On one-dimensional diffusion processes living in a bounded space interval

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Abstract. We prove that under some assumptions a one-dimensional Itô equation has a strong solution concentrated on a finite spatial interval, and the pathwise uniqueness holds.

Introduction. In the present paper we will consider a diffusion satisfying the stochastic integral Itô equation

\begin{equation}
X(t) = X(0) + \int_0^t a(s, X(s)) \, ds + \int_0^t b(s, X(s)) \, dW(s)
\end{equation}

where \( W(t) \) is a given one-dimensional Wiener process on a probability space \((\Omega, \mathcal{F}, P)\).

It is known ([1], p. 372) that if \( b(t, r_i) = 0 \leq (-1)^i a(t, r_i) \), \( i = 0, 1, t \geq 0 \), and if \( a \) and \( b \) are sufficiently regular, then (1) has a unique solution \( X(t) \) concentrated on the interval \([r_0, r_1]\).

In this paper we consider strong solutions of (1) ([3], p. 149). An example of a stochastic integral equation which has a solution but has no strong solution is due to H. Tanaka ([3], p. 152). We will give some sufficient conditions in order that (1) has a unique (in the sense of pathwise uniqueness) strong solution \( X(t) \), satisfying \( X(t) \in (\alpha(t), \beta(t)) \) for \( t \geq 0 \), where \( \alpha \) and \( \beta \) are given sufficiently regular real-valued functions defined for \( t \geq 0 \).

Existence and pathwise uniqueness of the strong solution of equation (1) on a finite spatial interval. First we give some sufficient conditions in order that a strong solution \( X(t) \) of the stochastic equation

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exists and satisfies the additional condition
\[ |X(t)| < 1 \quad \text{for } t \geq 0. \]

We will need the following theorem ([1], Theorem 3.11, p. 300 in the case \( d = 1 \)):

**Theorem 1.** Let \( a : [0,\infty) \times \mathbb{R} \to \mathbb{R} \) and \( b : [0,\infty) \times \mathbb{R} \to \mathbb{R} \) be locally bounded and Borel measurable. Suppose that for each \( T > 0 \) and \( N \geq 1 \) there exist constants \( K_T \) and \( K_{T,N} \) such that

1) \[ |b(t,x)|^2 \leq K_T(1 + x^2), \quad xa(t,x) \leq K_T(1 + x^2), \quad 0 \leq t \leq T, \ x \in \mathbb{R}, \]

2) \[ |b(t,x) - b(t,y)| \vee |a(t,x) - a(t,y)| \leq K_{T,N}|x - y|, \quad 0 \leq t \leq T, \ |x| \vee |y| \leq N. \]

Given a 1-dimensional Brownian motion \( W \) and an independent \( \mathbb{R} \)-valued random variable \( \xi \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( E[|\xi|^2] < \infty \), there exists a process \( X \) with \( X(0) = \xi \) a.s. such that \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}, W, X)\) is a solution of the stochastic integral equation (1), where \( \mathcal{F}_t = \mathcal{F}^W_t \vee \sigma(\xi) \) (\( \sigma(\xi) \) denotes the minimal \( \sigma \)-algebra with respect to which \( \xi \) is measurable).

Let \( \Phi(t,x) \) be a monotone (in \( x \)) continuous function, defined for \( t \in [0,T], x \in (-1,1) \), for which the derivatives \( \Phi_t(t,x) \), \( \Phi_x(t,x) \) and \( \Phi_{xx}(t,x) \) exist and are continuous. For each \( t \in [0,T] \) there exists a function \( \Psi(t,x) \) inverse to \( \Phi(t,x) \), i.e. \( \Phi(t,\Psi(t,x)) = x, \Psi(t,\Phi(t,x)) = x \). If \( \xi(t) \) satisfies (1) and \( |\xi(t)| < 1 \) for \( t \in [0,T] \), then applying Itô’s formula ([2], Theorem 4, p. 24) we conclude that the process \( X(t) = \Phi(t,\xi(t)) \) satisfies the equation

\[ dX(t) = m(t,X(t)) \, dt + \sigma(t,X(t)) \, dW(t), \]

where

\[ m(t,x) = \frac{\partial \phi}{\partial t}(t,\Psi(t,x)) + \frac{\partial \phi}{\partial x}(t,\Psi(t,x))a(t,\Psi(t,x)) + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t,\Psi(t,x))b^2(t,\Psi(t,x)), \]

\[ \sigma(t,x) = \frac{\partial \phi}{\partial x}(t,\Psi(t,x))b(t,\Psi(t,x)). \]

Let

\[ p(x) = \int_0^x \frac{ds}{\sqrt{1 + s^2}}, \]

\[ \Phi(x) = p^{-1}\left( \ln \frac{1 + x}{1 - x} \right). \]
Note that $\Phi$ is an increasing one-to-one mapping from $(-1, 1)$ onto $\mathbb{R}$. Define

$$\Psi(x) = \Phi^{-1}(x) = \frac{e^{p(x)} - 1}{e^{p(x)} + 1}.$$  

**Theorem 2.** Assume that a 1-dimensional Wiener process $W(t)$ and an independent $\mathbb{R}$-valued random variable $X_0$ on a probability space $(\Omega, \mathcal{F}, P)$ are given, $|X_0| < 1$ with probability 1. Let the coefficients $a(t, x)$ and $b(t, x)$ of (1) be defined, Borel measurable and locally bounded for $t \geq 0$, $|x| \leq 1$. Suppose further that

1) for each $T > 0$ there exists a constant $K_T$ such that
$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K_T|x - y|$$
$t \in [0, T], |x| \leq 1, |y| \leq 1$,

2) $b(t, \mp 1) = 0$ for $0 \leq t \leq T$,

3) $a(t, 1) \leq 0, a(t, -1) \geq 0$ for $0 \leq t \leq T$,

4) $E(\Phi(X_0))^2 < \infty$.

Then there exists a process $X(t)$ with $X(0) = X_0$ a.s. such that $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, W, X(t))$ is a solution of the stochastic integral equation (1), where $\mathcal{F}_t = \mathcal{F}_t^W \lor \sigma(X_0)$, and $|X(t)| < 1$ for $0 \leq t \leq T$ a.s. If $X_1(t)$ and $X_2(t)$ are two solutions of (1) with $P(X_i(0) = X_0) = 1$ and $|X_i(t)| < 1$ a.s. for $i = 1, 2$ and for $t \in [0, T]$, then
$$P\left\{ \sup_{0 \leq t \leq T} |X_1(t) - X_2(t)| = 0 \right\} = 1.$$

**Proof.** By 1) and 2) we have $|b(t, x)| = |b(t, x) - b(t, 1)| \leq K_T|x - 1|$. Thus
$$\left| \frac{b(t, x)}{x} \right| \leq K_T \quad \text{for } 0 \leq t \leq T, |x| < 1.$$  

Analogously
$$\left| \frac{b(t, x)}{x + 1} \right| \leq K_T \quad \text{for } 0 \leq t \leq T, |x| < 1.$$  

From 1) and 3) we have
$$\frac{a(t, x)}{x + 1} = \frac{a(t, x) - a(t, -1)}{x + 1} + \frac{a(t, -1)}{x + 1} \geq \frac{a(t, x) - a(t, -1)}{x + 1} \geq \frac{-|a(t, x) - a(t, -1)|}{x + 1}.$$  

Hence
$$\frac{a(t, x)}{x + 1} \geq -K_T \quad \text{for } 0 \leq t \leq T, |x| < 1.$$
Analogously

\[ \frac{a(t,x)}{1-x} - \frac{a(t,1)}{1-x} \leq K_T \quad \text{for } 0 \leq t \leq T, |x| < 1. \]

Consider the equation (1) with the drift coefficient \( m(t,x) \) and the diffusion coefficient \( \sigma(t,x) \) given by the formulas (2) and (3); \( \Phi \) and \( \Psi \) are given by (5) and (6). We will prove that they satisfy all assumptions of Theorem 1. By (6)

\[ \Psi'(x) = \frac{2e^{p(x)}}{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}, \]

\[ \Psi''(x) = \frac{2e^{p(x)}[(1 - e^{p(x)})\sqrt{1 + x^2} - e^{p(x)} + 1]}{(1 + x^2)^{3/2}[e^{p(x)} + 1]^3}. \]

Since \( \Phi \circ \Psi = \text{id} \), we have

\[ \Phi'(\Psi(x)) = \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}}. \]

Differentiating the identity \( \Phi'(\Psi(x))\Psi'(x) = 1 \), we obtain \( \Phi''(\Psi(x)) = -\Psi''(x)\{\Psi'(x)\}^{-3} \). Thus

\[ m(t,x) = a(t,\Psi(x)) \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}} \]

\[ \quad - \frac{1}{2} b^2(t,\Psi(x)) \left( \frac{b(t,\Psi(x))}{\Psi'(x)} \right)^2 \Psi''(x) \frac{\Psi''(x)}{\Psi'(x)}, \]

\[ \sigma(t,x) = b(t,\Psi(x)) \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}}. \]

If \( x \geq 0 \), then \( p(x) \geq 0 \) and by (7) and (12) we obtain

\[ |\sigma(t,x)| \leq K_T \frac{e^{p(x)} + 1}{e^{p(x)}} \sqrt{1 + x^2} \leq 2K_T \sqrt{1 + x^2}. \]

If \( x \leq 0 \), then \( p(x) \leq 0 \) and by (8) and (12) we have

\[ |\sigma(t,x)| \leq K_T \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}} \leq 2K_T \sqrt{1 + x^2}. \]

Thus \( \sigma(t,x) \) satisfies Condition 1) of Theorem 1.
If $x \geq 0$, then by (10)
\begin{equation}
(13) \quad xa(t, \Psi(x)) \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}} = \frac{a(t, \Psi(x))}{1 - \Psi(x)} x \sqrt{1 + x^2(1 + e^{-p(x)})} \leq 2K_T(1 + x^2).
\end{equation}

If $x \leq 0$, then by (9)
\begin{equation}
(14) \quad xa(t, \Psi(x)) \frac{\sqrt{1 + x^2(e^{p(x)} + 1)^2}}{2e^{p(x)}} = \frac{a(t, \Psi(x))}{1 + \Psi(x)} x \sqrt{1 + x^2(e^{p(x)} + 1)} \\
\leq -K_T x \sqrt{1 + x^2(e^{p(x)} + 1)} - K_T (-x) \sqrt{1 + x^2(e^{p(x)} + 1)} \leq 2K_T(1 + x^2).
\end{equation}

Next
\begin{equation}
(15) \quad -\frac{1}{2} \Psi''(x) = \frac{1}{2} \Psi(x) - \frac{x}{2(1 + x^2)}.
\end{equation}

Since $b(t, \Psi(x))/\Psi'(x) = \sigma(t, x)$ satisfies Condition 1) of Theorem 1, by (13)–(15) we conclude that $m(t, x)$ satisfies Condition 1) of Theorem 1. Condition 2) of Theorem 1 also holds.

Thus, there exists a process $Y(t)$ satisfying (1) with the coefficients $m(t, x)$ and $\sigma(t, x)$ with the initial condition $Y(0) = \Phi(0, X_0)$. Using Itô’s formula, we prove that the process $X(t) = \Psi(t, Y(t))$ satisfies the equation
\begin{align*}
\quad dX(t) &= a_1(t, X(t))dt + b_1(t, X(t))dW(t), \\
\quad a_1(t, x) &= \Psi'(\Phi(x))m(t, \Phi(x)) + \frac{1}{2}\Psi''(\Phi(x))\sigma^2(t, \Phi(x)), \\
\quad b_1(t, x) &= \Psi'\Phi(x))\sigma(t, \Phi(x)).
\end{align*}

Applying formulas (2), (3) and the identity $\Psi \circ \Phi = \text{id}$, we obtain
\begin{equation*}
\quad a_1(t, x) = a(t, x)(\Psi \circ \Phi)'(x) + \frac{1}{2}b^2(t, x)(\Psi \circ \Phi)''(x) = a(t, x).
\end{equation*}

Analogously,
\begin{equation*}
\quad b_1(t, x) = b(t, x)(\Psi \circ \Phi)'(x) = b(t, x).
\end{equation*}

Thus $X(t)$ is a strong solution of (1) with the initial condition $X(0) = \Phi(0, Y(0)) = \Phi(0, \Phi(0, X_0)) = X_0$. Moreover, $|X(t)| < 1$ for $t \geq 0$ a.s. Let $X_1(t)$ and $X_2(t)$ be two solutions of (1) with $P(X_i(0) = X_0) = 1$ and $|X_i(t)| < 1$ for $t \in [0, T]$, $i = 1, 2$. Extend $b$ to be zero outside $[-1, 1]$ and set $a(t, x) = a(t, -1)$, $x < -1$, and $a(t, x) = a(t, 1)$, $x > 1$. Then from Theorem 3.7 of [1], p. 297, we conclude that $P\{X_1(t) = X_2(t) \text{ for } 0 \leq t \leq T\} = 1$, that is to say, the pathwise uniqueness holds. The proof is finished.

If the coefficients of (1) satisfy the assumptions of Theorem 2 and additionally $a(t, x)$ and $b(t, x)$ are continuous in both arguments, then ([2], Theorem 2, p. 68 and [2], p. 66) the solution of (1) is a diffusion with diffusion coefficient $b^2(t, x)$ and drift coefficient $a(t, x)$.
Let \( f(t, x) \) be a real function defined in \( G = \{(t, x) : 0 \leq t \leq T, \alpha(t) \leq x \leq \beta(t)\} \), where \( \alpha, \beta \in C^1[0, T] \). Assume that \( f(t, x) \) is \( C^3 \) in some open neighbourhood of \( G \) and \( (\partial f/\partial x)(t, x) > 0 \) in \( G \). Moreover, suppose \( f(t, \cdot) \) is a one-to-one mapping from \( (\alpha(t), \beta(t)) \) onto \((-1, 1)\) for \( t \in [0, T] \). Let \( g(t, \cdot) \) denote the inverse of \( f(t, \cdot) \), i.e.,

\[
g(t, f(t, x)) = x \quad \text{for } t \in [0, T].
\]

From Theorem 2 follows:

**Corollary 1.** Assume that a 1-dimensional Wiener process \( W(t) \) and an independent \( \mathbb{R} \)-valued random variable \( X_0 \) on a probability space \((\Omega, \mathcal{F}, P)\) are given, and \( X_0 \in (\alpha(0), \beta(0)) \) a.s. Let \( a(t, x) \) and \( b(t, x) \) be measurable in \( G \). Suppose the following assumptions are satisfied:

1. \(|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|\) for \((t, x), (t, y) \in G\),
2. \(b(t, \alpha(t)) = b(t, \beta(t)) = 0\) for \(t \in [0, T]\),
3. \(a(t, \alpha(t)) \geq a(t, \beta(t)) \leq \beta(t)\) for \( t \in [0, T]\),
4. \(E[\Phi(0, (0, X_0))]^2 < \infty\).

Then there exists a process \( X(t) \) satisfying the conditions:

(A) \( X(t) = X_0 \) for \( t = 0 \),
(B) \( X(t) \in (\alpha(t), \beta(t)) \) a.s. for \( t \in [0, T]\).
(C) \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, W, X(t))\) is a solution of (1), where \( \mathcal{F}_t = \mathcal{F}^W_t \vee \sigma(X_0) \).

**Proof.** Define

\[
a_1(t, x) = \frac{\partial f}{\partial t}(t, g(t, x)) + \frac{\partial f}{\partial x}(t, g(t, x))a(t, g(t, x)) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, g(t, x))b^2(t, g(t, x)),
\]

\[
b_1(t, x) = \frac{\partial f}{\partial x}(t, g(t, x))b(t, g(t, x)).
\]

We will show that \( a_1(t, x) \) and \( b_1(t, x) \) satisfy all the assumptions of Theorem 2.

Since \( f \) and \( g \) are \( C^3 \), by 1) the coefficients \( a_1(t, x) \) and \( b_1(t, x) \) satisfy Condition 1) of Theorem 2. Since \( g(t, -1) = (t, 0) = \beta(t), f(t, \beta(t)) = 1 \) and \( f_x(t, x) = 2 \) imply Conditions 2)–4) of Theorem 2, respectively.

Thus, by Theorem 2, there exists a solution \( X_1(t) \) of (1) with the coefficients \( a_1(t, x) \) and \( b_1(t, x) \) satisfying \( X_1(0) = f(0, X_0), |X_1(t)| < 1 \) a.s. for \( t \in [0, T]\). In the same way as in Theorem 2 we prove that the process \( X(t) = g(t, X_1(t)) \) is a solution of (1) with the coefficients \( a(t, x) \) and \( b(t, x) \). Moreover, \( X(t) \) satisfies Conditions (A)–(C).
If $X(t)$ and $\overline{X}(t)$ are two solutions of (1) satisfying (A)–(C), then by Theorem 2
\[ P\left\{ \sup_{0 \leq t \leq T} |X(t) - \overline{X}(t)| = 0 \right\} = P\left\{ \sup_{0 \leq t \leq T} |f(t, X(t)) - f(t, \overline{X}(t))| = 0 \right\} = 1. \]
The corollary is proved.

If the conditions of Corollary 1 are fulfilled and additionally $a(t, x)$ and $b(t, x)$ are continuous in both arguments, then ([2], Theorem 2, p. 68 and [2], p. 66) $X(t)$ is a diffusion with diffusion coefficient $b^2(t, x)$ and drift coefficient $a(t, x)$.

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References


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