On starlikeness of certain integral transforms

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Abstract. Let $A$ denote the class of normalized analytic functions in the unit disc $U = \{ z : |z| < 1 \}$. The author obtains fixed values of $\delta$ and $\varrho$ ($\delta \approx 0.308390864 \ldots$, $\varrho \approx 0.0903572 \ldots$) such that the integral transforms $F$ and $G$ defined by

$$F(z) = \int_0^z \frac{f(t)}{t} \, dt \quad \text{and} \quad G(z) = \frac{2}{z} \int_0^z g(t) \, dt$$

are starlike (univalent) in $U$, whenever $f \in A$ and $g \in A$ satisfy $\text{Re} f'(z) > -\delta$ and $\text{Re} g'(z) > -\varrho$ respectively in $U$.

1. Introduction. Let $A$ denote the class of analytic functions $f$ defined in the unit disc $U = \{ z : |z| < 1 \}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Let $S^*$ be the usual class of starlike (univalent) functions in $U$, i.e.

$$S^* = \{ f \in A : \text{Re}[zf'(z)/f(z)] > 0, \, z \in U \}$$

and let $R(\beta) = \{ f \in A : \text{Re} f'(z) > \beta, \, z \in U \}$, $R(0) \equiv R (\beta < 1)$.

In a recent paper [8], Singh and Singh proved that if $f \in R$, then $F \in S^*$, where

$$F(z) = \int_0^z \frac{f(t)}{t} \, dt,$$

and in [3], Mocanu considered the Libera transform $G$ defined by

$$G(z) = \frac{2}{z} \int_0^z g(t) \, dt$$

and showed that $g \in R$ implies $G \in S^*$.

In this note, we improve both the above results by showing that the same conclusions hold under a much weaker condition on $f$ and $g$ respectively.

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2. Preliminaries. We need the following lemma to prove our results.

**Lemma A.** Let $\Omega$ be a set in the complex plane $\mathbb{C}$. Suppose that the function $m : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfies the condition

$$m(ix, iy, z) \notin \Omega$$

for all real $x, y \leq -1 + x^2/2$ and all $z \in U$. If the function $p$ is analytic in $U$ with $p(0) = 1$ and if $m(p(z), zp'(z); z) \in \Omega$, $z \in U$, then $\text{Re} p(z) > 0$ in $U$.

A more general form of this lemma may be found in [2].

3. Main results. From the result of Hallenbeck and Ruscheweyh [1] (see also [2]) we find that if $P$ is analytic in $U$ with $P(0) = 1$ then

$$\text{Re}[P(z) + zP'(z)] > \beta \text{ implies } P(z) < \beta + (1 - \beta)l(z), z \in U,$$

and

$$\text{Re}[P(z) + \frac{1}{2}zP'(z)] > \beta \text{ implies } P(z) < \beta + (1 - \beta)L(z), z \in U,$$

where $\prec$ stands for usual subordination, and $l$ and $L$ defined by $l(z) = -1 - (2/z) \log(1 - z)$ and $L(z) = 2((l(z) - 1)/z) - 1$ are convex (univalent) in $U$. In view of the fact that the coefficients in the series expansions of $l$ and $L$ are positive, we easily deduce that $\text{Re} l(z) > 2\ln 2 - 1$ and $\text{Re} L(z) > 3 - 4\ln 2$ in $U$. Also it is easily seen that $l(U) \subset \{w \in \mathbb{C} : |\arg w| < \pi/3\}$. Using a result of Mocanu et al. [4] we can replace $\pi/3$ by $\theta$, where $\theta$ lies between 0.911621904 and 0.911621907. This combined with the result of Robertson [7] yields that $l(U) \subset \Omega_1 \cap \Omega_2 \cap \Omega_3$ with $\Omega_1 = \{w \in \mathbb{C} : \text{Re} w > 2\ln 2 - 1\}$, $\Omega_2 = \{w \in \mathbb{C} : |\arg w| < 0.911621907\}$ and $\Omega_3 = \{w \in \mathbb{C} : |\text{Im} w| < \theta\}$.

**Theorem 1.** If $\delta = (2\ln 2 - 1)(3 - 2\ln 2)/[3 - (2\ln 2 - 1)(3 - 2\ln 2)] = 0.262...$ and $f \in R(-\delta)$ then the function $F$ defined by (1) is starlike in $U$.

**Proof.** From (1) we deduce

$$zf''(z) + F'(z) = f'(z), \quad z \in U.$$

Let $P(z) = F'(z)$ and $Q(z) = F(z)/z$. Since $\text{Re} f'(z) > -\delta$ in $U$, by using (4) and (6) we find that $\text{Re} F'(z) > -\delta + (1 + \delta)(2\ln 2 - 1)$ for $z \in U$. Again by using (4) this in turn implies $\text{Re} Q(z) > 2\delta > 0$, $z \in U$.

Now if we set $p(z) = zF'(z)/F(z)$ then $p$ is analytic in $U$, $p(0) = 1$ and $f'(z) = m(p(z), zp'(z); z)$, where $m(u, v; z) = (u^2 + v^2)Q(z)$. To prove $F \in S^*$, it is enough to show $\text{Re} p(z) > 0$ in $U$. Since $\text{Re} f'(z) > -\delta$ in $U$, by (6), we get $\text{Re} m(p(z), zp'(z); z) > -\delta$ in $U$. Now for all real $x, y \leq -1 + x^2/2$ and $z \in U$, we have $\text{Re} m(ix, iy; z) = (y - x^2)\text{Re} Q(z) \leq -1 + 3x^2 \text{Re} Q(z)/2) \leq -\delta$ and so applying Lemma A with $\Omega = \{w \in \mathbb{C} : \text{Re} w > -\delta\}$, we get $\text{Re} p(z) > 0$ in $U$. Hence the theorem.
For the proof of Theorem 2 we prove the following lemmas. Theorem 2 further improves Theorem 1.

**Lemma 1.** If \( g \in R(\beta) \) then \( G \) defined by (2) belongs to \( R(\beta + (1 - \beta)(3 - 4 \ln 2)) \) \((\beta < 1)\).

**Proof.** From (2) we deduce
\[
\begin{align*}
 zG'(z) + G(z) &= 2g(z), \\
 zG''(z) + 2G'(z) &= 2g'(z).
\end{align*}
\]
Since \( g \in R(\delta) \), by using (5), we obtain \( G'(z) < \beta + (1 - \beta)L(z) \), \( z \in U \), and so \( \text{Re} G'(z) > \beta + (1 - \beta)(3 - 4 \ln 2), z \in U \). Here \( L(z) \) is as defined earlier. This proves Lemma 1.

**Lemma 2.** Let \( M = 2(2 \ln 2 - 1)(1 - \ln 2), \theta = 0.911621907, N = \tan \theta, \) \( a = 4(1 + M)^2 - \frac{4}{3}N^2(2 \ln 2 - 1)^4 - 4, b = -4(1 - 2M)(1 + M) - \frac{2}{3}(2 \ln 2 - 1)^4N^2, \) \( c = (1 - 2M)^2 - \frac{2}{3}(2 \ln 2 - 1)^2N \) and \( \varrho = (-b - (b^2 - 4ac)^{1/2})/(2a) \). Suppose that \( Q \) is a complex function with \( Q(0) = 1 \) satisfying
\[
Q(U) \subset E_1 \cap E_2 \cap E_3 \quad \text{where}
\]
\[
\begin{align*}
 E_1 &= \{ w \in \mathbb{C} : \text{Re} w > 1 - 2M(1 + \varrho) \}, \\
 E_2 &= \{ w \in \mathbb{C} : |\arg (w - (1 - 2[2 \ln 2 - 1](\varrho + 1)))| < \theta \}, \\
 E_3 &= \{ w \in \mathbb{C} : |\text{Im} w| < 2(2 \ln 2 - 1)(\varrho + 1)\pi \}.
\end{align*}
\]
If \( p \) is analytic in \( U \) with \( p(0) = 1 \) and if
\[
\text{Re} Q(z)[zp'(z) + p^2(z) + p(z)] > -2\varrho, \quad z \in U,
\]
then \( \text{Re} p(z) > 0 \) in \( U \).

Throughout the paper \( M, \theta, N, a, b, c, \varrho, \) and \( E \)'s are all as defined above.

**Proof of Lemma 2.** If we let \( m(u, v; z) = Q(z)(v + u^2 + u)/2 \) then for all \( x, y \leq -(1 + x^2)/2 \) and \( z \in U \), we have
\[
\text{Re} m(ix, y; z) = [(y - x^2) \text{Re} Q(z) - x \text{Im} Q(z)]/2 \leq -[3x^2 \text{Re} Q(z) + 2x \text{Im} Q(z) + \text{Re} Q(z)]/4.
\]
Thus \( \text{Re} m(ix, y; z) \leq -\varrho \) if \( Q \) satisfies
\[
((X - 2\varrho)^2/(2a)^2] - [Y^2/(3(2a)^2)] \geq 1
\]
where \( X = \text{Re} Q(z) \) and \( Y = \text{Im} Q(z) \).
Since $Q$ satisfies (9), to prove (10) it is enough to show that the point $(X_0, Y_0)$ with $X_0 = 1 - 2M(g + 1)$ and $Y_0 = 2(2 \ln 2 - 1)^2 (g + 1) \tan \theta$ lies on the hyperbola $[(X - 2g)^2/(2g)^2] - [Y^2/(3(2g)^2)] = 1$. Thus by substituting this value in this hyperbola, we get, by a simple calculation,

$$a \varrho^2 + b \varrho + c = 0.$$  

Hence by hypothesis, we deduce that $\text{Re}(ix, y, z) \leq -\varrho$ for all $z \in U$. Now by Lemma A, with $P = \{w \in \mathbb{C} : \text{Re} w > -\varrho\}$, we obtain $\text{Re} p(z) > 0$ in $U$. This completes the proof of Lemma 2.

**Remark 1.** If we let $g'$ and $g''$ be the roots of the quadratic equation (12) then the approximate calculations show that

$g' = (-b - (b^2 - 4ac)^{1/2})/(2a) \approx 0.09032572 \ldots,$

$g'' = (-b + (b^2 - 4ac)^{1/2})/(2a) \approx 1.2113303378 \ldots$

(Here $a \approx 2.071919132 \ldots$, $b \approx -2.701014071$, $c \approx 0.227066802 \ldots$ and $b^2 - 4ac = \frac{80}{81} \tan^2 \theta(2 \ln 2 - 1)^2 + 16(1 - 2M)^2 \approx [2.326718893 \ldots]^2$.)

**Theorem 2.** Let $f$ be as defined in Lemma 2, i.e., $f \approx 0.09032572 \ldots$ and $g \in R(-\varrho)$. Then the Libera transform $G$ defined by (2) is in $S^*$.  

**Proof.** Since $g \in R(-\varrho)$, by using Lemma 1 we obtain $G \in R(\beta)$ with

$$\beta = -\varrho + (1 + \varrho)(3 - 4 \ln 2) = 1 - 2(2 \ln 2 - 1)(\varrho + 1).$$

Now using (4) and the fact that $G \in R(\beta)$ we get

$$\langle G(z)/z \rangle < \beta + (1 - \beta)l(z), \quad z \in U,$$

where $l(z) = -1 - (2/z) \log(1 - z)$. By (13), a simple calculation yields $\beta + (1 - \beta)(2 \ln 2 - 1) = 1 - 2M(1 + \varrho)$. This, from (14) and the observation made earlier, shows that the complex function $Q$ defined by $Q(z) = G(z)/z$ satisfies (9). If we set $p(z) = zG'(z)/G(z)$, by using (2) we obtain

$$zG''(z) + 2G'(z) = 2g'(z).$$

Since $\text{Re} g'(z) > -\varrho$ in $U$, by using (15) we easily get

$$\text{Re}\{Q(z)[zp'(z) + p^2(z) + p(z)]\} > -2\varrho, \quad z \in U,$$

and by Lemma 2 we deduce $\text{Re} p(z) > 0$ in $U$, which shows that $G \in S^*$. Hence the theorem.

The following theorem can be proved along similar lines and so we omit its proof.

**Theorem 3.** If $h \in A$ satisfies $\text{Re}\{h'(z)h(z)/z\} > -\varrho$ in $U$ then the function $H$ defined by $H(z) = \int_0^1 (h(t)/t) \, dt$ is starlike in $U$.

**Remark 2.** In [6], the author showed that for $f \in A$ and $1/6 \leq \beta < 1,$ $\text{Re}[h'(z)h(z)/z] > \beta(3\beta - 1)/2$ implies $\text{Re}(f(z)/z) > \beta$ in $U$. 
Remark 3. For $\alpha \geq 0$ and $\beta < 1$, let $R(\alpha, \beta)$ be the class of functions $f$ in $A$ satisfying $\text{Re}[f'(z) + \alpha zf''(z)] > \beta$ for $z$ in $U$. From a result of Ponnusamy and Karunakaran [5], we have $R(\alpha, \beta) \subset R(\alpha', \beta + (\alpha - \alpha')(1 - \beta)/(2 + \alpha))$ for all $\alpha > \alpha' \geq 0$. This relation and Theorem 2 show that

$$R(\alpha, (-2\rho(2 + \alpha) + 1 - 2\alpha)/5) \subset S^*$$

for all $\alpha \geq 1/2$.

As an immediate consequence of the above observation, we have

Theorem 4. If $f \in A$ satisfies $\text{Re}[f'(z)] > (-2\rho(2 + \alpha) + 1 - 2\alpha)/5$, $z \in U$, for $\alpha \geq 1/2$, then the function $F$ defined by

$$F(z) = \alpha z^{1 - 1/\alpha} \int_0^z f(t)^{1/\alpha - 2} \, dt$$

is in $S^*$.

Corollary. If $f \in A$ satisfies $\text{Re}[f'(z)] > -(6\rho + 1)/5 \approx 0.3083908 \ldots$ for $z$ in $U$, then the function $F$ defined by (1) is starlike in $U$.

The above corollary improves Theorem 1.

Remark 4. For $g$ defined by $g(z) = z(2 + z)/2(1 - z)$ (and hence $g$ satisfies $\text{Re}[zg'(z)/g(z)] > -1/2$ in $U$) it is well known that the corresponding Libera transform $G$ is starlike in $U$. On the other hand, a simple calculation shows that $g \in R(-1/8)$. Hence the natural problem which arises is to find the best possible $\rho'$ ($> \rho$) such that $g \in R(-\rho')$ implies $G$ is starlike in $U$.

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References


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