ANNALES
POLONICI MATHEMATICI
LVI.3 (1992)

Regularity of solutions of parabolic equations with coefficients depending on t and parameters

by T. WINIARSKA (Kraków)

Abstract. The main object of this paper is to study the regularity with respect to the parameter h of solutions of the problem $du/dt + A_h(t)u(t) = f_h(t)$, $u(0) = u_h^0$. The continuity of u with respect to both h and t has been considered in [6].

1. Introduction. In this paper, we consider the family of parabolic problems

$$\frac{du}{dt}(t) + A_h(t)u(t) = f_h(t) \quad \text{for } t \in (0,T], \tag{1}$$

$$u_h(0) = u_h^0, \tag{2}$$

with $h \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$. It is well known that, under certain assumptions, the solution of the problem (1), (2) is given by the formula

$$u_h(t) = U_h(t,0)u_h^0 + \int_0^t U_h(t,s)f_h(s)\,ds, \tag{3}$$

where U_h is the fundamental solution of equation (1), for a fixed $h \in \Omega$. The problem of continuity of the mapping

$$\Omega \times [0,T] \ni (h,t) \to u_h(t) \in X, \tag{4}$$

with u_h given by (3), was considered in [6].

The main object of this paper is to study the differentiability of (4) with respect to h.

Similar problems are considered in [7], [8] but for differential equations with A_h independent of t.

2. Preliminaries. Let X, Y be Banach spaces and let Ω be an open subset of $\mathbb{R}^n$. To simplify notation we shall assume that Ω is an open interval in $\mathbb{R}$.

1991 *Mathematics Subject Classification*: 35B30.

We denote by $B(X,Y)$ the space of bounded linear operators from X to Y. The space $B(X,X)$ is denoted by $B(X)$. The space of closed linear operators from X to X will be denoted by $\mathcal{C}(X)$. For $A\in\mathcal{C}(X)$ the resolvent set of A will be denoted by $P(A)$.

Let D be a normed vector space such that there exist a Banach space Z and a bijective bounded operator $\mathcal{T}:Z\to D$. Similarly to [8], we shall consider the space

$$SB(D,Y):=\{A:D\to Y \mid A \text{ is linear and } A\mathcal{T}\in B(Z,Y)\}\,. \tag{5}$$

The definition of $SB(D,Y)$ is independent of the choice of $(Z,\mathcal{T})$. The space

$$\mathcal{M}_{\mathcal{T}}:=\{A:[0,T]\to SB(D,Y) \mid \text{the mapping } [0,T]\ni t\to A(t)\mathcal{T}\in B(Z,Y) \text{ is continuous}\}$$

is a Banach space with the norm

$$\|A\|_{\mathcal{T}}:=\sup\{\|A(t)\mathcal{T}\| \mid t\in[0,T]\}\,.$$

If $(Z',\mathcal{T}')$ is another pair as needed in (5) then $\mathcal{M}_{\mathcal{T}}=\mathcal{M}_{\mathcal{T}'}$ with equivalent norms. Thus, instead of $\mathcal{M}_{\mathcal{T}}$ we may write $\mathcal{M}$ or $\mathcal{M}(D,Y)$.

Accordingly, a mapping $\Omega\ni h\to A_h\in\mathcal{M}$ is *differentiable* (*continuous*) at $h_0\in\Omega$ if $\Omega\ni h\to A_h\in\mathcal{M}_{\mathcal{T}}$ is differentiable (continuous) at h_0. We have

$$A'_{h_0}(t)=\left(\frac{d}{dh}(A_h(t)\mathcal{T})\Big|_{h=h_0}\right)\mathcal{T}^{-1}\quad \text{for } t\in[0,T]\,.$$

The operator A'_{h_0} is independent of $(Z,\mathcal{T})$. Higher order differentiability and the $\mathcal{C}^k$-classes are now defined in the standard way.

The Banach space of all continuous mappings from $[0,T]$ into Y, with the topology of uniform convergence, is denoted by $\mathcal{C}([0,T];Y)$.

We shall consider a family $(A_h(t))_{(h,t)\in\Omega\times[0,T]}$ of closed linear operators from X to X defined, for each $(h,t)\in\Omega\times[0,T]$, on a dense linear subspace $D(A_h(t))=D$ of X.

ASSUMPTION Z_1. *There exist a Banach space Z and a bijective mapping $\mathcal{T}:Z\to D$ such that $\mathcal{T}\in B(Z,X)$ and the mapping*

$$\Omega\times[0,T]\ni(h,t)\to A_h(t)\mathcal{T}\in B(Z,X)$$

is continuous.

If Assumption Z_1 is fulfilled then $A_h\in\mathcal{M}_{\mathcal{T}}$ for all $h\in\Omega$ and the mapping $\Omega\ni h\to A_h\in\mathcal{M}_{\mathcal{T}}$ is continuous, and vice versa.

ASSUMPTION Z_2. *There exist a Banach space Z, a continuous linear bijective mapping $\mathcal{T}:Z\to D$ and $\alpha\in(0,1]$ such that the mapping*

$$[0,T]\ni t\to A_h(t)\mathcal{T}\in B(Z,X)$$

is Hölder continuous with exponent α, *i.e. there exists* $\widetilde{L} > 0$ *such that*

$$\|A_h(t)\mathcal{T} - A_h(s)\mathcal{T}\| \le \widetilde{L}|t-s|^\alpha$$

for $h \in \Omega$, $0 \le s \le T$ *and* $0 \le t \le T$.

Assumption Z_3. $A_h(t) \in G(C_0)$ *for* $(h,t) \in \Omega \times [0,T]$, *where*

$$G(C_0) = \{A \in \mathcal{C}(X) \mid \overline{D(A)} = X,\ [0,\infty) \subset P(-A),\ \|(A+\xi)^{-k}\| \le C_0\xi^{-k} \text{ for } \xi > 0,\ k = 1,2,\ldots \text{ and } \|A\exp(-tA)\| \le C_0 t^{-1} \text{ for } t > 0\}\,.$$

Let $\Delta = \{(t,s) \mid 0 \le s \le t \le T\}$.

Definition 1. A mapping

$$U_h : \Delta \ni (t,s) \to U_h(t,s) \in B(X) \tag{6}$$

is said to be a *fundamental solution* of (1) if

1) for every $x \in X$ the mapping $\Delta \ni (t,s) \to U_h(t,s)x \in X$ is continuous,
2) $U_h(t,r)U_h(r,s) = U_h(t,s)$ for $0 \le s \le r \le t \le T$,
3) $U_h(s,s) = I$ for $s \in [0,T]$,
4) for every $x \in X$ the mapping (6) is differentiable with respect to t and

$$\frac{\partial}{\partial t}U_h(t,s)x = A_h(t)U_h(t,s)x\,,$$

5) for every $x \in D$ the mapping (6) is differentiable with respect to s and

$$\frac{\partial}{\partial s}U_h(t,s)x = -U_h(t,s)A_h(s)x\,.$$

Under Assumptions Z_1–Z_3 we may define (for details see e.g. [5], Chap. 5, and [6])

$$R_1^h(t,s) := -(A_h(t) - A_h(s))\exp(-(t-s)A_h(s))\,,$$

$$R_m^h(t,s) := \int_s^t R_1^h(t,\tau)R_{m-1}^h(\tau,s)\,d\tau \quad \text{for } m = 2,3,\ldots\,,$$

$$R^h(t,s) := \sum_{m=1}^{\infty} R_m^h(t,s)\,,$$

$$W^h(t,s) := \int_s^t \exp(-(t-\tau)A_h(\tau))R^h(\tau,s)\,d\tau\,,$$

$$U_h(t,s) := \exp(-(t-s)A_h(s)) + W^h(t,s)\,, \tag{7}$$

where $\exp(-tA_h(s))$ is the strongly continuous semigroup with the infinitesimal generator $A_h(s)$ for $h \in \Omega$, $s \in [0,T]$.

Since sufficient conditions for U_h given by (7) to be a fundamental solution of (1) are known (see e.g. [5]), we do not discuss them here.

Assumption Z_4. *U_h given by* (7) *is a fundamental solution of* (1) *for* $h \in \Omega$.

We shall use the following two theorems:

Theorem 1. *Suppose Assumptions Z_1–Z_4 are fulfilled.*

(i) ([5], Th. 5.2.2) *If, for any $h \in \Omega$, there exists a solution u_h of the problem* (1), (2) *and the mapping $[0,T] \ni t \to f_h(t) \in X$ is continuous, then u_h is given by* (3).

(ii) ([6], Th. 1) *If the mappings*

$$\Omega \ni h \to u_h^0 \in X\,, \tag{8}$$

$$\Omega \times [0,T] \ni (h,t) \to f_h(t) \tag{9}$$

are continuous, then the mapping (4), *with u_h given by* (3), *is continuous.*

Theorem 2 ([4], Th. 4 and Th. 5, p. 301). *Let k be a nonnegative integer. If, for any $h \in \Omega$,*

(a) *the mapping $\Omega \ni h \to A_h \in \mathcal{M}$, is k times differentiable and its k-th derivative is Hölder continuous,*

(b) *the mapping $[0,T] \ni t \to f_h(t) \in X$ is k times differentiable and its k-th derivative is Hölder continuous,*

(c) $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0\} \subset P(-A_h(t))$ *and* $\exists C > 0$ *such that*

$$\|(A_h(t) + \lambda I)^{-1}\| \leq C\frac{1}{|\lambda| + 1}\,,$$

then

1) *there exists a solution u_h of the problem* (1), (2),
2) *u_h is given by* (3),
3) *u_h is of class $\mathcal{C}^1$ in $[0,T]$ and $\mathcal{C}^{k+1}$ in $(0,T]$.*

3. Differentiability with respect to h. For $h\in\Omega$, let u_h be a solution of (1), (2) and let $h_0 \in \Omega$. The function w_h defined by

$$w_h(t) = \frac{u_h(t) - u_{h_0}(t)}{h - h_0} \quad \text{for } h \neq h_0$$

is, for $h \neq h_0$, a solution of the problem

$$\frac{dw_h}{dt}(t) + A_h(t)w_h(t) = F_h(t) \quad \text{for } t \in (0,T]\,, \tag{10}$$

$$w_h(0) = w_h^0\,, \tag{11}$$

where

$$F_h(t) = \begin{cases} \dfrac{f_h(t) - f_{h_0}(t)}{h - h_0} - \dfrac{A_h(t) - A_{h_0}(t)}{h - h_0}u_{h_0}(t) & \text{for } h \neq h_0,\\ f'_{h_0}(t) - A'_{h_0}(t)u_{h_0}(t) & \text{for } h = h_0, \end{cases}$$

$$w_h^0 = \begin{cases} \dfrac{u_h^0 - u_{h_0}^0}{h - h_0} & \text{for } h \neq h_0, \\ u_{h_0}^{0\prime} & \text{for } h = h_0, \end{cases}$$

and "$'$" denotes differentiation with respect to h.

PROPOSITION 1. *Under the assumptions of Theorem* 1, *if the mappings*

$$(12) \quad \Omega \ni h \to f_h \in \mathcal{C}([0,T];X), \quad \Omega \ni h \to A_h \in \mathcal{M}, \quad \Omega \ni h \to u_h^0 \in X$$

are differentiable at h_0 *and the mapping*

$$(13) \qquad [0,T] \ni t \to A_{h_0}(t) u_{h_0}(t) \in X$$

is continuous, *then the mapping*

$$(14) \qquad \Omega \ni h \to u_h \in \mathcal{C}([0,T];X)$$

is differentiable at h_0 *and its derivative at* h_0 *is given by*

$$(15) \qquad u'_{h_0}(t) = U_{h_0}(t,0) w_{h_0}^0 + \int_0^t U_{h_0}(t,s) F_{h_0}(s)\, ds\,.$$

Proof. Since

$$\frac{A_h(t) - A_{h_0}(t)}{h - h_0} u_{h_0}(t) = \frac{A_h(t) - A_{h_0}(t)}{h - h_0} \mathcal{T}(A_{h_0}(t)\mathcal{T})^{-1} A_{h_0}(t) u_{h_0}(t)$$

and the convergence in $\mathcal{M}$ is uniform with respect to t, the mapping $\Omega \times [0,T] \ni (h,t) \to F_h(t)$ is continuous. By Theorem 1 the mapping

$$\Omega \times [0,T] \to \widetilde{w}_h(t) := U_h(t,0) w_h^0 + \int_0^t U_h(t,s) F_h(s)\, ds$$

is continuous and $\widetilde{w}_h = w_h$ for $h \neq h_0$. Thus, (4) is differentiable with respect to h at h_0, and its derivative at h_0 is given by (15).

If u_{h_0} is a solution of (1), (2) for $h = h_0$, and f is continuous, then (13) is continuous iff u_{h_0} is of class $\mathcal{C}^1$ in $[0,T]$. For some theorems on regularity of u_h with respect to t we refer the reader to [4] and [3]. Combining Theorem 2 with Proposition 1 we have

THEOREM 3. *If the assumptions of Theorem* 1 *are fulfilled*, $u_h^0 \in D$ *for* $h \in \Omega$, *the mappings* (12) *are differentiable*, $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0\} \subset P(-A_h(t))$, $\exists C > 0$ *such that*

$$\|(A_h(t) + \lambda I)^{-1}\| \leq C \frac{1}{|\lambda| + 1}\,,$$

and there exist $K > 0$ *and* $\delta \in (0,1]$ *such that*

$$\|f_h(t) - f_h(\tau)\| \leq K |t - \tau|^{\delta}\,,$$

then

1) *there exists a solution* u_h *of the problem* (1), (2),
2) *the solution* u_h *is given by* (3),
3) *the mapping* (14) *is differentiable and its derivative is given by*

$$u_h'(t) = U_h(t,0)u_h^{0\,\prime} + \int\limits_0^t U_h(t,s)f_h^1(s)\,ds\,,$$

where

$$f_h^1(s) = f_h'(s) - A_h'(s)u_h(s) \quad \text{for } h \in \Omega\,. \tag{16}$$

4. Higher order regularity. In this section we assume that the assumptions of Theorem 1 are fulfilled, the mappings (12) are differentiable in Ω, there exists a solution u_h of the problem (1), (2) and that, for every $h \in \Omega$, it is of class $\mathcal{C}^1$ in $[0,T]$.

Let f_h^1 be defined by (16).

LEMMA 1. *If* v_h *is a solution of the problem*

$$\frac{dv}{dt}(t) + A_h(t)v(t) = f_h^1(t) \quad \text{for } t \in (0,T]\,,$$
$$v_h(0) = u_h^{0\,\prime}\,,$$

then

$$v_h(t) = U_h(t,0)u_h^{0\,\prime} + \int\limits_0^t U_h(t,s)f_h^1(s)\,ds \tag{17}$$

and therefore $v_h = u_h'$ *for* $h \in \Omega$. *Moreover*, *if the mapping*

$$\Omega \times [0,T] \ni (h,t) \to f_h^1(t) \in X \tag{18}$$

is continuous, *then the mapping* (14) *is of class* $\mathcal{C}^1$.

Proof. Since, for a given $h \in \Omega$, f_h' is continuous (because the convergence in $\mathcal{C}([0,T];X)$ is uniform) and

$$A_h'(t)u_h(t) = [A_h'(t)\mathcal{T}] \circ [(A_h(t)\mathcal{T})^{-1}] \circ [A_h(t)u_h(t)]$$

gives also the continuity of the mapping $t \to A_h'(t)u_h(t)$, f_h^1 is continuous in $[0,T]$. Therefore, by Theorem 1(i), we have (17). By Theorem 1(ii) and since the mapping (18) is continuous, the mapping (14) is of class $\mathcal{C}^1$.

LEMMA 2. *If*, *for* $h \in \Omega$, u_h *is a solution of the problem* (1), (2) *of class* $\mathcal{C}^1$ *in* $[0,T]$ *and* $\mathcal{C}^2$ *in* $(0,T]$, f_h *and* A_h *are differentiable with respect to* t, *and the mappings*

$$\Omega \ni h \to f_h(0) - A_h(0)u_h^0 \in X\,,$$
$$\Omega \times [0,T] \ni (h,t) \to \frac{df_h(t)}{dt} - \frac{dA_h(t)}{dt}u_h(t) \in X$$

are continuous, then the mapping

$$\Omega \ni h \to \frac{du_h}{dt} \in \mathcal{C}([0,T];X)$$

is continuous.

P r o o f. Since u_h is of class $\mathcal{C}^1$ in $[0,T]$ and $\mathcal{C}^2$ in $(0,T]$, du_h/dt is a solution of the problem

$$\frac{d\omega_h}{dt}(t) + A_h(t)\omega_h(t) = \frac{df_h(t)}{dt} - \frac{dA_h(t)}{dt}u_h(t) \quad \text{for } t \in (0,T]\,,$$
$$\omega_h(0) = f_h(0) - A_h(0)u_h^0\,.$$

Now Theorem 1 completes the proof.

THEOREM 4. *If the assumptions of Lemmas* 1 *and* 2 *are fulfilled then the mapping* (4) *is of class* $\mathcal{C}^1$.

P r o o f. This is an immediate consequence of Lemmas 1 and 2.

The method presented here is the key to the inductive construction of theorems on the higher order regularity of the solution of the problem (1), (2) with respect to the parameter h.

References

[1] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 1981.
[2] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1966.
[3] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.
[4] P. E. Sobolevskiĭ, *On equations of parabolic type in a Banach space*, Trudy Moskov. Mat. Obshch. 10 (1986), 297–350 (in Russian).
[5] H. Tanabe, *Equations of Evolution*, Pitman, 1979.
[6] T. Winiarska, *Parabolic equations with coefficients depending on t and parameters*, Ann. Polon. Math. 51 (1990), 325–339.
[7] —, *Evolution equation with parameter*, Univ. Iag. Acta Math. 28 (1987), 219–227.
[8] —, *Differential Equations with Parameters*, Monograph 68, Technical University of Cracow, 1988.
[9] K. Yosida, *Functional Analysis*, Springer, 1980.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF CRACOW
WARSZAWSKA 24
31-155 KRAKÓW, POLAND

Reçu par la Rédaction le 20.6.1991