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A simulation of integral and derivative of the solution of a stochastic integral equation

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Abstract. A stochastic integral equation corresponding to a probability space $(\Omega, \Sigma_\omega, P_\omega)$ is considered. This equation plays the role of a dynamical system in many problems of stochastic control with the control variable $u(\cdot) : \mathbb{R}^1 \to \mathbb{R}^m$. One constructs stochastic processes $\eta^{(1)}(t)$, $\eta^{(2)}(t)$ connected with a Markov chain and with the space $(\Omega, \Sigma_\omega, P_\omega)$. The expected values of $\eta^{(i)}(t)$ $(i = 1, 2)$ are respectively the expected value of an integral representation of a solution $x(t)$ of the equation and that of its derivative $x'_u(t)$.

1. Introduction. Given a probability space $(\Omega, \Sigma_\omega, P_\omega)$. Let $L^2(\Omega)$ be the Hilbert space of all real random variables defined on $(\Omega, \Sigma_\omega, P_\omega)$ which have finite second moment:

$$L^2(\Omega) = \Big\{\xi : \Omega \to \mathbb{R}^1 \,\Big|\, \|\xi\|_{L^2(\Omega)} := \Big[\int_\Omega \xi^2(\omega) P_\omega(d\omega)\Big]^{1/2} = [E_\omega\{\xi^2(\omega)\}]^{1/2} < \infty\Big\}. \tag{1.1}$$

We consider the stochastic equation

$$x(t) = \int_a^b K(t,\tau)x(\tau)\,\mu(d\tau) + g(t, u(\cdot)) \quad (t \in \langle a, b\rangle \ (\mathrm{mod}\,\mu)) \tag{1.2}$$

where $\langle a, b\rangle$ is a closed or open interval, $-\infty \le a < b < \infty$, μ is Lebesgue measure on $\mathbb{R}^1$, $u(t) \in \mathbb{R}^m$, $x(t) = (x_1(t), \ldots, x_n(t))^T$, $x_i(t) \in L^2(\langle a, b\rangle)$.

We suppose that the given functions in (1.2) satisfy the following conditions:

(A) The vector-valued function $u(t) = (u_1(t), \ldots, u_m(t))^T : \langle a, b\rangle \to (\underline{u}, \overline{u}) \subset \mathbb{R}^m$ is deterministic (where $(\underline{u}, \overline{u}) = \{(u_1, \ldots, u_m) : \underline{u}_i <$

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$u_i < \overline{u}_i, i = 1, \ldots, m\}$). The function $\overline{g}(t) = (\overline{g}_1(t), \ldots, \overline{g}_n(t))^T := g(t, u(\cdot); \omega)$ is a Hilbert valued n-variate process with parameter $t \in \langle a, b\rangle$, integrable on (a, b) (see [8]).

(B) The matrix-valued function $K(t, \tau; \omega) := (K_{ij}(t, \tau; \omega))_{n\times n}$ is a Hilbert valued $(n \times n)$-variate process with parameters $(t, \tau) \in \langle a, b\rangle \times \langle a, b\rangle$.

Under some assumptions, there exists a unique solution of (1.2) which depends on $u(\cdot) : x(t) = x(t, u(\cdot); \omega)$ and is a Hilbert valued n-variate process with parameter $t \in \langle a, b\rangle$, i.e. (see [8]) $x_i(t) = x_i(t, u(\cdot)) \in L^2(\Omega)$. In many general problems of stochastic control, we deal with the state equation of the form (1.2) (see [15], [5], [11], [12]) with $K(t, \tau) = 0$ for $\tau > t \geq a$; $x(t)$ is the state variable, $u(t)$ is the control variable.

Using gradient methods to solve the corresponding stochastic control problems, we must determine the random gradient $\Phi_u(x, u, t; \omega)$ of some average cost $E\{\int_a^b \Phi\, dt\}$. This problem reduces to determining the expected value:

$$\overline{x}(t) := E_\omega\{x(t, u(\cdot); \omega)\}\,, \tag{1.3}$$

$$\overline{x}'_u(t) := E_\omega\{x'_u(t, u(\cdot); \omega)\}\,, \tag{1.4}$$

where the components of the matrix x'_u :

$$\frac{\partial x_i(t, u(\cdot); \omega)}{\partial u_j} = \frac{\partial x_i(t, u; \omega)}{\partial u_j}\bigg|_{u_j = u_j(\cdot)}$$

are the mean quadratic derivatives (m.q.d.) of the stochastic process $x_i(t, u_1, \ldots, u_m; \omega)$ with respect to the parameter u_j (see [8] or (3.1b)). Further, we have to determine the following vector and matrix:

$$\langle\varphi^{(1)}, x\rangle := E_\omega\Big\{ \int_a^b \varphi^{(1)}(t; \omega) x(t, u(\cdot); \omega)\, \mu(dt)\Big\}\,, \tag{1.5}$$

$$\langle\varphi^{(2)}, x'_u\rangle := E_\omega\Big\{ \int_a^b \varphi^{(2)}(t; \omega) x'_u(t, u(\cdot); \omega)\, \mu(dt)\Big\}\,. \tag{1.6}$$

Here $\varphi^{(i)}$ $(i = 1, 2)$ satisfy the following condition:

(C_i) $\varphi^{(i)}(t; \omega)$ is a Hilbert valued $(n_i \times n)$-variate process with components having second moment integrable on $\langle a, b\rangle$.

When $n_1 = n$ and $\varphi^{(1)}(t; \omega)$ is the unit matrix of order n, problem (1.5) reduces to determining the expected value of the integral of the solution of (1.2) on $\langle a, b\rangle$. We can also investigate the similar problem for the integral of the derivatives of the solution.

In particular, when $n = m = 1$, $u(t) = t$, and $K(t,\tau)$ and $g(t, u(\cdot)) = g(u(t)) = g(t)$ are deterministic, (1.2) takes the form of a Fredholm equation of the second type in $L^2(a,b)$. Then an unbiased estimator (u.e.) and an asymptotic u.e. of (1.3), (1.5) are obtained in [14], [10], [9].

When $g(t) = E_\omega\{\eta(t;\omega)\}$ and $K(t,\tau) = E_\omega\{\chi(t,\tau;\omega)\}$, analogous estimators are constructed basing on a realization of the processes $\eta^{(t)}$, $\chi(t,\tau)$ (see [7], [6], [4], [1]). An asymptotic u.e. and an u.e. of (1.4), (1.5) are also constructed for $\chi(t,\tau) \geq 0$ (see [3]).

In this paper, we provide an u.e. of (1.3)–(1.6) for the stochastic integral equation (1.2).

By the above method we can apply the Monte Carlo method to the numerical calculation of the quantities (1.3)–(1.6).

2. A random model connected with an integral transformation of a solution. Let $L^2_{1\times 1} = L^2_{1\times 1}(\Omega \times (a,b))$ be the class of Hilbert valued processes with second moment integrable on $\langle a,b\rangle$:

$$(2.1)\quad L^2_{1\times 1} = \Big\{ f : \langle a,b\rangle \to L^2(\Omega) \,\Big|\, \|f\|^2_{L^2_{1\times 1}} = \int_a^b E_\omega\{f^2(t)\}\,\mu(dt) < \infty \Big\}.$$

Then $L^2_{1\times 1}$ is a Hilbert space (see [2]).

Let $L^2_{n\times s} = L^2_{n\times s}(\Omega \times (a,b))$ be the class of Hilbert valued $(n\times s)$-variate processes with components having second moment integrable on (a,b):

$$(2.2)\quad L^2_{n\times s} = \Big\{ F = (F_{ij})_{n\times s} \,\Big|\, F_{ij} : \langle a,b\rangle \to L^2(\Omega),$$
$$\int_a^b \|F_{ij}(t)\|^2_{L^2(\Omega)}\,\mu(dt) = \int_a^b \int_\Omega F^2_{ij}(t;\omega) P_\omega(d\omega)\,\mu(dt) < \infty \Big\}.$$

We can represent $L^2_{n\times s}$ in the form $L^2_{n\times s} = L^2_{1\times 1} \oplus \ldots \oplus L^2_{1\times 1}$ ($n\times s$ summands). Hence $L^2_{n\times s}$ is also a Hilbert space (see [2]) with the norm

$$(2.2')\qquad \|F\|_{L^2_{n\times s}} = \Big(\sum_{i=1}^{n} \sum_{j=1}^{s} \|F_{ij}\|^2_{L^2_{n\times s}} \Big)^{1/2} \qquad (\forall F \in L^2_{n\times s}).$$

Let $\mathcal{K}$ and $\mathcal{K}_+$ be the following integral operators:

$$(2.3)\qquad [\mathcal{K} f](t) = \int_a^b K(t,\tau) f(\tau)\mu(d\tau) \qquad (f \in L^2_{n\times 1}),$$

$$(2.4)\qquad [\mathcal{K}_+ f](t) = \int_a^b K_+(t,\tau) f(\tau)\mu(d\tau) \qquad (f \in L^2_{n\times 1}),$$

where for each matrix $H = (H_{ij})_{\overline{n}\times\overline{m}}$ we write $H_+ = (|H_{ij})_{\overline{n}\times\overline{m}}$. We suppose that:

Then we have (see [8])

$$P_\theta\Big(\bigcup_{k=0}^{\infty}\Theta_k\Big) = 1\,. \tag{2.13}$$

It follows from (2.10), (2.12) that $\bigcup_{k=0}^{\infty}\Theta_k \subset \Theta$ and from (2.13) we obtain

$$P_\theta\Big(\Theta \setminus \bigcup_{k=0}^{\infty}\Theta_k\Big) = 0\,. \tag{2.14}$$

Moreover, from (2.12), it is easy to see that

$$\Theta_k \cap \Theta_{k'} = \emptyset \qquad (\forall k \neq k')\,. \tag{2.15}$$

Notice that the space

$$(\Omega^\infty, \Sigma_{\overline{\omega}}, P_{\overline{\omega}}) := \Big(\prod_{i=0}^{\infty}\Omega, \bigotimes_{i=0}^{\infty}\Sigma_\omega; \prod_{i=0}^{\infty}P_\omega\Big)$$

is a probability space. Its elementary events take the form $\overline{\omega} := (\omega_0, \dots, \omega_k, \dots)$, $\omega_i \in \Omega$ $(i = 0, 1, 2, \dots)$. Consider the product probability space $(\Theta \times \Omega^\infty, \Sigma_{\theta,\,\overline{\omega}}, P_{\theta,\,\overline{\omega}})$ where

$$\begin{aligned} \Sigma_{\theta,\,\overline{\omega}} &= \Sigma_\theta \otimes \Sigma_{\overline{\omega}}, \qquad & P_{\theta,\,\overline{\omega}} &= P_\theta \times P_{\overline{\omega}}\,, \\ \Sigma_{\overline{\omega}} &= \bigotimes_{i=0}^{\infty}\Sigma_\omega, \qquad & P_{\overline{\omega}} &= \prod_{i=0}^{\infty}P_\omega \end{aligned} \tag{2.16}$$

and a mapping $\eta^{(1)} :\ \Theta \times \Omega \to \mathbb{R}^{n_1}$ defined by

$$\begin{aligned} \eta^{(1)}(\theta;\overline{\omega}) &= F^{(1)}(\theta,\overline{\omega};\varphi^{(1)},\overline{g}) \\ &:= \sum_{k=0}^{\infty}\mathbf{1}_{\Theta_k}(\theta)\frac{\varphi^{(1)}(\theta_0;\omega_0)}{p_0(\theta_0)q(\theta_k)}\prod_{i=0}^{k}\frac{K(\theta_{i-1},\theta_i;\omega_{i-1})}{p(\theta_{i-1},\theta_i)}\overline{g}(\theta_k;\omega_k) \\ &\qquad\qquad ((\theta,\overline{\omega}) \in \Theta \times \Omega)\,, \end{aligned} \tag{2.17}$$

with the convention that

$$\frac{K(\theta_{-1},\theta_0;\omega_{-1})}{p(\theta_{-1},\theta_0)} := 1\,,$$

The measurability of $\varphi^{(1)}, K, g$ (see assumptions (C_1), (B), (B_1), (A)) and (2.5), (2.7), (2.17) yield the $\Sigma_{\theta,\,\overline{\omega}}$-measurability of $\eta^{(1)}(\theta;\overline{\omega})$ on $\Theta \times \Omega^\infty$, i.e. $\eta^{(1)}(\theta;\overline{\omega})$ is a random n_1-variate vector defined on $(\Theta \times \Omega^\infty, \Sigma_{\theta,\,\overline{\omega}}, P_{\theta,\,\overline{\omega}})$. We suppose that the random processes $\varphi(t;\omega)$, $d(t;\omega)$ fulfil the following condition:

(D′) For given $(\theta_0, \dots, \theta_k)$, the random variables $\varphi(\theta_0;\omega)$, $K(\theta_{j-1},\theta_j;\omega)$ $(j = 1, \dots, k)$, $d(\theta_k;\omega)$ are mutually independent $(\forall k = 1, 2, \dots)$.

Now we put

$$\eta_i(k;\theta,\overline{\omega}) := \sum_{j=1}^{n}\left[\frac{\varphi^{(1)}(\theta_0;\omega_0)}{p_0(\theta_0)q(\theta_k)}\prod_{l=0}^{k}\frac{K(\theta_{l-1},\theta_l;\omega_{l-1})}{p(\theta_{l-1},\theta_l)}\right]_{ij}\overline{g}_j(\theta_k;\omega_k) \quad (i=1,\ldots,n) \tag{2.24}$$

where for each matrix $A=(a_{ij})$ we write $[A]_{ij}=a_{ij}$. Then from Fubini's theorem it follows that

$$\begin{aligned} Y_i(k) &:= \int\limits_{\Theta_k\times\Omega^\infty}|\eta_i(k;\theta,\overline{\omega})|P_{\theta,\,\overline{\omega}}\,(d\theta\times d\overline{\omega}) \\ &\le \sum_{j=1}^{n}\int\limits_{\Theta_k\times\Omega^\infty}\Bigg|\left[\frac{\varphi^{(1)}(\theta_0;\omega_0)}{p_0(\theta_0)q(\theta_k)}\right. \\ &\qquad \times\left.\prod_{l=0}^{k}\frac{K(\theta_{l-1},\theta_l;\omega_{l-1})}{p(\theta_{l-1},\theta_l)}\right]_{ij}\overline{g}_j(\theta_k;\omega_k)\,P_\theta(d\theta)\,P_{\overline{\omega}}(d\overline{\omega})\,. \end{aligned} \tag{2.25}$$

From (D), (2.25), (2.8)–(2.11), (2.19) we have (with $(k+1)$-fold integration)

$$\begin{aligned} Y_i(k) &\le E_\omega\Big\{\int_a^b\ldots\int_a^b\sum_{j=1}^{n}\Big|\Big[\varphi^{(1)}(\theta_0;\omega) \\ &\qquad \times\prod_{l=0}^{k}K(\theta_{l-1},\theta_l;\omega)\Big]_{ij}\overline{g}_j(\theta_k;\omega)\Big|\,\mu(d\theta_0)\ldots\mu(d\theta_k)\Big\} \\ &\le E_\omega\Big\{\Big[\int_a^b\ldots\int_a^b\varphi_+^{(1)}(\theta_0;\omega) \\ &\qquad \times\prod_{l=0}^{k}K_+(\theta_{l-1},\theta_l;\omega)\overline{g}_+(\theta_k;\omega)\Big]_i\,\mu(d\theta_0)\ldots\mu(d\theta_k)\Big\} \end{aligned} \tag{2.26}$$

Therefore (see (2.4))

$$Y_i(k) \le E_\omega\Big\{\int_a^b[\varphi_+^{(1)}(\theta_0;\omega)\mathcal{K}_+^k\overline{g}_+(\theta_0;\omega)]_i\,\mu(d\theta_0)\Big\}\,. \tag{2.27}$$

Using Hölder's inequality, by (2.25), (2.27), (2.2′) and (C_1), (B_1), (A) we obtain

$$\int\limits_{\Theta_k\times\Omega^\infty}|\eta_i(k;\theta,\overline{\omega})|P_{\theta,\,\overline{\omega}}\,(d\theta\times d\overline{\omega}) \le \|\varphi_+^{(1)}\|_{L^2_{n_1\times n}}\|\mathcal{K}_+^k\overline{g}_+\|_{L^2_{n\times 1}} < \infty\,. \tag{2.28}$$

THEOREM 1. *Suppose that assumptions* (A), (B), (B_1), (C_1) *are satisfied and* $\varphi(t) = \varphi^{(1)}(t;\omega)$ *and* $d(t) = \overline{g}(t;\omega)$ *fulfil condition* (D). *Then the equation* (1.2) *has a solution* $x(t) = x(t, u(\cdot);\omega)$ *in* $L^2_{n\times 1}$. *Moreover, the expected value* $E_{\theta,\,\overline{\omega}}\{\eta^{(1)}(\theta;\overline{\omega})\}$ *exists and is finite and*

$$M_{\theta,\,\overline{\omega}}\{\eta^{(1)}(\theta;\overline{\omega})\} = \langle \varphi^{(1)}, x\rangle\,. \tag{2.18}$$

Proof. For each vector $h = (h_1, \dots, h_n)^T \in \mathbb{R}^n$, we set $[h]_i := h_i$. Therefore, from (B_1), it follows that

$$\Big| \int_a^b [K(t,\tau;\omega)f(\tau;\omega)]_i\ \mu(d\tau)\Big| \le \int_a^b [K_+(t,\tau;\omega)f_+(\tau;\omega)]_i\ \mu(d\tau) < \infty\,, \tag{2.19}$$

$i = 1, \dots, n$), ($\forall f \in L^2_{n\times 1}$, $t \in \langle a, b\rangle \pmod{\mu}$, $\omega \in \Omega \pmod{P_\omega}$); further, from (2.2)–(2.4) we have

$$\begin{aligned}\|\mathcal{K}f\|_{L^2_{n\times 1}} &\le \|\mathcal{K}_+f_+\|_{L^2_{n\times 1}} \le \|\mathcal{K}_+\|\,\|f_+\|_{L^2_{n\times 1}} \\ &= \|\mathcal{K}_+\|\,\|f\|_{L^2_{n\times 1}} \qquad (\forall f \in L^2_{n\times 1})\,,\end{aligned} \tag{2.20}$$

i.e. the operator $\mathcal{K} : L^2_{n\times 1} \to L^2_{n\times 1}$ is bounded. Put

$$\begin{aligned}\mathcal{S}^{(k)} &:= \sum_{j=0}^{k} \mathcal{K}^j\overline{g} = \overline{g} + \mathcal{K}\mathcal{S}^{(k-1)}, &\qquad \mathcal{S}^{(0)} &= \overline{g}\,,\\ \mathcal{S}_+^{(k)} &:= \sum_{j=0}^{k} \mathcal{K}_+^j\overline{g}_+ = \overline{g}_+ + \mathcal{K}_+\mathcal{S}_+^{(k-1)}, & \mathcal{S}_+^{(0)} &= \overline{g}_+\,.\end{aligned} \tag{2.21}$$

Then for some natural number p, it follows from (2.2)–(2.4) and (2.19) that

$$\|\mathcal{S}^{(k+p)} - \mathcal{S}^{(k)}\|_{L^2_{n\times 1}} \le \|\mathcal{S}_+^{(k+p)} - \mathcal{S}_+^{(k)}\|_{L^2_{n\times 1}} \qquad (\forall k = 1, 2, \dots)\,. \tag{2.22}$$

As the Neumann series $\sum_{j=0}^{\infty} \mathcal{K}_+^j\overline{g}_+ = \mathrm{l.i.m.}_{k\to\infty}\mathcal{S}_+^{(k)}$ converges in $L^2_{n\times 1}$ (by (A), (B_1)), $\{\mathcal{S}_+^{(k)}\}_k$ is a Cauchy sequence. Hence, by (2.22) we deduce the convergence of $\sum_{j=0}^{\infty} \mathcal{K}^j\overline{g} = \mathrm{l.i.m.}_{k\to\infty}\mathcal{S}^{(k)}$ in the Hilbert space $L^2_{n\times 1}$. Therefore from the continuity of $\mathcal{K}$, we have

$$x = \sum_{j=0}^{\infty} \mathcal{K}^j\overline{g} \tag{2.23}$$

(x is a solution in $L^2_{n\times 1}$ of (1.2)).

From (2.17), (2.24), (2.27) and (B_1), (C_1) we have

$$(2.29)\quad \sum_{k=0}^{\infty} \int_{\Theta_k \times \Omega^\infty} |[\eta^{(1)}(\theta;\overline{\omega})]_i| P_{\theta,\,\overline{\omega}}(d\theta \times d\overline{\omega})$$

$$= \sum_{k=0}^{\infty} \int_{\Theta_k \times \Omega^\infty} |\eta_i(k;\theta,\overline{\omega})| P_{\theta,\,\overline{\omega}}(d\theta \times d\overline{\omega}) = \sum_{k=0}^{\infty} Y_i(k)$$

$$\leq \int_{\Omega} \int_a^b \Big[\varphi_+^{(1)}(\theta_0;\omega) \sum_{k=0}^{\infty} (\mathcal{K}_+^k \overline{g}_+)(\theta_0;\omega)\Big]_i \mu(d\theta_0) P_\omega(d\omega) < \infty\,.$$

Hence, by (2.14), (2.17), (2.24), (2.28) it is easy to deduce the existence and finiteness of the expected value

$$(2.30)\quad E_{\theta,\,\overline{\omega}}\{[\eta^{(1)}(\theta;\overline{\omega})]_i\} = \sum_{k=0}^{\infty} \int_{\Theta_k \times \Omega^\infty} \eta_i(k;\theta,\overline{\omega}) P_{\theta,\overline{\omega}}(d\theta \times d\omega) < \infty\,.$$

Then, using Fubini's theorem, from the convergence of the series (2.29) and from (2.23), (1.5), we get

$$E_{\theta,\,\overline{\omega}}\{[\eta^{(1)}(\theta;\overline{\omega})]_i\} = E_\omega\Big\{\Big[\int_a^b \varphi^{(1)}(\theta_0;\omega) \sum_{k=0}^{\infty} \mathcal{K}^k \overline{g}(\theta_0;\omega)\, \mu(d\theta_0)\Big]_i\Big\}$$

$$= E_\omega\Big\{\Big[\int_a^b \varphi^{(1)}(\theta_0;\omega) x(\theta_0;\omega)\, \mu(d\theta_0)\Big]_i\Big\}$$

$$= [\langle \varphi^{(1)}, x\rangle]_i \qquad (i = 1, \dots, n_1)\,.$$

This completes the proof.

Corollary 1. *Suppose assumptions* (A), (B), (B_1) *are satisfied and* $\varphi(t) = K(t_0, t;\omega)$ ($\forall t_0 \in (a,b)$) *and* $d(t) = \overline{g}(t;\omega)$ *fulfil condition* (D). *Then the expected value* $E_{\theta,\,\overline{\omega}}\{\xi^{(1)}(t_0;\theta,\omega)\}$ *exists and is finite and*

$$(2.31)\qquad E_{\theta,\,\overline{\omega}}\{\xi^{(1)}(t_0;\theta,\overline{\omega})\} = \overline{x}(t_0) \qquad (\forall t_0 \in \langle a,b\rangle \ (\mathrm{mod}\,\mu))\,,$$

where $x(t) = x(t, u(\cdot);\omega)$ *is a solution of* (1.2) *and*

$$(2.32)\qquad \xi^{(1)}(t_0;\theta,\overline{\omega}) = F^{(1)}(\theta,\overline{\omega}; K(t_0,\cdot), \overline{g}) + \overline{g}(t_0;\omega_0)\,.$$

Proof. From (2.20), (2.2′), it is easy to deduce

$$\int_a^b \int_{\Omega} \Big(\int_a^b K_{ij}(t_0,\tau;\omega) f(\tau;\omega)\, \mu(d\tau)\Big)^2 P_\omega(d\omega)\, \mu(dt) < \infty$$

$$(\forall t_0 \in \langle a,b\rangle \ (\mathrm{mod}\,\mu);\ \forall f(\cdot) \in L^2_{n\times 1})\,.$$

Then (see p. 380 of [2]) for a fixed $t_0 \in \langle a, b\rangle$, we easily see that $\varphi^{(1)}(t;\omega) = K(t_0, t;\omega)$ satisfies condition (C_1). By Theorem 1, we have

$$(2.33) \qquad \langle K(t_0,\cdot), x\rangle = E_{\theta,\,\overline{\omega}}\{F^{(1)}(\theta, \overline{\omega}; K(t_0,\cdot), \overline{g})\}$$
$$(\forall t_0 \in \langle a, b\rangle \ (\mathrm{mod}\ \mu) \quad (\text{see } (2.17),\ (2.18)))$$

Further, since $x(t)$ is a solution of (1.2), therefore we have

$$(2.34) \qquad E_\omega\{x(t_0)\} = \langle K(t_0,\cdot), x\rangle + E_\omega\{\overline{g}(t_0;\omega)\} \ (\forall t_0 \in \langle a, b\rangle \ (\mathrm{mod}\ \mu)).$$

But $E_\omega\{\overline{g}(t_0;\omega)\} = E_{\theta,\,\overline{\omega}}\{\overline{g}(t_0;\omega_0)\}$. Hence, from (2.34), (2.33), (1.3) we deduce (2.31).

This completes the proof.

3. A random model connected with an integral transformation of the derivative of the solution. We consider a Hilbert valued n-variate process with parameters $(t,u) = (t, u_1, \ldots, n_m)^T$ having the m.q.d. with respect to the parameters $u = (u_1, \ldots, u_m)^T \in (\underline{u}, \overline{u})$:

$$(3.1) \qquad f'_u(t,u) = (f'_{u_1}(t,u), \ldots, f'_{u_m}(t,u)), \qquad \text{where}$$

$$(3.1\text{a}) \qquad f'_{u_j}(t,u) = (\partial f_1(t,u)/\partial u_j, \ldots, \partial f_n(t,u)/\partial u_j)^T.$$

Here, $\partial f_i(t,u)/\partial u_j$ is the m.q.d. of the Hilbert valued (scalar) process $f_i(t, u_1, \ldots, u_m)$ with parameter $u_j \in (\underline{u}_j, \overline{u}_j)$ (see [8]). Now, we set

$$(3.1\text{b}) \qquad f'_u(t, u(\cdot);\omega) = f'_u(t, u;\omega)|_{u=u(\cdot)}$$

and consider the equation

$$(3.2) \qquad x(t,u) = \int_a^b K(t,\tau)x(\tau,u)\,\mu(d\tau) + g(t,u)$$
$$(\forall t \in \langle a, b\rangle \ (\mathrm{mod}\ \mu),\ \forall u \in (\underline{u}, \overline{u})).$$

Suppose that:

(A_1) $g(t,u)$ is a Hilbert valued n-variate process with parameters $(t,u) \in \langle a, b\rangle \times (\underline{u}, \overline{u})$ and $g(\cdot, u) \in L^2_{n\times 1}$ ($\forall u \in (\underline{u}, \overline{u})$).

(A_2) For a fixed $t \in \langle a, b\rangle$, the stochastic process $g(t,u) = (g_1(t,u), \ldots$ $\ldots, g_n(t,u))^T$ has a m.q.d. with respect to the parameters $u = (u_1, \ldots, u_m) \in (\underline{u}, \overline{u})$ so that

$$\int_a^b E_\omega\{\partial g_i(t, u(\cdot);\omega)/\partial u_j\}\,\mu(dt) < \infty.$$

Moreover, for a fixed $t \in \langle a, b\rangle$, $\partial g_i(t,u)/\partial u_j$ ($i = 1, \ldots, n$, $j = 1, \ldots, m$) are Hilbert valued processes, mean quadratic continuous in $u \in (\underline{u}, \overline{u})$.

(B_2) The inverse operator $(I-\mathcal{K})^{-1}: L^2_{n\times 1} \to L^2_{n\times 1}$ exists and is bounded, where $I: L^2_{n\times 1} \to L^2_{n\times 1}$ is the identity operator.

From assumptions (A_2), (B_2), (C_2), we can construct a random $(n_2 \times m)$-variate $\eta^{(2)}(\theta;\overline{\omega})$ defined on $(\Theta \times \Omega^\infty, \Sigma_{\theta,\overline{\omega}}, P_{\theta,\overline{\omega}})$ like $\eta^{(1)}(\theta;\overline{\omega})$:

$$\begin{aligned}\eta^{(2)}(\theta;\overline{\omega}) &= F^{(2)}(\theta,\overline{\omega};\varphi^{(2)}) \\ &= \sum_{k=0}^{\infty} \mathbf{1}_{\Theta_k}(\theta)\frac{\varphi^{(2)}(\theta_0;\omega_0)}{p_0(\theta_0)q(\theta_k)}\prod_{i=0}^{k}\frac{K(\theta_{i-1},\theta_i;\omega_{i-1})}{p(\theta_{i-1},\theta_i)} g_u(\theta_k,u(\cdot);\omega_0)\,. \end{aligned} \tag{3.3}$$

THEOREM 2. *Suppose that assumptions* (A_1), (A_2), (B), (B_1), (B_2), (C_2) *are satisfied and* $\varphi(t) = \varphi^{(2)}(t;\omega)$, $d(t) = g'_u(t,u(\cdot);\omega)$ *fulfil condition* (D). *Then*:

(i) *For all* $u \in (\underline{u},\overline{u})$ *the equation* (3.2) *has a unique solution and* $x(\cdot,u) \in L^2_{n\times 1}$.

(ii) *For all* $(t,u) \in \langle a,b\rangle \times (\underline{u},\overline{u})$, *the m.q.d.* $x'_u(t,u)$ *exists such that* $x'_u(\cdot,u) \in L^2_{n_2\times m}$.

(iii) *The expected value of the random* $(n_2\times n)$*-variate vector* $\eta^{(2)}(\theta;\omega)$ *exists and is finite and we have*

$$E_{\theta,\overline{\omega}}\{\eta^{(2)}(\theta;\overline{\omega})\} = \langle \varphi^{(2)}, x'_u\rangle\,. \tag{3.4}$$

Proof. From (B_1), (A_1) it follows that the solution $x(t,u)$ of (3.2) exists and is represented by a formula like (2.23). Hence, by (B_2) we can write it uniquely as

$$x(t,u) = [(I-\mathcal{K})^{-1}g(\cdot,u)](t) \qquad (\forall u \in (\underline{u},\overline{u}))\,. \tag{3.5}$$

The conclusion (i) is proved.

Since $(I-\mathcal{K})^{-1}$ is linear and bounded, from (A_2), (3.5) we deduce (see [13]) that the m.q.d. $x'_u(t,u)$ exists and

$$x'_{u_j}(t,u) = [(I-\mathcal{K})^{-1}g'_{u_j}(\cdot;u)](t) = \Big[\sum_{n=0}^{\infty}\mathcal{K}^{(n)}g'_{u_j}(\cdot;u)\Big](t) \qquad (j=1,\dots,m)\,. \tag{3.6}$$

Hence, from (A_2), (B_2) we get (ii). By (3.6), it is easy to see that $x'_u(t,u(\cdot))$ is the solution of the equation

$$x'_{u_j}(t,u(\cdot)) = \int_a^b K(t,\tau)x'_{u_j}(\tau,u(\cdot))\,\mu(d\tau) + g'_{u_j}(t,u(\cdot))\,. \tag{3.7}$$

From (A_2), (B), (B_1), (C_2), (D) it follows that the assumptions of Theorem 1 are satisfied for (3.7).

Then the random vector

$$([\eta^{(2)}(\theta;\overline{\omega})]_{1j},\ldots,[\eta^{(2)}(\theta;\overline{\omega})]_{n_2 j})^T = F^{(1)}(\theta,\overline{\omega};\varphi^{(2)},g'_{u_j}) \qquad (j=1,\ldots,m),$$

has finite expected value, and (see (2.18), (3.1), (1.6), (3.3))

$$E_{\overline{\omega}}\{[\eta^{(2)}(\theta;\overline{\omega})]_{ij}\} = [\langle\varphi^{(2)},x'_{u_j}\rangle]_i = [\langle\varphi^{(2)},x'_u\rangle]_{ij} \qquad (i=1,\ldots,n_2),$$

i.e. we get (iii).

This completes the proof.

By Theorem 2, we can deduce the following result like in the proof of Corollary 1:

COROLLARY 2. *Suppose that assumptions* (A_1), (A_2), (B), (B_1), (B_2) *are satisfied, and* $\varphi(t) = K(t_0,t;\omega)$ ($\forall t_0 \in \langle a,b\rangle$), $d(t) = g'_u(t,u(\cdot);\omega)$ *fulfil condition* (D). *Then the expected value* $E_{\theta,\overline{\omega}}\{\xi^{(2)}(t_0;\theta,\overline{\omega})\}$ *exists and is finite and* $E_{\theta,\overline{\omega}}\{\xi^{(2)}(t_0;\theta,\overline{\omega})\} = E_\omega\{x'_u(t_0)\} = \overline{x}'_u(t_0)$ ($\forall t \in \langle a,b\rangle \pmod{\mu}$), *where* $\xi^{(2)}(t_0;\theta,\overline{\omega}) = F^{(2)}(\theta,\overline{\omega};K(t_0,\cdot)) + g'_u(t,u(\cdot);\omega_0)$.

Remark. From (2.19), from Schwarz's and Hölder's inequalities and from Banach's theorem (see p. 159 of [13]) it is easy to see that (B), (B_1), (B_2) can be replaced by the following assumption:

(B^*) There exists a natural number l so that

$$K^{(l)} := \operatorname{vraisup}\Big\{\int_a^b \cdots \int_a^b \prod_{i=1}^{l} \|K(t_{i-1},t_i)\|^2_{\mathbb{R}^{n\times n}}\,\mu(dt_0)\ldots\mu(dt_l)\Big\} < 1$$

(with $l+1$ integrals),

$$K^{(1)} := \operatorname{vraisup}\Big\{\int_a^b\int_a^b \|K(t,\tau;\omega)\|^2_{\mathbb{R}^{n\times n}}\,\mu(d\tau)\,\mu(dt)\Big\} < \infty,$$

where

$$\|K(t,\tau;\omega)\|^2_{\mathbb{R}^{n\times n}} = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}[K(t,\tau;\omega)]^2_{ij}\Big)^{1/2}.$$

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