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Structure of mixing and category of complete mixing for stochastic operators

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Abstract. Let T be a stochastic operator on a σ-finite standard measure space with an equivalent σ-finite infinite subinvariant measure λ. Then T possesses a natural "conservative deterministic factor" Φ which is the Frobenius–Perron operator of an invertible measure preserving transformation φ. Moreover, T is mixing ("sweeping") iff φ is a mixing transformation. Some stronger versions of mixing are also discussed. In particular, a notion of ${}^*L^1$-s.o.t. mixing is introduced and characterized in terms of weak compactness. Finally, it is shown that most stochastic operators are completely mixing and that the same holds for convolution stochastic operators on l.c.a. groups.

1. Introduction. Let (X, Σ, m) be a σ-finite measure space. A linear operator T on $L^1(m)$ is called *stochastic* if T takes probability densities to probability densities. The adjoint T' is then a Markov operator on $L^\infty(m)$. If $\varphi : X \to X$ is a nonsingular transformation then the stochastic operator T_φ such that $T'_\varphi h = h \circ \varphi$ is called the *Frobenius–Perron operator* of φ. On a standard measure space the Frobenius–Perron operators are exactly the extreme points of the convex set $\mathcal{S} = \mathcal{S}(X, \Sigma, m)$ of all stochastic operators.

We consider (strong) mixing properties of stochastic operators and for the most part restrict our attention to the set of operators admitting an infinite invariant (or subinvariant) equivalent measure λ. If T is such an operator then the action of T on $L^1(\lambda)$ will be denoted by P (see the notion of λ-representation below). Various gradations of mixing depend on the mode of convergence of P^n (see Section 2). If $P^n \to 0$ in the weak operator topology on $L^2(\lambda)$ then T (and P) is simply called *mixing* [4], [11], or "sweeping" [9]. Using some ideas of Foguel [4] we analyze the structure of P and prove that the mixing property depends solely on the "conservative deterministic factor" Φ of P (Thm. 1). In particular, a stochastic operator with a trivial deterministic factor must be mixing.

Another, stronger notion is that of complete mixing (and its version

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of norm complete mixing) introduced in Krengel and Sucheston [10] and investigated by Lin [11] in the general case (see also [1]). Interrelations between various mixing properties are indicated in Section 3 (Thm. 2). We have also introduced $^*L^1$-s.o.t. mixing, a notion essentially stronger than the complete mixing of the adjoint, characterized in terms of weak compactness (Prop. 2).

The topological size of the set of mixing operators was analyzed in [7] for the operator norm topology (o.n.t.), strong operator topology (s.o.t.), and weak operator topology (w.o.t.) in $\mathcal{S}$; the reader is also referred to [3] for historic background. In Section 4 we prove that the norm completely mixing operators form a residual set in s.o.t. both in $\mathcal{S}$ and in the set of stochastic operators admitting a fixed σ-finite invariant (or subinvariant) measure. Briefly, most stochastic operators are completely mixing (Thms. 3 and 4). This strengthens some results in [7]. The last section is devoted to random walks on second countable locally compact abelian groups. We observe that most random walks are completely mixing (Thm. 5).

Generally our notation is taken from [7] and [4]. Most of the basic facts presented in this section can be found in [4].

It is well known that every positive operator in $\mathcal{L}(L^1(m))$ can be uniquely extended by monotonicity to act on arbitrary nonnegative measurable functions (or absolutely continuous measures). As in [7], if λ is a σ-finite equivalent measure ($\lambda \sim m$) then we write

$$\mathcal{S}_{\leq\lambda} = \{T \in \mathcal{S} : T\lambda \leq \lambda\}, \qquad \mathcal{S}_{\lambda} = \{T \in \mathcal{S} : T\lambda = \lambda\},$$

and λ is called *subinvariant* (resp. *invariant*) with respect to T. Clearly $\mathcal{S}_{\leq\lambda} = \mathcal{S}_{\lambda}$ whenever λ is finite.

For any $T \in \mathcal{S}_{\leq\lambda}$ the action of T on the densities with respect to λ is given by the λ-*representation* P of T, where P is a stochastic operator on $L^1(\lambda)$ defined by

$$Pf = \frac{dm}{d\lambda} T\left(\frac{d\lambda}{dm} f\right) \qquad (f \in L^1(\lambda)).$$

Clearly P acts as a positive contraction on any $L^p(\lambda)$, $1 \leq p \leq \infty$. We denote by P^* the $L^2(\lambda)$-adjoint of P as well as its monotone extensions to all the spaces $L^p(\lambda)$.

It is easy to see that an operator $P \in \mathcal{L}(L^1(\lambda))$ is the λ-representation of some $T \in \mathcal{S}_{\leq\lambda}$ (resp. $T \in \mathcal{S}_{\lambda}$) iff $P \in \mathcal{P}_{\leq\lambda}$ (resp. $P \in \mathcal{P}_{\lambda}$), where

$$\mathcal{P}_{\leq\lambda} = \{P \in \mathcal{L}(L^1(\lambda)) : P \geq 0,\ P1 \leq 1,\ P^*1 = 1\},$$
$$\mathcal{P}_{\lambda} = \{P \in \mathcal{L}(L^1(\lambda)) : P \geq 0,\ P1 = 1,\ P^*1 = 1\}.$$

For $T \in \mathcal{S}$ let $X = C(T) \cup D(T)$ be the Hopf decomposition into the conservative and dissipative part of T. It is well known that $T'\chi_{D(T)} \leq \chi_{D(T)}$

and $C(T_{C(T)}) = C(T)$. A set $A \in \Sigma$ is called *T-invariant* if $T'\chi_A = \chi_A$. A function f will be called *invariant* with respect to T if $f \in L^1(m)$ and $Tf = f$.

Finally, we recall the Harris decomposition $T^n = Q_n + R_n$, where Q_n is a positive kernel operator and there is no nonzero kernel operator K with $0 \le K \le R_n$. An operator $T \in \mathcal{S}$ is called *Harris* if it is conservative and Q_n is not the zero operator for some $n \ge 1$.

2. Mixing, factorization theorem. Our definition of mixing is the same as in [7] and in accord with [4] and [11]. Let $T \in \mathcal{S}_{\le\lambda}$ and $P \in \mathcal{P}_{\le\lambda}$ be the λ-representation of T. Then T (or P) is called *mixing* if

$$\text{(M)} \qquad \lim \int_B P^n \chi_A d\lambda = \lambda(A)\lambda(B)/\lambda(X)$$

whenever $\lambda(A) + \lambda(B) < \infty$. In other words, T is mixing if the iterations P^n converge in the w.o.t. of $\mathcal{L}(L^2(\lambda))$ to the operator E_λ defined by $E_\lambda f = \int f d\lambda/\lambda(X)$ if $\lambda(X) < \infty$, and $E_\lambda = 0$ if $\lambda(X) = \infty$. We remark that, for $\lambda(X) = \infty$, mixing operators are also called "sweeping with respect to sets of finite measure" and if the convergence of iterates is weakened to the Cesàro convergence

$$\text{(C)} \qquad (1/n)\sum_{i<n} \int_B P^i \chi_A d\lambda \to 0 \qquad (\lambda(A) + \lambda(B) < \infty)\,,$$

then P is called "Cesàro sweeping with respect to the family of λ-finite sets" (see [9]). It is easy to see e.g. by the ergodic theorem ([4], Thm. B, Ch. VII) that (C) holds iff P has no nonzero invariant functions in $L^1(\lambda)$.

For infinite λ, if $T \in \mathcal{S}_{\le\lambda}$ is mixing then T has no nonzero invariant functions. Nevertheless, there exist nonergodic mixing operators, so (M) is essentially stronger than (C). Indeed, it follows from [15] (Thms. 2.2 and 2.3) that a generic Frobenius–Perron operator in $\mathcal{S}_\lambda$ is ergodic and conservative—hence satisfies (C)—without being mixing. Concrete examples of ergodic nonmixing transformations can easily be obtained by modifying the classical Chacon transformation.

Now we characterize mixing in terms of a "deterministic factor" of $P \in \mathcal{P}_{\le\lambda}$. As in [4], Ch. VIII, let

$$K = \{f \in L^2(\lambda) : \|P^n f\|_2 = \|(P^*)^n f\|_2 = \|f\|_2,\ n = 1, 2, \ldots\}\,.$$

Then K is a closed sublattice of $L^2(\lambda)$ and the operator P is unitary on K. Note that for every f from the orthogonal complement of K, $P^n f \to 0$ and $(P^*)^n f \to 0$ weakly in $L^2(\lambda)$. The family $\Sigma_1(P) = \{A \in \Sigma : \chi_A \in K\}$ is a subring of Σ on which P and P^* act as automorphisms. Moreover, K is the closed span of $\{\chi_A : A \in \Sigma_1\}$ and if $X_1 \in \Sigma$ is minimal in Σ with the property that $A \subset X_1 \pmod{\lambda}$ for every $A \in \Sigma_1$ then X_1 is P-invariant.

The set X_1 will be referred to as the *deterministic part* of P. Moreover, if $A \subset X_2 = X \backslash X_1$ and A has finite measure, then χ_A is orthogonal to K whence $P^n \chi_A \to 0$ weakly, which shows that P is always mixing on the nondeterministic part X_2. This allows us to consider the restriction of P to X_1. Note that the deterministic part of the restricted operator P_{X_1} is also equal to X_1.

Now the following proposition, which is implicit in [4], easily follows:

PROPOSITION 1. *Assume (X, Σ) is a standard Borel space and λ is σ-finite. Let $T \in \mathcal{S}_{\le\lambda}$ and let P be the λ-representation of T. Then the space*

$$K' = \{f \in L^2(\lambda_{C(P)}) : \|P^n f\|_2 = \|(P^*)^n f\|_2 = \|f\|_2,\ n = 1, 2, \ldots\}$$

coincides with $L^2(X', \Sigma', \lambda')$ where $X' = X_1 \cap C(P)$, Σ' is the σ-algebra generated by $\Sigma_1(P)$ restricted to X' and λ' is the (σ-finite) restriction of λ to Σ'. Moreover, the following diagram commutes:

$$\begin{array}{ccc} L^2(\lambda) & \xrightarrow{P} & L^2(\lambda) \\ \Pi \downarrow & & \downarrow \Pi \\ L^2(\lambda') & \xrightarrow{\Phi} & L^2(\lambda') \end{array}$$

where Πf is the conditional expectation of $\chi_{C(P)} f$ given Σ', and $\Phi g = g \circ \varphi^{-1}$ where φ is an invertible measure preserving transformation of (X', Σ', λ'). ■

We remark that if P is a Harris operator (in particular, if P is given by an integral kernel) then by Thm. D of Ch. V in [4], and by [5], the measure space (X', Σ', λ') is atomic. If, in addition, P admits no invariant functions then each atom of λ' must be a wandering set. Consequently, $X' \subseteq D(P) \cap C(P)$ so it is trivial. On the other extreme, if P is the Frobenius–Perron operator induced by a conservative invertible measure preserving transformation of (X, Σ, λ) then clearly $(X, \Sigma, \lambda) = (X', \Sigma', \lambda')$ and $\Phi = P$. In general, the transformation φ of (X', Σ', λ') can be viewed as the *conservative deterministic factor* of P (or T).

Under the assumption of Proposition 1 we have

THEOREM 1. *P is mixing iff Φ is mixing.*

Proof. It is clear that P is mixing iff P_{X_1} is mixing. Also, P is mixing iff $P_{C(P)}$ is mixing since $P_{D(P)}$ is always mixing. Moreover, P is always mixing on the orthogonal complement of K. Now observe that $P_{C(P)}$ on K is equal to Φ. ■

3. Stronger version of mixing for infinite measure. In this section we fix an equivalent σ-finite infinite measure λ and consider the set $\mathcal{P}_{\le\lambda}$. An operator $P \in \mathcal{P}_{\le\lambda}$ may possess stronger mixing properties than the mixing considered above. They are defined according to the mode of convergence of P^n to zero. Some stronger versions of mixing have been discussed in the literature ([8], [10], [13], [11], [12], [1], [7]). Below we introduce another kind of mixing and review the relations between various mixing properties.

As in [7], we say that P is *L^p-s.o.t. mixing* if $P^n \to 0$ in L^p-s.o.t. (Here and below in this section we only consider L^p spaces for the measure λ.) Note that L^2-s.o.t. mixing is equivalent to the condition (ii) in Thm. 3.3 of [11] (see the proof below). Similarly, we say that P is *$^*L^p$-s.o.t. mixing* if $(P^*)^n \to 0$ in L^p-s.o.t. According to [10] (see also [11]), P is called *completely mixing* if for every $f \in L^1_0$ (i.e. $f \in L^1$ with integral zero), $P^n f \to 0$ weakly in L^1. It follows from [10] that complete mixing implies ergodicity and is equivalent to the apparently stronger norm complete mixing defined by $\|P^n f\|_1 \to 0$ ($f \in L^1_0$). A more general definition of norm complete mixing for any T in $\mathcal{S}$ is considered in Section 4. Analogously, P will be called (norm) **completely mixing* if for every $f \in L^1_0$, $\|(P^*)^n f\|_1 \to 0$ (this is condition (a) in Thm. 3.3 of [11]).

THEOREM 2. *The following implications hold*:

$$\begin{array}{r}
L^\infty\text{-}s.o.t.\ mixing \\
\downarrow \\
complete\ mixing \to L^2\text{-}s.o.t.\ mixing \\
\downarrow \\
{}^*L^1\text{-}s.o.t.\ mixing \to {}^*complete\ mixing \to {}^*L^2\text{-}s.o.t.\ mixing \to mixing
\end{array}$$

Proof. $^*L^1$-s.o.t. mixing implies *complete mixing by definition. It follows from Thm. 3.3 of [11] that *complete mixing implies $\|(P^*)^n g\|_2 \to 0$ ($g \in L^2 \cap L^1_0$). To prove that *complete mixing implies $^*L^2$-s.o.t. mixing it suffices to observe that $L^2 \cap L^1_0$ is dense in L^2 for infinite λ. Analogously, complete mixing implies L^2-s.o.t. mixing. That L^∞-s.o.t. mixing implies L^2-s.o.t. mixing has been observed in [7]. Clearly, mixing is implied by L^2-s.o.t. mixing and by $^*L^2$-s.o.t. mixing. ∎

Remark. None of the implications can be reserved. The operator $Pf(x) = f(x+1)$ on $L^1(\mathbb{R})$ is mixing without being L^2-s.o.t. mixing or $^*L^2$-s.o.t. mixing. Random walks on l.c.a. groups are usually L^2-s.o.t. (and $^*L^2$-s.o.t.) mixing without being L^∞-s.o.t. mixing or complete mixing (see Section 5). A $^*L^1$-s.o.t. mixing operator must be dissipative while there exist conservative *complete mixing operators among e.g. symmetric random walks.

The notion of $^*L^1$-s.o.t. mixing can be characterized in terms of weak compactness. As in [4], 3.7, we denote by $C^1(P)$ the maximal part of $C(P)$ which is a countable union of P-invariant sets of finite measure.

LEMMA 1. *Let $P \in \mathcal{P}_{\le\lambda}$. Then P and P^* have the same sets of invariant functions. Moreover*, *P has no nonzero invariant functions iff $C^1(P) = \emptyset$.*

Proof. From $C(P) = C(P^*)$ we have $(P_{C(P)})^* = P^*_{C(P^*)}$. Furthermore, as every invariant function is zero on the dissipative part, we can limit ourselves to the case of conservative operators. Clearly, it suffices to prove that every P-invariant function f is P^*-invariant. The invariant functions form a sublattice of $L^1(\lambda)$ so we can assume that $f \ge 0$. Now f is measurable with respect to P-invariant sets (see [4], Thm. A in Ch. III, and 7.4 in [4]), so by a monotonicity argument, f is P^*-invariant.

If P has no nonzero invariant functions then, by 7.4 and 3.7 in [4], $C^1(P) = \emptyset$. To prove the converse it suffices to show that the supports of P-invariant functions are contained in $C^1(P)$. This follows immediately from the first part of the proof and the definition of $C^1(P)$. ∎

If P is a λ-representation of $T \in \mathcal{S}_{\le\lambda}$ then $P^* = T'$ and therefore P and T have the same invariant sets. In particular, $C^1(P) = \emptyset$ iff every T-invariant set has λ measure zero or infinity. On the other hand, P has no nonzero invariant functions iff T has no nonzero invariant functions.

Now by Lemma 1, the above remark, and Lemma 8.6 in [16] (see also [10]) we obtain

PROPOSITION 2. *Let P be a λ-representation of $T \in \mathcal{S}_{\le\lambda}$ and suppose the T-invariant sets are of measure zero or infinity. Then for every $f \in L^1(\lambda)$, $\|(P^*)^n f\|_1 \to 0$ iff the sequence $(P^*)^n f$ is conditionally weakly compact.* ∎

To conclude this section we remark that although it may seem reasonable to consider the whole range of L^p-s.o.t. mixing properties for $1 < p < \infty$, they all coincide with the L^2-s.o.t. mixing. We omit a standard proof. An analogous result for a finite measure λ has been proved by Lin ([11], Thm. 2.3).

It should be recalled that by Lin [12], if $P \in \mathcal{P}_{\le\lambda}$, $\lambda(X) = \infty$, and P is given by a transition probability then P is L^p-s.o.t. mixing iff the past σ-field of the associated 2-sided Markov shift with the initial distribution λ has no sets of positive finite measure.

4. Most stochastic operators are completely mixing. Throughout the present section $L^1(m)$ is a separable space. We shall consider two topologies in $\mathcal{S}$: the o.n.t. and the s.o.t. Recall that $\mathcal{S}$ is a Polish space for both.

Denote by $\mathcal{N}$ the set of *norm completely mixing* operators in $\mathcal{S}$, i.e. a stochastic operator T is in $\mathcal{N}$ iff

$$\|T^n f\|_1 \to 0 \quad \text{for every } f \in L^1_0(m)\,.$$

The following theorem shows that most operators in $\mathcal{S}$ are completely mixing.

THEOREM 3. *The set of norm completely mixing operators is a dense G_δ in $\mathcal{S}$ for both s.o.t. and o.n.t.*

Proof. Let $f_1, f_2, \ldots$ be a dense subset of $L_0^1(m)$. It is clear that

$$\mathcal{N} = \bigcap_n \bigcap_k \bigcap_p \bigcup_{r \geq p} \{T \in \mathcal{S} : \|T^r f_n\|_1 < 1/k\}$$

so $\mathcal{N}$ is a G_δ set for s.o.t. To prove the norm denseness we fix a function $0 \leq u \in L^1(m)$ with $\int u\, dm = 1$ and define

$$Ef = u \int f\, dm\,.$$

Clearly E is in $\mathcal{S}$ and so is $T_\alpha = \alpha T + (1-\alpha)E$ for every $T \in \mathcal{S}$ and $0 < \alpha < 1$. If $f \in L_0^1(m)$ then $T^k f \in L_0^1(m)$ whence $ET^k f = 0$ for $k \geq 0$. Consequently, $T_\alpha^n f = \alpha^n T^n f \to 0$ in L^1-norm so $T_\alpha \in \mathcal{N}$. Moreover, $\|T_\alpha - T\| \to 0$ as $\alpha \to 1$, which proves the denseness of $\mathcal{N}$ in $\mathcal{S}$. ∎

Now we consider the local size of the family of completely mixing operators by looking at $\mathcal{N} \cap \mathcal{S}_\lambda$, where λ is a fixed equivalent σ-finite measure. We denote by $\mathcal{C}_\lambda$ the set of λ-representations of (norm) completely mixing operators in $\mathcal{S}_\lambda$. Equivalently, $P \in \mathcal{C}_\lambda$ iff $P \in \mathcal{P}_\lambda$ and $\|P^n f\| \to 0$ for every $f \in L_0^1(\lambda)$ (here and below $\| \;\|$ denotes the norm in $L^1(\lambda)$). We recall that $\mathcal{S}_\lambda$ as well as $\mathcal{P}_\lambda$ are Polish spaces for s.o.t. and the natural correspondence $T \to P$ between $\mathcal{S}_\lambda$ and $\mathcal{P}_\lambda$ is an s.o.t. homeomorphism.

LEMMA 2. *For every s.o.t. neighborhood $\mathcal{V}$ in $\mathcal{P}_\lambda$ and every $F \in \Sigma$ with $\lambda(F) < \infty$ there exist $P \in \mathcal{V}$ and $A \in \Sigma$ such that $\lambda(A) < \infty$, $F \subset A$, and $P\chi_A = \chi_A$.*

Proof. We may assume that

$$\mathcal{V} = \{P \in \mathcal{P}_\lambda : \|P\chi_{E_i} - P_0\chi_{E_i}\| < \varepsilon\,,\ i = 1, \ldots, p\}$$

where $\varepsilon > 0$, $P_0 \in \mathcal{P}_\lambda$, and $E_1, \ldots, E_p$ are disjoint sets of finite positive λ measure. Now we may find nonnegative functions $g_1, \ldots, g_p$ that approximate $P_0\chi_{E_1}, \ldots, P_0\chi_{E_p}$ so that

(a) $\lambda\{g_i \neq 0\} < \infty$,
(b) $\int g_i\, d\lambda = \lambda(E_i)$,
(c) $\|g_i - P_0\chi_{E_i}\| < \varepsilon$,
(d) $\sum g_i \leq 1$.

Indeed, note that $\sum_i P_0\chi_{E_i} \leq 1$. Now if $\lambda\{P_0\chi_{E_i} \neq 0\} < \infty$ then simply let $g_i = P_0\chi_{E_i}$. Otherwise, let $g_i = \chi_{A_i} P_0\chi_{E_i} + c_i\chi_{B_i}$ where the coefficients $0 < c_i \leq 1$ and the sets A_i, B_i are chosen in such a way that $A_1 \cup \ldots \cup A_p$, $B_1, \ldots, B_p$ are pairwise disjoint of finite measure and $c_i\lambda(B_i) = \|\chi_{A_i^c} P_0\chi_{E_i}\| < \varepsilon/2$ $(i = 1, \ldots, p)$.

We now let $E = E_1 \cup \ldots \cup E_p$ and A any set of finite measure containing

$$F \cup E \cup \bigcup_{i=1}^{p} \{g_i \neq 0\}\,.$$

We define

$$Pf = \sum_i \frac{g_i}{\lambda(E)} \int_{E_i} f\,d\lambda + \frac{\chi_A - \sum g_i}{\lambda(A\backslash E)} \int_{A\backslash E} f\,d\lambda + \chi_{A^c} f$$

(if $E = X$ then omit the second term on the right; if $\lambda(E^c) > 0$, we may always choose A with $\lambda(A\backslash E) > 0$). It is easy to check that $P \in \mathcal{P}_\lambda$ and $P\chi_{E_j} = g_j$. Therefore $P \in \mathcal{V}$ by (c). Clearly $P\chi_A = \chi_A$, which ends the proof of the lemma. ■

THEOREM 4. *The set of completely mixing operators is a dense G_δ in $\mathcal{S}_\lambda$ with s.o.t.*

Proof. It suffices to prove that $\mathcal{P}_\lambda \backslash \mathcal{C}_\lambda$ is a countable union of closed nowhere dense sets in $\mathcal{P}_\lambda$. To this end we choose a dense sequence $f_1, f_2, \ldots$ in $L^1_0(\lambda)$ satisfying $\lambda\{f_i \neq 0\} < \infty$ $(i \geq 1)$. Now let

$$\mathcal{A}(n,k) = \bigcap_r \{P \in \mathcal{P}_\lambda : \|P^r f_n\| \geq 1/k\}$$

and note that

$$\mathcal{P}_\lambda \backslash \mathcal{C}_\lambda = \bigcup_n \bigcup_k \mathcal{A}(n,k)\,.$$

Since $\mathcal{A}(n,k)$ is closed, it suffices to show that it has empty interior.

Given any open neighborhood $\mathcal{V}$ in $\mathcal{P}_\lambda$ we apply the lemma above to find $P \in \mathcal{V}$ and $A \in \Sigma$ with $0 < \lambda(A) < \infty$ and $\{f_n \neq 0\} \subset A$ such that $P\chi_A = \chi_A$. Define

$$E_A f = \frac{\int_A f\,d\lambda}{\lambda(A)} \chi_A$$

and $Qf = E_A f + \chi_{A^c} f$. Clearly $Q \in \mathcal{P}_\lambda$. Moreover, for every $0 < \alpha < 1$ we have $P_\alpha = \alpha P + (1-\alpha)Q \in P_\lambda$. Note that if $f \in L^1(\lambda)$ and $\{f \neq 0\} \subset A$ then $PE_A f = E_A P f = E_A f$, so $P^r_\alpha f = \alpha^r P f + (1-\alpha^r) E_A f$. In particular, for $n = 1, 2, \ldots$, $\|P^r_\alpha f_n\| = \alpha^r \|Pf\| \to 0$, which implies $P_\alpha \notin \mathcal{A}(n,k)$. Since $P_\alpha \in \mathcal{V}$ for α sufficiently close to 1, the neighborhood $\mathcal{V}$ cannot be contained in $\mathcal{A}(n,k)$. ■

5. Remarks on random walks. In this section we consider "space homogeneous" stochastic operators. Let $X = G$ be a second countable locally compact group with Borel σ-algebra Σ and left Haar measure $m = \lambda$. Any probability measure μ on G defines a left convolution operator T_μ, where $(T_\mu f)(x) = (\mu * f)(x) = \int f(y^{-1}x)\,d\mu(y)$ for $f \in L^1(\lambda)$. Clearly $T_\mu \in \mathcal{S}_\lambda$

and the associated Markov process is the (left) random walk determined by μ. We note that if $T \in \mathcal{S}$ then by Wendel's theorem (see [6], 35.5), T is a convolution operator iff T is (right) translation invariant, which means that $TR_a = R_a T$ $(a \in G)$, where $R_a f(x) = f(xa)$.

It has been shown by J. Rosenblatt [14] that there exists a completely mixing convolution on G iff G is amenable. There seems to be no characterization of completely mixing random walks on general amenable groups in terms of the support of the measure. On the other hand, the abelian case is well understood. The following facts concerning l.c.a. groups are well known (references can be found in [14]).

Let G be abelian and μ a probability measure on G. Then

(a) T_μ *is ergodic iff* μ *is not concentrated on a closed proper subgroup* (Choquet–Deny);

(b) T_μ *is norm completely mixing iff* μ *is not concentrated on a coset of a closed proper subgroup.*

It has also been observed in [7] that

(c) T_μ *is* L^2*-s.o.t. mixing iff* $|\widehat{\mu}| < 1$ *a.e. on* $\widehat{G}$.

In particular, we have the following example which should be compared with Thm. 2.

Example. *Let μ be a probability measure on $G = \mathbb{R}$. Then*

(i) T_μ *is mixing iff* $\mu \neq \delta_0$;

(ii) T_μ *is* L^2*-s.o.t. mixing iff* μ *is not a point mass*;

(iii) T_μ *is norm completely mixing iff* μ *is not concentrated on a bilateral arithmetic progression.*

Assertions (i) and (ii) follow readily from (c) and the fact that $\{\gamma : |\widehat{\mu}(\gamma)| = 1\}$ is a closed subgroup of the dual group. Clearly, similar conditions hold in $\mathbb{R}^n$.

Now we show that the generic convolution operator on an abelian group is completely mixing. We first observe that the set of convolution operators is w.o.t. closed in $\mathcal{S}$ (and in $\mathcal{S}_\lambda$), so it is s.o.t. Baire. Indeed, by Wendel's theorem $\bigcap_{a \in G} \{T \in \mathcal{S} : TR_a = R_a T\}$ is the set of all convolution operators in $\mathcal{S} = \mathcal{S}(G, \Sigma, m)$.

Theorem 5. *Let G be a second countable locally compact abelian group. The norm completely mixing convolution operators form a dense G_δ subset of the set of convolution stochastic operators for both s.o.t. and o.n.t.*

Proof. The G_δ assertion follows from Thm. 3, so it suffices to prove the norm denseness. Let ν be a fixed probability measure with $\operatorname{supp} \nu = G$. If T_μ is a stochastic convolution operator, then for any $0 < \alpha < 1$ the operator

$\alpha T_\mu + (1-\alpha)T_\nu = T_{\alpha\mu+(1-\alpha)\nu}$ is norm completely mixing by (b) and norm approximates T_μ as α tends to 1. ∎

Recently, W. Bartoszek [2] generalized the above theorem to amenable groups.

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