zz}\Phi(X)_{z}^{-1})^2 = 0,
\]
where \( \Phi(X)_z \) denotes the partial derivative of \( \Phi(X) \) in the direction of \( z \).

In particular, Proposition 13 tells us that the group \( U(2, 2) \), used to describe Möbius transformations in Minkowski space, has a generalized Schwarzian derivative associated with it.

Proposition 13 leads us to the following definition.

**Definition 5.** Suppose that \( V \) is a domain in \( H_n \) and \( h : V \hookrightarrow H_n \) is a \( C^3 \) diffeomorphism, and for some direction \( z \in H \setminus \{0\} \) the element \( h(X)_z \) is invertible. Then
\[
h(X)_{zzz}h(X)_{z}^{-1} - \frac{3}{2}(h(X)_{zz}h(X)_{z}^{-1})^2
\]
is called the \( U(n, n) \) Schwarzian derivative of \( h(X) \) in the direction of \( z \). We denote it by
\[
\{SU(n, n), h(X)\}_z.
\]

By similar arguments to those used to deduce Theorem 4 we have

**Theorem 5.** Suppose that \( (A B) \in U(n, n) \), \( V \) is a domain in \( H_n \) and \( h : V \hookrightarrow H_n \) is a \( C^3 \) diffeomorphism. Suppose that for some direction \( z \in H_n \setminus \{0\} \) the element \( h(X)_z \) is invertible. Then
\[
\{SU(n, n), (Ah(X) + B)(h(X) + D)^{-1}\}
\]
\[
= (h(X)C^T + DT)^{-1}\{SU(n, n), h(X)\}_z(h(X)C^T + DT).
\]
5. Besides the groups $V(n)$ and $U(n,n)$ we can also associate a Schwarzian with the real symplectic group

$$\text{Sp}(n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{R}(n) \right\},$$

described in [7, 9], and elsewhere. $\text{Sp}(n,\mathbb{R})$ can be seen as the closure of the subgroup of $\mathbb{R}(2n)$ with generators the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : A, B \in \mathbb{R}(n) \right\}.$$

By similar arguments to those used in Section 3 we find that for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n,\mathbb{R})$ the matrix $CX + D$ is invertible on an open, dense subset of $S_n = \{ X \in \mathbb{R}(n) : X^T = X \}$. Moreover, $(AX + B)(CX + D)^{-1} \in S_n$ on this set.

**Definition 6.** Suppose that $V$ is a domain in $S_n$ and $h : V \hookrightarrow S_n$ is a $C^3$ diffeomorphism. Suppose also for some direction $z \in S_n \setminus \{0\}$ the element $h(X)z$ is invertible. Then

$$h(X)zzz = \frac{3}{2}\{h(X)zzz(h(X)^{-1}z)^2$$

is called the $\text{Sp}(n,\mathbb{R})$ Schwarzian derivative of $h(X)$ in the direction of $z$. We denote it by $\{S_{\text{Sp}(n,\mathbb{R})}, h(X)\}_z$.

**Theorem 6.** Suppose that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n,\mathbb{R})$. Then

$$\{S_{\text{Sp}(n,\mathbb{R})}, (Ah(X) + B)(Ch(X) + D)^{-1}\}_z$$

$$= (h(X)C^T + D^T)^{-1}\{S_{\text{Sp}(n,\mathbb{R})}, h(X)\}_z (h(X)C^T + D^T).$$

If $h(X) = X$ for all $X \in S_n$ then

$$\{S_{\text{Sp}(n,\mathbb{R})}, (AX + B)(CX + D)^{-1}\}_z = 0. \blacksquare$$

By similar arguments we may introduce a Schwarzian derivative and an analogue of Theorems 5 and 6 for the quaternionic group

$$H(n, n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H(2n) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right) \right\},$$

where $\cdot^\dagger$ here denotes quaternionic conjugation.

6. In this final section we briefly describe how the results of the previous two sections carry through to the group $V(p,q)$.

First suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(p,q)$. Then it follows from the description of $V(p,q)$ given in Section 3 that $(cx + d)(\bar{x} + d)$ is real-valued, non-zero.
on an open dense subset of $\mathbb{R}^{p,q}$. Consequently, $(ax + b)(cx + d)^{-1}$ is well defined on this set. Moreover, it follows from our characterization of $V(p,q)$ that $(ax + b)(cx + d)^{-1}$ is a M"obius transformation on $\mathbb{R}^{p,q}$. It is now straightforward to construct a Schwarzian derivative on $\mathbb{R}^{p,q}$ and to obtain an analogue of Theorems 5 and 6 in this setting.

References


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