Regularity of solutions of parabolic equations with coefficients depending on \(t\) and parameters

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Abstract. The main object of this paper is to study the regularity with respect to the parameter \(h\) of solutions of the problem 
\[
\frac{du}{dt}(t) + A_h(t)u(t) = f_h(t), \quad u(0) = u_0^h.
\]
The continuity of \(u\) with respect to both \(h\) and \(t\) has been considered in [6].

1. Introduction. In this paper, we consider the family of parabolic problems
\[
\begin{align*}
(1) \quad \frac{du}{dt}(t) + A_h(t)u(t) &= f_h(t) \quad \text{for } t \in (0, T], \\
(2) \quad u_h(0) &= u_0^h,
\end{align*}
\]
with \(h \in \Omega\), where \(\Omega\) is an open subset of \(\mathbb{R}^n\). It is well known that, under certain assumptions, the solution of the problem (1), (2) is given by the formula
\[
(3) \quad u_h(t) = U_h(t, 0)u_0^h + \int_0^t U_h(t, s)f_h(s)\,ds,
\]
where \(U_h\) is the fundamental solution of equation (1), for a fixed \(h \in \Omega\). The problem of continuity of the mapping
\[
(4) \quad \Omega \times [0, T] \ni (h, t) \mapsto u_h(t) \in X,
\]
with \(u_h\) given by (3), was considered in [6].

The main object of this paper is to study the differentiability of (4) with respect to \(h\).

Similar problems are considered in [7], [8] but for differential equations with \(A_h\) independent of \(t\).

2. Preliminaries. Let \(X, Y\) be Banach spaces and let \(\Omega\) be an open subset of \(\mathbb{R}^n\). To simplify notation we shall assume that \(\Omega\) is an open interval in \(\mathbb{R}\).

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We denote by $B(X,Y)$ the space of bounded linear operators from $X$ to $Y$. The space $B(X,X)$ is denoted by $B(X)$. The space of closed linear operators from $X$ to $X$ will be denoted by $C(X)$. For $A \in C(X)$ the resolvent set of $A$ will be denoted by $\rho(A)$.

Let $D$ be a normed vector space such that there exist a Banach space $Z$ and a bijective bounded operator $T : Z \to D$. Similarly to [8], we shall consider the space
\begin{equation}
SB(D,Y) := \{ A : D \to Y \mid A \text{ is linear and } A T \in B(Z,Y) \}.
\end{equation}

The definition of $SB(D,Y)$ is independent of the choice of $(Z,T)$. The space $\mathcal{M}_T := \{ A : [0,T] \to SB(D,Y) \mid \text{the mapping } [0,T] \ni t \to A(t)T \in B(Z,Y) \text{ is continuous} \}$ is a Banach space with the norm $\|A\|_T := \sup \{\|A(t)T\| \mid t \in [0,T] \}$.

If $(Z', T')$ is another pair as needed in (5) then $\mathcal{M}_T = \mathcal{M}_{T'}$ with equivalent norms. Thus, instead of $\mathcal{M}_T$ we may write $\mathcal{M}$ or $M(D,Y)$.

Accordingly, a mapping $\Omega \ni h \to A_h \in \mathcal{M}$ is differentiable (continuous) at $h_0 \in \Omega$ if $\Omega \ni h \to A_h \in \mathcal{M}_T$ is differentiable (continuous) at $h_0$. We have
\begin{equation}
A_{h_0}'(t) = \left. \left( \frac{d}{dh} (A_h(t)T) \right) \right|_{h=h_0} T^{-1} \quad \text{for } t \in [0,T].
\end{equation}

The operator $A_{h_0}'$ is independent of $(Z,T)$. Higher order differentiability and the $C^k$-classes are now defined in the standard way.

The Banach space of all continuous mappings from $[0,T]$ into $Y$, with the topology of uniform convergence, is denoted by $C([0,T];Y)$.

We shall consider a family $(A_h(t))_{(h,t) \in \Omega \times [0,T]}$ of closed linear operators from $X$ to $X$ defined, for each $(h,t) \in \Omega \times [0,T]$, on a dense linear subspace $D(A_h(t)) = D$ of $X$.

**Assumption $Z_1$.** There exist a Banach space $Z$ and a bijective mapping $T : Z \to D$ such that $T \in B(Z,X)$ and the mapping $\Omega \times [0,T] \ni (h,t) \to A_h(t)T \in B(Z,X)$ is continuous.

If Assumption $Z_1$ is fulfilled then $A_h \in \mathcal{M}_T$ for all $h \in \Omega$ and the mapping $\Omega \ni h \to A_h \in \mathcal{M}_T$ is continuous, and vice versa.

**Assumption $Z_2$.** There exist a Banach space $Z$, a continuous linear bijective mapping $T : Z \to D$ and $\alpha \in (0,1]$ such that the mapping $[0,T] \ni t \to A_h(t)T \in B(Z,X)$
is Hölder continuous with exponent $\alpha$, i.e. there exists $\tilde{L} > 0$ such that
\[ \|A_h(t)T - A_h(s)T\| \leq \tilde{L}|t - s|^\alpha \]
for $h \in \Omega, 0 \leq s \leq T$ and $0 \leq t \leq T$.

**Assumption Z3.** $A_h(t) \in G(C_0)$ for $(h, t) \in \Omega \times [0, T]$, where
\[ G(C_0) = \{ A \in C(X) \mid D(A) = X, [0, \infty) \subset P(-A), \|(A + \xi)^{-k}\| \leq C_0 \xi^{-k} \text{ for } \xi > 0, k = 1, 2, \ldots \text{ and } \|A \exp(-tA)\| \leq C_0 t^{-1} \text{ for } t > 0 \} . \]

Let $\Delta = \{(t, s) \mid 0 \leq s \leq t \leq T\}$.

**Definition 1.** A mapping
\[ (6) \quad U_h : \Delta \ni (t, s) \to U_h(t, s) \in B(X) \]
is said to be a fundamental solution of (1) if
1) for every $x \in X$ the mapping $\Delta \ni (t, s) \to U_h(t, s)x \in X$ is continuous,
2) $U_h(t, r)U_h(r, s) = U_h(t, s)$ for $0 \leq s \leq r \leq t \leq T$,
3) $U_h(s, s) = I$ for $s \in [0, T]$,
4) for every $x \in X$ the mapping (6) is differentiable with respect to $t$ and
\[ \frac{\partial}{\partial t} U_h(t, s)x = A_h(t)U_h(t, s)x , \]
5) for every $x \in D$ the mapping (6) is differentiable with respect to $s$ and
\[ \frac{\partial}{\partial s} U_h(t, s)x = -U_h(t, s)A_h(s)x . \]

Under Assumptions $Z_1$–$Z_3$ we may define (for details see e.g. [5], Chap. 5, and [6])
\[ R^h_1(t, s) := -(A_h(t) - A_h(s)) \exp(-(t - s)A_h(s)) , \]
\[ R^h_m(t, s) := \int_s^t R^h_1(t, \tau) R^h_{m-1}(\tau, s) \, d\tau \quad \text{for } m = 2, 3, \ldots , \]
\[ R^h(t, s) := \sum_{m=1}^\infty R^h_m(t, s) , \]
\[ W^h(t, s) := \int_s^t \exp(-(t - \tau)A_h(\tau)) R^h(\tau, s) \, d\tau , \]
(7) $U_h(t, s) := \exp(-(t - s)A_h(s)) + W^h(t, s)$,
where $\exp(-(tA_h(s)))$ is the strongly continuous semigroup with the infinitesimal generator $A_h(s)$ for $h \in \Omega, s \in [0, T]$.

Since sufficient conditions for $U_h$ given by (7) to be a fundamental solution of (1) are known (see e.g. [5]), we do not discuss them here.
Assumption Z₄. $U_h$ given by (7) is a fundamental solution of (1) for $h \in \Omega$.

We shall use the following two theorems:

**Theorem 1.** Suppose Assumptions Z₁–Z₄ are fulfilled.

(i) ([5], Th. 5.2.2) If, for any $h \in \Omega$, there exists a solution $u_h$ of the problem (1), (2) and the mapping $[0, T] \ni t \mapsto f_h(t) \in X$ is continuous, then $u_h$ is given by (3).

(ii) ([6], Th. 1) If the mappings

\begin{align*}
\Omega \ni h &\mapsto u_0^h \in X, \\
\Omega \times [0, T] \ni (h, t) &\mapsto f_h(t)
\end{align*}

are continuous, then the mapping (4), with $u_h$ given by (3), is continuous.

**Theorem 2** ([4], Th. 4 and Th. 5, p. 301). Let $k$ be a nonnegative integer.

(a) the mapping $\Omega \ni h \mapsto A_h \in \mathcal{M}$, is $k$ times differentiable and its $k$-th derivative is Hölder continuous,

(b) the mapping $[0, T] \ni t \mapsto f_h(t) \in X$ is $k$ times differentiable and its $k$-th derivative is Hölder continuous,

(c) $\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \subset P(-A_h(t))$ and $\exists C > 0$ such that

$$||(A_h(t) + \lambda I)^{-1}|| \leq C \frac{1}{|\lambda| + 1},$$

then

1) there exists a solution $u_h$ of the problem (1), (2),

2) $u_h$ is given by (3),

3) $u_h$ is of class $C^1$ in $[0, T]$ and $C^{k+1}$ in $(0, T]$.

3. Differentiability with respect to $h$. For $h \in \Omega$, let $u_h$ be a solution of (1), (2) and let $h_0 \in \Omega$. The function $w_h$ defined by

$$w_h(t) = \frac{u_h(t) - u_{h_0}(t)}{h - h_0} \quad \text{for } h \neq h_0$$

is, for $h \neq h_0$, a solution of the problem

\begin{align*}
\frac{dw_h(t)}{dt}(t) + A_h(t)w_h(t) &= F_h(t) \quad \text{for } t \in (0, T], \\
w_h(0) &= w_0^h,
\end{align*}

where

$$F_h(t) = \begin{cases} 
\frac{f_h(t) - f_{h_0}(t)}{h - h_0} - \frac{A_h(t) - A_{h_0}(t)}{h - h_0} u_{h_0}(t) & \text{for } h \neq h_0, \\
\frac{f_{h_0}(t)'}{h - h_0} - A_{h_0}(t) u_{h_0}(t) & \text{for } h = h_0,
\end{cases}$$
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\[ w_h^0 = \begin{cases} 
  u_0^0 - u_{h_0}^0 & \text{for } h \neq h_0, \\
  \frac{u_0'}{u_{h_0}'} & \text{for } h = h_0,
\end{cases} \]

and \( \omega^n \) denotes differentiation with respect to \( h \).

**Proposition 1.** Under the assumptions of Theorem 1, if the mappings
\[(12) \quad \Omega \ni h \mapsto f_h \in C([0,T];X), \quad \Omega \ni h \mapsto A_h \in \mathcal{M}, \quad \Omega \ni h \mapsto u_h^0 \in X \]
are differentiable at \( h_0 \) and the mapping
\[(13) \quad [0,T] \ni t \mapsto A_{h_0}(t)u_{h_0}(t) \in X \]
is continuous, then the mapping
\[(14) \quad \Omega \ni h \mapsto u_h \in C([0,T];X) \]
is differentiable at \( h_0 \) and its derivative at \( h_0 \) is given by
\[(15) \quad u_h'(t) = U_{h_0}(t,0)w_h^0 + \int_0^t U_{h_0}(t,s)F_h(s) \, ds. \]

**Proof.** Since
\[
\frac{A_h(t) - A_{h_0}(t)}{h - h_0} u_{h_0}(t) = \frac{A_h(t) - A_{h_0}(t)}{h - h_0} T(A_{h_0}(t)T)^{-1} A_{h_0}(t)u_{h_0}(t)
\]
and the convergence in \( \mathcal{M} \) is uniform with respect to \( t \), the mapping \( \Omega \times [0,T] \ni (h,t) \mapsto F_h(t) \) is continuous. By Theorem 1 the mapping
\[
\Omega \times [0,T] \ni (h,t) \mapsto \tilde{w}_h(t) := U_{h_0}(t,0)w_h^0 + \int_0^t U_{h_0}(t,s)F_h(s) \, ds
\]
is continuous and \( \tilde{w}_h = w_h \) for \( h \neq h_0 \). Thus, (4) is differentiable with respect to \( h \) at \( h_0 \), and its derivative at \( h_0 \) is given by (15).

If \( u_{h_0} \) is a solution of (1), (2) for \( h = h_0 \), and \( f \) is continuous, then (13) is continuous iff \( u_{h_0} \) is of class \( C^1 \) in \([0,T]\). For some theorems on regularity of \( u_h \) with respect to \( t \) we refer the reader to [4] and [3]. Combining Theorem 2 with Proposition 1 we have

**Theorem 3.** If the assumptions of Theorem 1 are fulfilled, \( u_h^0 \in D \) for \( h \in \Omega \), the mappings (12) are differentiable, \( \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \} \subset P(-A_h(t)), \exists C > 0 \) such that
\[
\|(A_h(t) + \lambda I)^{-1}\| \leq C \frac{1}{|\lambda| + 1},
\]
and there exist \( K > 0 \) and \( \delta \in (0,1] \) such that
\[
\|f_h(t) - f_h(\tau)\| \leq K|t - \tau|^\delta,
\]
then
1) there exists a solution $u_h$ of the problem (1), (2),
2) the solution $u_h$ is given by (3),
3) the mapping (14) is differentiable and its derivative is given by
   
   $u_h'(t) = U_h(t, 0)u_0^h + \int_0^t U_h(t, s)f^1_h(s)\,ds$
   
   where
   
   $f^1_h(s) = f'_h(s) - A'_h(s)u_h(s)$ for $h \in \Omega$.

4. Higher order regularity. In this section we assume that the
   assumptions of Theorem 1 are fulfilled, the mappings (12) are differentiable
   in $\Omega$, there exists a solution $u_h$ of the problem (1), (2) and that, for every
   $h \in \Omega$, it is of class $C^1$ in $[0, T]$.

   Let $f^1_h$ be defined by (16).

   **Lemma 1.** If $v_h$ is a solution of the problem
   
   \[
   \frac{dv}{dt}(t) + A_h(t)v(t) = f^1_h(t) \quad \text{for } t \in (0, T],
   \]
   
   $v_h(0) = u_0^h$,

   then
   
   \[
   v_h(t) = U_h(t, 0)u_0^h + \int_0^t U_h(t, s)f^1_h(s)\,ds
   \]
   
   and therefore $v_h = u'_h$ for $h \in \Omega$. Moreover, if the mapping
   
   \[
   \Omega \times [0, T] \ni (h, t) \rightarrow f^1_h(t) \in X
   \]

   is continuous, then the mapping (14) is of class $C^1$.

   **Proof.** Since, for a given $h \in \Omega$, $f'_h$ is continuous (because the convergence in $C([0, T]; X)$ is uniform) and

   \[
   A'_h(t)u_h(t) = [A'_h(t)T]\circ[(A_h(t)T)^{-1}]\circ[A_h(t)u_h(t)]
   \]

   gives also the continuity of the mapping $t \rightarrow A'_h(t)u_h(t)$, $f^1_h$ is continuous in $[0, T]$. Therefore, by Theorem 1(i), we have (17). By Theorem 1(ii) and since the mapping (18) is continuous, the mapping (14) is of class $C^1$.

   **Lemma 2.** If, for $h \in \Omega$, $u_h$ is a solution of the problem (1), (2) of class $C^1$ in $[0, T]$ and $C^2$ in $(0, T]$, $f_h$ and $A_h$ are differentiable with respect to $t$, and the mappings

   \[
   \Omega \ni h \rightarrow f_h(0) - A_h(0)u_0^h \in X,
   \]

   \[
   \Omega \times [0, T] \ni (h, t) \rightarrow \frac{df_h(t)}{dt} - \frac{dA_h(t)}{dt}u_h(t) \in X
   \]
are continuous, then the mapping
\[ \Omega \ni h \mapsto \frac{du_h}{dt} \in C([0,T]; X) \]
is continuous.

Proof. Since \( u_h \) is of class \( C^1 \) in \([0,T]\) and \( C^2 \) in \((0,T)\), \( du_h/dt \) is a solution of the problem
\[
\frac{d\omega_h}{dt}(t) + A_h(t)\omega_h(t) = \frac{df_h(t)}{dt} - \frac{dA_h(t)}{dt} u_h(t) \quad \text{for } t \in (0,T),
\]
\[ \omega_h(0) = f_h(0) - A_h(0)u_0. \]
Now Theorem 1 completes the proof.

Theorem 4. If the assumptions of Lemmas 1 and 2 are fulfilled then the mapping (4) is of class \( C^1 \).

Proof. This is an immediate consequence of Lemmas 1 and 2.

The method presented here is the key to the inductive construction of theorems on the higher order regularity of the solution of the problem (1), (2) with respect to the parameter \( h \).

References