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A simple formula showing L^1 is a maximal overspace for two-dimensional real spaces

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Abstract. It follows easily from a result of Lindenstrauss that, for any real two-dimensional subspace v of L^1, the relative projection constant $\lambda(v; L^1)$ of v equals its (absolute) projection constant $\lambda(v) = \sup_X \lambda(v; X)$. The purpose of this paper is to recapture this result by exhibiting a simple formula for a subspace V contained in $L^\infty(\nu)$ and isometric to v and a projection P_∞ from $C \oplus V$ onto V such that $\|P_\infty\| = \|P_1\|$, where P_1 is a minimal projection from $L^1(\nu)$ onto v. Specifically, if $P_1 = \sum_{i=1}^2 U_i \otimes v_i$, then $P_\infty = \sum_{i=1}^2 u_i \otimes V_i$, where $dV_i = 2v_i\, d\nu$ and $dU_i = -2u_i\, d\nu$.

1. Introduction and preliminaries

Notation. For any two Banach spaces E and X, with $E \subset X$, set $\lambda(E; X) = \inf_P \|P\|$, where P runs through all projections of X onto E. The number $\lambda(E; X)$ is called the *relative projection constant of E with respect to X*. The number $\lambda(E) = \sup_X \lambda(E; X)$ is called the (*absolute*) *projection constant of E*. Any X for which $\lambda(E; X) = \lambda(E)$ is called a *maximal overspace* for E.

It follows easily from a result of Lindenstrauss ([5], Theorem 3) that, for any two-dimensional real subspace v of L^1, the relative projection constant $\lambda(v; L^1)$ of v equals its (absolute) projection constant $\lambda(v)$; that is, L^1 is a maximal overspace for v. The purpose of this paper is to recapture this result by exhibiting a simple formula for a subspace V contained in $L^\infty(\nu)$ and isometric to v and a projection P_∞ from $C \oplus V$ onto V such that $\|P_\infty\| = \|P_1\|$, where P_1 is a minimal projection from $L^1(\nu)$ onto v, via the following procedure. First we note the simple fact that v is also (isometric to) a subspace of $L^1(\overline{\mathbb{R}}, \nu)$, where ν is some finite measure and $\lambda(v, L^1) \ge \lambda(v, L^1(\overline{\mathbb{R}}, \nu))$. Secondly it is shown that $\lambda(v) = \lambda(v, L^1(\overline{\mathbb{R}}, \nu))$. Specifically, by use of recent work [2] on minimal L^1 projections, we show that there exists a minimal

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projection $P_1 = \sum_{i=1}^{2} U_i \otimes v_i$ from $L^1(\overline{\mathbb{R}}, \nu)$ onto $v = [v_1, v_2]$ and a projection $P_\infty = \sum_{i=1}^{2} u_i \otimes V_i$ from $C \oplus V \subset L^\infty(\overline{\mathbb{R}}, \nu)$ onto $V = [V_1, V_2]$, such that V is isometric to v and $\|P_\infty\| = \|P_1\|$, where $dV_i = 2v_i d\nu$ and $dU_i = -2u_i d\nu$. This procedure was first demonstrated in [1], where $v = [1, t]$ is the two-dimensional subspace of lines in $L^1[-1, 1]$. In this case P_1 was already known [4] and $P_\infty : C[-1, 1] \to V = [t, 1 - t^2]$ was determined in [1].

It is known that any two-dimensional real Banach space is isometric to a subspace of the Lebesgue space $L^1[-1, 1]$ (see, e.g., [5] and [6]). More generally, let $v = [v_1, v_2]$ denote any two-dimensional real subspace of $L^1(S, \Sigma, \mu)$, where it will be assumed that (S, Σ, μ) is an arbitrary complete measure space. In the following we will construct $V = [V_1, V_2] \subset L^\infty$ such that v is isometric to V and such that $\lambda(V; C \oplus V) = \lambda(v; L^1)$. From the well-known facts that $\lambda(V; X) = \lambda(V)$ for $C \subset X \subset L^\infty$ and $\lambda(v) = \lambda(V)$, we can then conclude that L^1 is a maximal overspace for v; i.e., $\lambda(v; L^1) = \lambda(v)$.

LEMMA 1. *Any two-dimensional subspace v of $L^1(S, \Sigma, \mu)$ is isometric to a subspace $\widehat{v}$ of lines in $L^1(\overline{\mathbb{R}}, \nu)$ for some finite measure ν on $\overline{\mathbb{R}}$, and $\lambda(v; L^1(S, \Sigma, \mu)) \ge \lambda(\widehat{v}; L^1(\overline{\mathbb{R}}, \nu))$.*

Proof. Let $Z = \{s \in S : v_1(s) = 0\}$. Define $(\widehat{1}, \widehat{t}\,) = (1, t)$, $t \in \mathbb{R}$, and $(\widehat{1}, \widehat{t}\,)(\pm\infty) = (0, \pm 1)$. Then, for arbitrary (a_1, a_2),

$$\begin{aligned}
&\|(a_1, a_2) \cdot (v_1, v_2)\|_{L^1(S, \Sigma, \mu)} \\
&\quad = \int_S |a_1 v_1(s) + a_2 v_2(s)| \, d\mu(s) \\
&\quad = \int_{S-Z} \left| a_1 + a_2 \frac{v_2(s)}{v_1(s)} \right| |v_1(s)| \, d\mu(s) + |a_2| \int_Z |v_2(s)| \, d\mu(s) \\
&\quad = \int_{\mathbb{R}} |a_1 + a_2 t| \, d\nu(t) + |a_2| \nu\{-\infty\} = \|(a_1, a_2) \cdot (\widehat{1}, \widehat{t}\,)\|_{L^1(\overline{\mathbb{R}}, \nu)},
\end{aligned}$$

where

$$\nu\{t\} = \begin{cases} \displaystyle\int_{\{s \in S - Z : v_2(s)/v_1(s) = t\}} |v_1(s)| \, d\mu(s) & \text{for } |t| < \infty, \\ \displaystyle\int_Z |v_2(s)| \, d\mu(s) & \text{for } t = -\infty, \\ 0 & \text{for } t = \infty. \end{cases}$$

This demonstrates the desired isometry.

Finally, any projection from $L^1(\overline{\mathbb{R}}, \nu)$ onto $\widehat{v}$ clearly induces a projection onto v, of the same norm, from the subspace of $L^1(S, \Sigma, \mu)$ consisting of functions constant on the level sets of $(v_2/v_1)(s)$. Hence

$$\lambda(v; L^1(S, \Sigma, \mu)) \ge \lambda(\widehat{v}; L^1(\overline{\mathbb{R}}, \nu)). \ \blacksquare$$

Thus, assume without loss in the sequel that $v = [\vec{v}\,] = [\widehat{1}, \widehat{t}\,] \subset L^1(\overline{\mathbb{R}}, \nu)$, and denote the norm of an element x in $L^1(\overline{\mathbb{R}}, \nu)$ by $\|x\|_1$. Also, denote the sphere in $\mathbb{R}^2$ given by $\|\vec{a}\| = \|\vec{a} \cdot \vec{v}\|_1 = \varrho$, for $\vec{a} \in \mathbb{R}^2$, by $\mathcal{S}(\varrho)$. The following result is from the theory of finite-rank minimal projections in L^1, as applied to the present two-dimensional situation ([2], Theorems 2 and 3 and Lemma 4; see especially the geometric interpretation in §3).

Theorem A ([2]). *There exists a minimal projection $P_1 = \sum_{i=1}^{2} U_i \otimes v_i$ from $L^1(\overline{\mathbb{R}}, \nu)$ onto v such that, for some non-singular matrix M and all $r \in \overline{\mathbb{R}}$, $\vec{U}(r)$ is a point on $\mathcal{S}(\|P_1\|)$ such that a tangent line at $\vec{U}(r)$ is perpendicular (in the Euclidean sense) to the direction given by $M\vec{v}(r)$.*

Since M is non-singular and $M\vec{v}(r)$, $r \in \overline{\mathbb{R}}$, defines a line not passing through the origin, it follows that the directions given by $M\vec{v}(r)$ describe monotonically an arc subtending π radians (as r moves from $-\infty$ to ∞). As a consequence of this and the perpendicularity of a tangent line at $\vec{U}(r)$ with the direction of $M\vec{v}(r)$, note, for future reference, that we may assume $\vec{U}(\infty) = -\vec{U}(-\infty)$. Also, note that $\tan^{-1}[U_2(r)/U_1(r)]$ is monotonic in $r \in \overline{\mathbb{R}}$ (assume without loss, increasing).

Let

$$\vec{V}_0(r) = \int_{\overline{\mathbb{R}}} \vec{v}(s) \operatorname{sgn}^+[\vec{U}(r) \cdot \vec{v}(s)]\, d\nu(s), \qquad r \in \overline{\mathbb{R}}\,, \tag{1}$$

where

$$\operatorname{sgn}^+(t) = \begin{cases} -1, & t < 0\,, \\ +1, & 0 \le t\,. \end{cases}$$

Hence the "Lebesgue function"

$$L_{P_1}(r) = \vec{U}(r) \cdot \vec{V}_0(r) = \int_{\overline{\mathbb{R}}} |\vec{U}(r) \cdot \vec{v}(s)|\, d\nu(s) = \|\vec{U}(r) \cdot \vec{v}\|_1 = \|P_1\|$$

for all $r \in \overline{\mathbb{R}}$.

We now seek to define a function $r(t)$ so that $\vec{U}(r(t)) \cdot \vec{v}(t) = 0$. Because of possible "flat portions" on the sphere $\mathcal{S}(\|P_1\|)$, this cannot always be accomplished exactly when $\vec{v}(t)$ is perpendicular to a direction on a "flat portion", and hence we need the following more technical definition for $r(t)$. Let

$$a(r,t) = \frac{|\vec{U}(r) \cdot \vec{v}(t)|}{|\vec{U}(r)|\, |\vec{v}(t)|}\,;$$

i.e., $a(r,t)$ is the absolute value of the cosine of the angle between $\vec{U}(r)$ and $\vec{v}(t)$. Then let

$$a_0(t) = \inf_{r \in \overline{\mathbb{R}}} a(r,t)$$

and define

$$r(t)=\liminf_{a(r,t)\to a_0(t)} r\,. \tag{2}$$

Next define

$$\vec{V}(t)=\varepsilon(t)\vec{V}_0(r(t)^\sigma),\qquad t\in\overline{\mathbb{R}}\,, \tag{3}$$

where $\varepsilon(t)=\operatorname{sgn}^+[\vec{U}(\infty)\cdot\vec{v}(t)]$, and $\sigma=\sigma(t)=\pm$ is chosen so that $a(r(t)^\sigma,t)=a_0(t)$ if $0<a_0(t)$, and $\sigma=\sigma(t)$ is suppressed if $a_0(t)=0$. Note that $a_0(t)>0$ implies that $\vec{U}(r(t))$ lies on a "flat portion" of $\mathcal{S}(\|P_1\|)$.

LEMMA 2. *The two-dimensional space* $v=[\widehat{1},\widehat{t}]\subset L^1(\overline{\mathbb{R}},\nu)$ *is isometric to* $V=[V_1(t),V_2(t)]\subset L^\infty(\overline{\mathbb{R}},\nu)$, *where* $\vec{V}(t)=(V_1(t),V_2(t))$ *is given by* (3) *or*, *more simply*, *by* (4) *below.*

Proof. For arbitrary $\vec{a}\in\mathbb{R}^2$,

$$\begin{aligned}\|\vec{a}\cdot\vec{v}\|_1&=\int_{\overline{\mathbb{R}}}|\vec{a}\cdot\vec{v}(s)|\,d\nu(s)=\int_{\overline{\mathbb{R}}}\vec{a}\cdot\vec{v}(s)\operatorname{sgn}^+[\vec{a}\cdot\vec{v}(s)]\,d\nu(s)\\&=\sup_{\vec{b}\in\mathbb{R}^2}\Big|\int_{\overline{\mathbb{R}}}\vec{a}\cdot\vec{v}(s)\operatorname{sgn}^+[\vec{b}\cdot\vec{v}(s)]\,d\nu(s)\Big|\\&=\sup_{t\in\overline{\mathbb{R}}}\Big|\vec{a}\cdot\int_{\overline{\mathbb{R}}}\vec{v}(s)\operatorname{sgn}^+[\vec{U}(r(t))\cdot\vec{v}(s)]\,d\nu(s)\Big|\\&=\sup_{t\in\overline{\mathbb{R}}}\Big|\vec{a}\cdot\varepsilon(t)\int_{\overline{\mathbb{R}}}\vec{v}(s)\operatorname{sgn}^+[\vec{U}(r(t)^{\sigma(t)})\cdot\vec{v}(s)]\,d\nu(s)\Big|\,,\end{aligned}$$

since, as is easy to see by the construction of $\vec{U}(r(t))$, the points $\vec{U}(r(t))$, $t\in\overline{\mathbb{R}}$, cover all "non-flat portions" of the sphere $\mathcal{S}(\|P_1\|)$ lying in a half-space. Thus, we conclude that

$$\|\vec{a}\cdot\vec{v}\|_1=\sup_{t\in\overline{\mathbb{R}}}|\vec{a}\cdot\vec{V}(t)|=\|\vec{a}\cdot\vec{V}\|_\infty\,.$$

Finally, it is immediate that $V\subset L^\infty(\overline{\mathbb{R}},\nu)$, since $v\subset L^1(\overline{\mathbb{R}},\nu)$. ∎

LEMMA 3. $2\vec{v}$ *is the Radon–Nikodym derivative with respect to* ν *of the signed measure with cumulative distribution function given by* $\vec{V}(t)-\vec{V}(-\infty)$.

Proof. From (3) note that

$$\vec{V}(t)=\Big(\int_{[-\infty,t]}-\int_{(t,\infty]}\Big)\vec{v}\,d\nu=\int_{-\infty}^{t}2\vec{v}\,d\nu-\int_{-\infty}^{\infty}\vec{v}\,d\nu. \tag{4}$$ ∎

Note 1. It follows from (4) that $\vec{V}(\infty)=-\vec{V}(-\infty)$.

N o t e 2. If ν is absolutely continuous with respect to Lebesgue measure, then $V \subset C(\overline{\mathbb{R}})$.

2. Main result

THEOREM 1. *Let $P_1 = \sum_{i=1}^{2} U_i \otimes v_i$ be the projection from $L^1(\nu)$ onto v given in Theorem* A, *and let V be the two-dimensional subspace of $L^\infty(\nu)$ isometric to v and given in Lemma* 2. *Let $P_\infty = \sum_{i=1}^{2} u_i \otimes V_i$ be the operator from $C(\overline{\mathbb{R}}) \oplus V$ onto V given by*

$$\vec{u} = -\frac{1}{2}\frac{d\vec{U}}{d\nu}, \tag{5}$$

where $d\vec{U}/d\nu$ denotes the Radon–Nikodym derivative of the signed measure on $\overline{\mathbb{R}}$ with cumulative distribution function $\vec{U}(t) - \vec{U}(-\infty)$. Then P_∞ is a projection from $C(\overline{\mathbb{R}}) \oplus V$ onto V and $\|P_\infty\| = \|P_1\|$.

P r o o f. For simplicity of notation, suppress $\sigma = \sigma(t)$ throughout the following argument. Also in the following we will use the notation $\langle x(r), y(r)\rangle_r$ to stand for $\int_{\overline{\mathbb{R}}} x(r)y(r)\, d\nu(r)$ and to emphasize that r is the integration variable.

By use of the Lebesgue function $L_{P_\infty}(t)$ for P_∞ we have

$$L_{P_\infty}(t) = \langle \operatorname{sgn}[\vec{u}(r)\cdot\vec{V}(t)], \vec{u}(r)\rangle_r \cdot \vec{V}(t)\,.$$

First

$$\vec{u}(r)\cdot\vec{V}(t) = -\frac{1}{2}\frac{d\vec{U}(r)}{d\nu(r)}\cdot\vec{V}(t) = -\frac{1}{2}\frac{d\vec{U}(r)}{d\nu(r)}\cdot\varepsilon(t)\vec{V}_0(r(t))\,.$$

Next the Lebesgue function $L_{P_1}(r) = \vec{U}(r)\cdot\vec{V}_0(r)$ is constant for all $r \in \overline{\mathbb{R}}$, which implies that

$$d\vec{U}(r)\cdot\vec{V}_0(r) + \vec{U}(r)\cdot d\vec{V}_0(r) = 0\,.$$

From this it follows straightforwardly from the definition (2) of $r(t)$ and Lemma 3 ($\varepsilon(t)d\vec{V}_0(r(t)) = 2\vec{v}(t)\,d\nu(t)$) that

$$\operatorname{sgn}[\vec{u}(r)\cdot\vec{V}(t)] = \varepsilon(t)\operatorname{sgn}[\vec{U}(r)\cdot\vec{v}(t)]\,.$$

Hence,

$$\begin{aligned}\langle \operatorname{sgn}[\vec{u}(r)\cdot\vec{V}(t)], \vec{u}(r)\rangle_r &= \Big(-\int_{-\infty}^{r(t)} + \int_{r(t)}^{\infty}\Big)(-\tfrac{1}{2}d\vec{U}(r)) \\ &= \vec{U}(r(t)) - \tfrac{1}{2}[\vec{U}(-\infty) + \vec{U}(\infty)] = \vec{U}(r(t))\end{aligned}$$

for each fixed t. We conclude that

$$L_{P_\infty}(t) = \vec{U}(r(t))\cdot\vec{V}(t) = \vec{U}(r(t))\cdot\vec{V}_0(r(t)) = \|P_1\|\,.$$

Secondly, we show that $\langle V_i, u_j\rangle = \langle v_i, U_j\rangle$ as follows:

$$\langle V_i(t), u_j(t)\rangle_t = \left\langle V_i(t), -\frac{1}{2}\frac{dU_j(t)}{d\nu}\right\rangle_t = -\frac{1}{2}(V_iU_j)(t)\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} U_j(t)\,dV_i(t)\,.$$

But $\vec{V}(\infty)+\vec{V}(-\infty) = \vec{U}(\infty)+\vec{U}(-\infty) = \vec{0}$ and $dV_i(t) = 2v_i(t)\,d\nu(t)$, and we have the desired conclusion. ■

Note 3. Theorem A extends to any non-singular action (see [2]) on v (not necessarily the identity action, as in the case of projections). The proof of Theorem 1 above shows that L^1 is a maximal overspace for operators onto v with any specified non-singular action on v.

Note 4. L^1 is not, in general, a maximal overspace for three-dimensional subspaces as the following example shows. Consider $v = \ell_1^3$ (imbedded isometrically) in $L^1[0,1]$ as step functions with steps at $\{\frac{1}{3}, \frac{2}{3}\}$. Then $\lambda(v, L^1) = 1$ as is easily seen, while v is isometric to $V = [r_1, r_2, r_3]$, the span of the first three Rademacher functions in $L^\infty[0,1]$. But it is known that $\lambda(V) = \lambda(V, L^\infty) > 1$.

We will now obtain the result of [1] as an example of the results of this paper.

Example. Consider $L^1[-1,1]$ and $v = [\vec{v}(t)]$, $\vec{v}(t) = (1,t)$. Then ν is Lebesgue measure with support $[-1,1]$. From [2] and [4]

$$\vec{U}(r) = \frac{\|P_1\|}{2}\left[\frac{(1,mr)}{\sqrt{1+m^2r^2}} + (0,\operatorname{sgn}(r))\right], \quad r \in [-1,1]\,, \tag{6}$$

$M = \operatorname{diag}(1,m)$, where $\|P_1\| = -2m/\log(t_0)$, $m = (1-t_0^2)/2t_0 = (t_0^2 - t_0 - 1)\log(t_0)$. So, following the procedure of this paper, we extend $\vec{U}(r)$ to all $r \in \overline{\mathbb{R}}$ such that $\vec{U}(r)$ is on $\mathcal{S}(\|P_1\|)$ and the tangent line at $\vec{U}(r)$ is perpendicular (in the Euclidean sense) to the direction given by $M\vec{v}(r)$, $r \in \overline{\mathbb{R}}$. Hence we have $\vec{U}(r)$ extended to have the form (6) for all $r \in \overline{\mathbb{R}}$.

Then by (4)

$$\vec{V}(t) = \begin{cases} \int_{-1}^{t} (1,s)\,ds - \int_{t}^{1} (1,s)\,ds = (2t, t^2-1)\,, & t \in [-1,1]\,,\\ (2\operatorname{sgn}(t), 0), & |t| > 1\,. \end{cases}$$

By Lemma 2, v is isometric to $V = [\vec{V}] \subset C(\overline{\mathbb{R}})$.

By the theorem if we set

$$\vec{u}(r) = -\frac{1}{2}\frac{d\vec{U}(r)}{dr} = \frac{\|P_1\|}{2}\left[\frac{m}{2}\frac{(mr,-1)}{(1+m^2r^2)^{3/2}} + (0,-\delta_0)\right], \quad r \in \overline{\mathbb{R}}\,,$$

then $P_\infty = \sum_{i=1}^{2} u_i(r) \otimes V_i(t)$ is a projection from $C(\overline{\mathbb{R}})$ onto V such that $\|P_\infty\| = \|P_1\|$.

Note that in this example, if we restrict P_∞ to $C_{[-1,1]}(\overline{\mathbb{R}}) = \{f \in C(\overline{\mathbb{R}}) : f(t) = f(\mathrm{sgn}(t)),\ |t| \geq 1\}$, then $C_{[-1,1]}(\overline{\mathbb{R}})$ is isometric to $C[-1,1]$. Furthermore, $\|P_\infty|_{C_{[-1,1]}(\overline{\mathbb{R}})}\| = \|P_\infty\|$, and $P_\infty|_{C_{[-1,1]}(\overline{\mathbb{R}})}$ is easily identified with the projection in [1].

Remark. Using the fact that L^1 is a maximal overspace for two-dimensional real spaces is a basic preliminary step in [3].

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References

[1] B. L. Chalmers, *Absolute projection constant of the linear functions in a Lebesgue space*, Constr. Approx. 4 (1988), 107–110.

[2] B. L. Chalmers and F. T. Metcalf, *The determination of minimal projections and extensions in* L^1, Trans. Amer. Math. Soc. 329 (1992), 289–305.

[3] B. L. Chalmers, F. T. Metcalf, B. Shekhtman and Y. Shekhtman, *The projection constant of a two-dimensional real Banach space is no greater than* 4/3, submitted.

[4] C. Franchetti and E. W. Cheney, *Minimal projections in* $\mathcal{L}_1$*-spaces*, Duke Math. J. 43 (1976), 501–510.

[5] J. Lindenstrauss, *On the extension of operators with a finite-dimensional range*, Illinois J. Math. 8 (1964), 488–499.

[6] D. Yost, L_1 *contains every two-dimensional normed space*, Ann. Polon. Math. 49 (1988), 17–19.

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