On topological invariants of vector bundles

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Abstract. Let $E \to W$ be an oriented vector bundle, and let $X(E)$ denote the Euler number of $E$. The paper shows how to calculate $X(E)$ in terms of equations which describe $E$ and $W$.

Introduction. Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$, $n - k > 0$, be a $C^1$-map such that $W = F^{-1}(0)$ is compact and $\text{rank}[DF(x)] \equiv k$ at every $x \in W$. From the implicit function theorem $W$ is an $(n - k)$-dimensional $C^1$-manifold.

Let $G_1, \ldots, G_s : \mathbb{R}^n \to \mathbb{R}^m$, where $m = s + n - k$, be a family of $C^1$-vector functions, and assume that the vectors $G_1(x), \ldots, G_s(x)$ are linearly independent for every $x \in W$. Define

$$E = \{(x, y) \in W \times \mathbb{R}^m \mid y \perp G_i(x), \; i = 1, \ldots, s\}$$

Clearly $E$ is an $(n - k)$-dimensional vector bundle over $W$. In particular, if $s = k$ and $G_i = \text{grad} F_i$ then $E$ becomes $TW$. Later we shall describe how to orient $W$ and $E$.

Let $X(E)$ be the Euler number of the bundle $E$ (see [1], Chapter 5.2). The problem is how to calculate $X(E)$ in terms of $F$ and $G_1, \ldots, G_s$.

Let $S_R = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^s \mid \|x\|^2 + \|\lambda\|^2 = R^2\}$, and let $H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k$ be the map given by

$$H(x, \lambda) = \left(\sum_{i=1}^s \lambda_i G_i(x), F(x)\right),$$

where $\lambda = (\lambda_1, \ldots, \lambda_s)$.

Take $R > 0$ such that $W \subset \{x \in \mathbb{R}^n \mid \|x\| < R\}$. It is easy to see that $H|_{S_R} : S_R \to \mathbb{R}^m \times \mathbb{R}^k - \{0\}$. Since $n + s = m + k$, the topological degree

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deg($H|S_R$) of the map $H|S_R$ is well defined. We shall prove (see Theorem 4) that

$$X(E) = (-1)^{n(s+k)+k} \deg(H|S_R).$$

As a corollary we get a formula (see Theorem 5) which expresses the Euler characteristic $\chi(W)$ in terms of $F$. A very similar formula has been proved in [2]. The advantage of the present work is that it is usually easy to find the appropriate value of $R$. The same is not necessarily true in [2].

2. Preliminaries. We assume that every space $\mathbb{R}^n$, $n > 0$, has the canonical orientation corresponding to its canonical ordered basis.

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a $C^1$-map as above. For each $x \in W$ there is a natural inclusion $T_x W \subset \mathbb{R}^n$. Vectors $\xi_{k+1}, \ldots, \xi_n \in T_x W$ are said to be positively oriented if $\nabla F_1(x), \ldots, \nabla F_k(x)$, $\xi_{k+1}, \ldots, \xi_n$ form a positively oriented basis in $\mathbb{R}^n$. From now on we assume $W$ to be equipped with this orientation.

Let $G_1, \ldots, G_s : \mathbb{R}^n \to \mathbb{R}^m$, $m = s + n - k$, and the vector bundle $E$ over $W$ be as in the introduction. If $E(x)$ is the fibre of $E$ over $x \in W$ then there is a natural inclusion $E(x) \subset \mathbb{R}^m$. Vectors $v_{s+1}, \ldots, v_m \in E(x)$ are said to be positively oriented if $G_1(x), \ldots, G_s(x), v_{s+1}, \ldots, v_m$ form a positively oriented basis in $\mathbb{R}^m$. So $E$ is an $(n-k)$-dimensional oriented vector bundle.

Let

$$E' = \{(x, y) \in W \times \mathbb{R}^m \mid y \in \text{span}(G_1(x), \ldots, G_s(x))\}.$$

Then $E'$ is a trivial vector bundle over $W$ such that $E \oplus E'$ is trivial.

Let $p : W \to E$ be a $C^1$-section of $E$ such that $p(\overline{x}) = 0$, for some $\overline{x} \in W$. There are $C^1$-sections $v_{s+1}, \ldots, v_m : U \to E$ defined in some open neighbourhood $U$ of $\overline{x}$ in $W$ such that $v_{s+1}(\overline{x}), \ldots, v_m(\overline{x})$ are linearly independent and positively oriented in $E(\overline{x})$. The sections $v_{s+1}, \ldots, v_m$ define a trivialization of $E$ over $U$, and thus there are unique $C^1$-functions $t_{s+1}, \ldots, t_m : U \to \mathbb{R}$ such that $p = \sum_{i=s+1}^m t_i v_i$ over $U$. Let $(x_{k+1}, \ldots, x_n)$ be a positively oriented coordinate system in some neighbourhood of $\overline{x}$ in $W$.

**Definition.** $\text{ind}(p, \overline{x}) = \text{sign} \frac{\partial(t_{s+1}, \ldots, t_m)}{\partial(x_{k+1}, \ldots, x_n)}(\overline{x})$.

One can prove that the definition of $\text{ind}(p, \overline{x})$ does not depend on the choice of $v_{s+1}, \ldots, v_m$ and $(x_{k+1}, \ldots, x_n)$. Note that if the section $p$ is transversal to the zero-section at $\overline{x}$ then $\text{ind}(p, \overline{x})$ is the index of $p$ at $x$ (see [1], Chapter 5.2).
Let \( P = (P^1, \ldots, P^m) : \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^1 \)-vector function. There are sections \( p : W \to E, \ p' : W \to E' \) such that \( P|W = p + p' \). Let \( \tilde{H} : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k \) be given by
\[
\tilde{H}(x, \lambda) = (P(x) + \sum_{d=1}^{s} \lambda_d G_d(x), F(x)).
\]

**Lemma 1.** A point \( \vec{x} \in \mathbb{R}^n \) is in \( p^{-1}(0) \subset W \) if and only if there is a unique \( \vec{\lambda} \in \mathbb{R}^s \) such that \( \tilde{H}(\vec{x}, \vec{\lambda}) = 0 \).

**Proof.** \((\Rightarrow) \) If \( p(\vec{x}) = 0 \) then \( P(\vec{x}) = p'(\vec{x}) \in E'(\vec{x}) \), where \( E'(\vec{x}) \) is the fibre of \( E' \) over \( \vec{x} \). The vectors \( G_1(\vec{x}), \ldots, G_s(\vec{x}) \) form a basis in \( E'(\vec{x}) \), and thus there is a unique \( \vec{\lambda} = (\vec{\lambda}_1, \ldots, \vec{\lambda}_s) \in \mathbb{R}^s \) such that
\[
P(\vec{x}) + \sum_{d=1}^{s} \vec{\lambda}_d G_d(\vec{x}) = 0.
\]
Since \( \vec{x} \in W = F^{-1}(0) \), we get \( \tilde{H}(\vec{x}, \vec{\lambda}) = 0 \).

\((\Leftarrow) \) Clearly \( \vec{x} \in W \), \( P(\vec{x}) = p(\vec{x}) + p'(\vec{x}) \in \text{span}(G_1(\vec{x}), \ldots, G_s(\vec{x})) \), and so \( p(\vec{x}) = 0 \). \( \blacksquare \)

From now on we assume that \( \vec{x} \in p^{-1}(0) \). Let \( \vec{\lambda} \in \mathbb{R}^s \) be as in Lemma 1. Since \( n + s = m + k \), the derivative matrix \( D\tilde{H}(\vec{x}, \vec{\lambda}) \) is a square matrix.

**Lemma 2.** \( \text{ind}(p, \vec{x}) = (-1)^{n(s+k)+k} \text{sign det}[D\tilde{H}(\vec{x}, \vec{\lambda})] \).

**Proof.** We can find a coordinate system \((x_1, \ldots, x_n)\) in \( \mathbb{R}^n \) such that
\[
(1) \quad \frac{\partial F_i}{\partial x_j}(\vec{x}) = 0,
\]
for every \( 1 \leq i \leq k, j \geq k + 1 \). Let
\[
A = \left[ \frac{\partial F_i}{\partial x_j}(\vec{x}) \right]_{1 \leq i, j \leq k}.
\]
From (1) and from the fact that \( \text{rank}[DF(\vec{x})] = k \) we deduce that \( \text{det}[A] \neq 0 \).

For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we write \( x = (x', x'') \), where \( x' = (x_1, \ldots, x_k) \in \mathbb{R}^k \), \( x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k} \). From the implicit function theorem there is a germ of a \( C^1 \)-function \( \psi = (\psi_1, \ldots, \psi_k) : (\mathbb{R}^{n-k}, \vec{x}'') \to (\mathbb{R}^k, \vec{x}') \) such that
\[
(2) \quad F_i(\psi(x''), x'') \equiv 0, \quad 1 \leq i \leq k.
\]
Since graph \( \psi = W \) in some neighbourhood of \( \vec{x} \), we can treat \((x_{k+1}, \ldots, x_n)\) as a coordinate system in some neighbourhood of \( \vec{x} \) in \( W \). From (1)
\[
(3) \quad \text{the coordinate system } (x_{k+1}, \ldots, x_n) \text{ is positively oriented if and only if } \text{det}[A] > 0.
\]
From (1), (2), for every \(1 \leq i \leq k, j \geq k + 1\) we have
\[
\frac{\partial}{\partial x_j}[F_i(\psi(x''), x''')](x'') = \frac{\partial F_i}{\partial x_1}(x') \frac{\partial \psi_i}{\partial x_j}(x''') + \ldots + \frac{\partial F_i}{\partial x_k}(x') \frac{\partial \psi_k}{\partial x_j}(x''') = 0.
\]
The matrix \(A\) is non-singular, and therefore
\[
\frac{\partial \psi_i}{\partial x_j}(x'') = 0, \quad \text{for } 1 \leq i \leq k, j \geq k + 1.
\]
There are \(C^1\)-vector maps \(V_{s+1}, \ldots, V_m : \mathbb{R}^n \to \mathbb{R}^m\) defined in some neighbourhood of \(x\) such that \(V_{s+1}(x), \ldots, V_m(x)\) form a positively oriented basis in \(E(x)\). Write \(G_d = (G_{d_1}, \ldots, G_{d_m}), 1 \leq d \leq s, \) and \(V_d = (V_{d_1}, \ldots, V_{d_m}), s + 1 \leq d \leq m\). Since \(s < m\), after an orientation preserving change of coordinates in \(\mathbb{R}^m\) we may assume that
\[
G_d(x) = \delta_{d}, \quad \text{for } 1 \leq d \leq s, \quad V_d(x) = \delta_{d}, \quad \text{for } s + 1 \leq d \leq m,
\]
where \(\delta_{d}\) is the Kronecker delta.

There are \(C^1\)-functions \(T_1, \ldots, T_m : \mathbb{R}^n \to \mathbb{R}\) defined in a neighbourhood of \(x\) such that
\[
P = \sum_{d=1}^s T_d G_d + \sum_{d=s+1}^m T_d V_d.
\]
Since \(p(x) = 0\), we have \((T_1(x), \ldots, T_s(x)) = -\lambda\) and \(T_{s+1}(x) = \ldots = T_m(x) = 0\). Let \(\theta : (\mathbb{R}^{n-k}, x') \to (W, v)\) be given by \(\theta(x') = (\psi(x''), x''')\), and let \(p' = p \circ \theta, t_i = T_i \circ \theta, g_{d}' = G_{d} \circ \theta\) and \(v_{d}' = V_{d} \circ \theta\). Then
\[
(t_1(x''), \ldots, t_s(x'')) = -\lambda, \quad t_{s+1}(x'') = \ldots = t_m(x'') = 0.
\]
Let \(Z : \mathbb{R}^n \to \mathbb{R}\) be a \(C^1\)-function and let \(z : \mathbb{R}^{n-k} \to \mathbb{R}\) be given by \(z = Z \circ \theta\). From (4) we have
\[
\frac{\partial Z}{\partial x_j}(x'') = \sum_{i=1}^k \frac{\partial Z}{\partial x_i}(x) \frac{\partial \psi_i}{\partial x_j}(x''') + \frac{\partial Z}{\partial x_j}(x) = \frac{\partial Z}{\partial x_j}(x),
\]
for \(k + 1 \leq j \leq n\).
Take \(i \in \{s + 1, \ldots, m\}, j \in \{k + 1, \ldots, n\}\). Then
\[
p' = \sum_{d=1}^s t_d g_{d}' + \sum_{d=s+1}^m t_d v_{d}',
\]
and therefore, from (5) and (6),
\[
\frac{\partial p_i}{\partial x_j}(x'') = -\sum_{d=1}^s \lambda_d \frac{\partial g_{d}}{\partial x_j}(x') + \frac{\partial t_i}{\partial x_j}(x''),
\]
and so, from (7), we have
\[
\frac{\partial t_i}{\partial x_j}(\pi') = \frac{\partial P^i}{\partial x_j}(\pi) + \sum_{d=1}^s \lambda_d \frac{\partial G^i_d}{\partial x_j}(\pi).
\]

Let \( m_{ij} \) be the above expression, and let \( M = [m_{ij}]_{i+1 \leq i \leq m, k+1 \leq j \leq m} \). From (1) and (5) it is easy to see that the derivative matrix \( \tilde{D} \tilde{H}(x, \lambda) \) has the form
\[
\begin{bmatrix}
? & ? & I \\
? & M & 0 \\
A & 0 & 0
\end{bmatrix},
\]
where \( I \) is the \( s \times s \) identity matrix, so \( \det[\tilde{D} \tilde{H}(x, \lambda)] = (-1)^{n(s+k)+k} \times \det[M] \det[A] \). By (3),
\[
\text{ind}(p, \pi) = (-1)^{n(s+k)+k} \text{sign det}[\tilde{D} \tilde{H}(x, \lambda)].
\]

3. Main theorem. Let \( H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k \) be given by
\[
H(x, \lambda) = \left( \sum_{i=1}^s \lambda_i G_i(x), F(x) \right).
\]

**Lemma 3.** \( H^{-1}(0) = W \times \{0\} \).

**Proof.** If \((x, \lambda) \in H^{-1}(0)\) then \( F(x) = 0\), i.e. \( x \in W \). By our assumption, the vectors \( G_1(x), \ldots, G_s(x) \) are linearly independent, and so \( \lambda = 0 \).

Let \( B_R = \{ (x, \lambda) \mid \|x\|^2 + \|\lambda\|^2 < R^2 \} \) and \( S_R = \partial B_R \). Since \( W \) is compact, by the above lemma there is \( R > 0 \) such that \( H^{-1}(0) \subset B_R \). Hence \( H|S_R : S_R \to \mathbb{R}^m \times \mathbb{R}^k = \{0\} \). Let \( \text{deg}(H|S_R) \) be the topological degree of \( H|S_R \).

**Theorem 4.** \( X(E) = (-1)^{n(s+k)+k} \text{deg}(H|S_R) \).

**Proof.** Let \( D_R = \{ x \in \mathbb{R}^n \mid \|x\| < R \} \). For each \( C^1\)-map \( P : \mathbb{R}^n \to \mathbb{R}^m \) there are sections \( p : W \to E, p' : W \to E' \) such that \( P|W = p + p' \). For each \( \varepsilon > 0 \) we can choose \( P \) so that
\[
(1) \quad \sup_{x \in D_R} \|P(x)\| < \varepsilon,
\]
and
\[
(2) \quad \text{if } p(x) = 0 \text{ then } \text{ind}(p, x) \neq 0, \text{i.e. } p \text{ is transversal to the zero-section}.
\]

Let \( \tilde{H} = \tilde{H}(x, \lambda) = (P(x) + \sum_{i=1}^s \lambda_i G_i(x), F(x)) \). From (1) and Lemma 3 we can show (using Cramer’s rule) that \( \tilde{H}^{-1}(0) \) lies close to \( W \times \{0\} \) and thus, for small \( \varepsilon \), \( \tilde{H}^{-1}(0) \subset B_R \). The manifold \( W \) is compact and so, from (2) and Lemma 1, \( \tilde{H}^{-1}(0) \) is finite, say \( \tilde{H}^{-1}(0) = \{(x^1, \lambda^1), \ldots, (x^m, \lambda^m)\} \).
Then $p^{-1}(0) = \{x^1, \ldots, x^m\}$ and according to the definition of $X(E)$ (see [1], Chapter 5.2) and Lemma 2

$$X(E) = \sum_{j=1}^{m} \text{ind}(p, x^j) = (-1)^{n(s+k)+k} \sum_{j=1}^{m} \text{sign det}[D\tilde{H}(x^j, \lambda^j)].$$

Clearly the last sum equals $\deg(\tilde{H}|S_R)$, and since $H|_{S_R}$ and $\tilde{H}|_{S_R}$ are homotopic for $\varepsilon$ small enough, we conclude that

$$X(E) = (-1)^{n(s+k)+k} \deg(H|_{S_R}). \ ■$$

Clearly $TW = \{(x, y) \in W \times \mathbb{R}^n \mid y \perp \text{grad } F_i(x), i = 1, \ldots, k\}$. It is well known that $X(TW) = \chi(W)$, where $\chi(W)$ is the Euler characteristic of $W$. Let $H : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$ be given by

$$H(x, \lambda) = \left(\sum_{i=1}^{k} \lambda_i \text{grad } F_i(x), F(x)\right).$$

As above, there is $R > 0$ such that $H^{-1}(0) \subset B_R$ and so we have a continuous map $H|_{S_R} : S_R \to \mathbb{R}^n \times \mathbb{R}^k - \{0\}$. As a consequence of Theorem 4 we have

**Theorem 5.** $\chi(W) = (-1)^k \deg(H|_{S_R}). \ ■$

A very similar version of the above theorem has been proved in [2].

**Example 1.** Let $W = S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 - 1 = 0\}$, let $G = G(x) = (3 + x_1x_2 - x_1^2, x_1x_2 - x_2, x_1 - x_2x_3)$, and let $E_1 = \{(x, y) \in S^2 \times \mathbb{R}^3 \mid y \perp G(x)\}$. Then

$$H(x, \lambda) = (3\lambda + x_1x_2\lambda - x_3^2\lambda, x_1x_2\lambda - x_2\lambda, x_1\lambda - x_2x_3\lambda, x_1^2 + x_2^2 + x_3^2 - 1)$$

and $R = 2$. Thanks to a computer program written by Marek Izydorek and Sławomir Rybicki from the Mathematical Department of the Technical University of Gdańsk we have been able to calculate that $\deg(H|_{S_2}) = 0$, so $X(E) = 0$.

**Example 2.** Let $G = G(x) = (3x_1 + x_1x_2^2, 3x_2 + x_2x_3, 3x_3)$, and let $E_2 = \{(x, y) \in S^2 \times \mathbb{R}^3 \mid y \perp G(x)\}$. Then

$$H(x, \lambda) = (3x_1\lambda + x_1x_2^2\lambda, 3x_2\lambda + x_2x_3\lambda, 3x_3\lambda, x_1^2 + x_2^2 + x_3^2 - 1)$$

and $R = 2$. As above we have calculated that $\deg(H|_{S_2}) = -2$, so $X(E) = 2$.

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References