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On starlikeness of certain integral transforms

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Abstract. Let A denote the class of normalized analytic functions in the unit disc $U = \{z : |z| < 1\}$. The author obtains fixed values of δ and ϱ ($\delta \approx 0.308390864\ldots$, $\varrho \approx 0.0903572\ldots$) such that the integral transforms F and G defined by

$$F(z) = \int_0^z (f(t)/t)\,dt \quad \text{and} \quad G(z) = (2/z)\int_0^z g(t)\,dt$$

are starlike (univalent) in U, whenever $f \in A$ and $g \in A$ satisfy $\operatorname{Re} f'(z) > -\delta$ and $\operatorname{Re} g'(z) > -\varrho$ respectively in U.

1. Introduction. Let A denote the class of analytic functions f defined in the unit disc $U = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Let S^* be the usual class of starlike (univalent) functions in U, i.e.

$$S^* = \{f \in A : \operatorname{Re}[zf'(z)/f(z)] > 0,\ z \in U\}$$

and let $R(\beta) = \{f \in A : \operatorname{Re} f'(z) > \beta,\ z \in U\}$, $R(0) \equiv R$ $(\beta < 1)$.

In a recent paper [8], Singh and Singh proved that if $f \in R$, then $F \in S^*$, where

$$F(z) = \int_0^z (f(t)/t)\,dt\,, \tag{1}$$

and in [3], Mocanu considered the Libera transform G defined by

$$G(z) = (2/z)\int_0^z g(t)\,dt \tag{2}$$

and showed that $g \in R$ implies $G \in S^*$.

In this note, we improve both the above results by showing that the same conclusions hold under a much weaker condition on f and g respectively.

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2. Preliminaries. We need the following lemma to prove our results.

Lemma A. *Let Ω be a set in the complex plane $\mathbb{C}$. Suppose that the function $m : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfies the condition*

$$m(ix, y; z) \notin \Omega \tag{3}$$

for all real $x, y \le -(1+x^2)/2$ and all $z \in U$. If the function p is analytic in U with $p(0) = 1$ and if $m(p(z), zp'(z); z) \in \Omega$, $z \in U$, then $\operatorname{Re} p(z) > 0$ *in U.*

A more general form of this lemma may be found in [2].

3. Main results. From the result of Hallenbeck and Ruscheweyh [1] (see also [2]) we find that if P is analytic in U with $P(0) = 1$ then

$$\operatorname{Re}[P(z) + zP'(z)] > \beta \text{ implies } P(z) \prec \beta + (1-\beta)l(z),\ z \in U\,, \tag{4}$$

and

$$\operatorname{Re}[P(z) + \tfrac{1}{2}zP'(z)] > \beta \text{ implies } P(z) \prec \beta + (1-\beta)L(z),\ z \in U\,, \tag{5}$$

where $\prec$ stands for usual subordination, and l and L defined by $l(z) = -1-(2/z)\log(1-z)$ and $L(z) = 2((l(z)-1)/z)-1$ are convex (univalent) in U. In view of the fact that the coefficients in the series expansions of l and L are positive, we easily deduce that $\operatorname{Re} l(z) > 2\ln 2-1$ and $\operatorname{Re} L(z) > 3-4\ln 2$ in U. Also it is easily seen that $l(U) \subset \{w \in \mathbb{C} : |\arg w| < \pi/3\}$. Using a result of Mocanu *et al.* [4] we can replace $\pi/3$ by θ, where θ lies between 0.911621904 and 0.911621907. This combined with the result of Robertson [7] yields that $l(U) \subset \Omega_1 \cap \Omega_2 \cap \Omega_3$ with $\Omega_1 = \{w \in \mathbb{C} : \operatorname{Re} w > 2\ln 2 - 1\}$, $\Omega_2 = \{w \in \mathbb{C} : |\arg w| < 0.911621907\}$ and $\Omega_3 = \{w \in \mathbb{C} : |\operatorname{Im} w| < \theta\}$.

Theorem 1. *If $\delta = (2\ln 2-1)(3-2\ln 2)/[3-(2\ln 2-1)(3-2\ln 2)] = 0.262\ldots$ and $f \in R(-\delta)$ then the function F defined by* (1) *is starlike in U.*

Proof. From (1) we deduce

$$zF''(z) + F'(z) = f'(z), \qquad z \in U\,. \tag{6}$$

Let $P(z) = F'(z)$ and $Q(z) = F(z)/z$. Since $\operatorname{Re} f'(z) > -\delta$ in U, by using (4) and (6) we find that $\operatorname{Re} F'(z) > -\delta + (1+\delta)(2\ln 2-1)$ for $z \in U$. Again by using (4) this in turn implies $\operatorname{Re}[Q(z)] > 2\delta > 0$, $z \in U$.

Now if we set $p(z) = zF'(z)/F(z)$ then p is analytic in U, $p(0) = 1$ and $f'(z) = m(p(z), zp'(z); z)$, where $m(u, v; z) = (u^2+v)Q(z)$. To prove $F \in S^*$, it is enough to show $\operatorname{Re} p(z) > 0$ in U. Since $\operatorname{Re} f'(z) > -\delta$ in U, by (6), we get $\operatorname{Re} m(p(z), zp'(z); z) > -\delta$ in U. Now for all real x, $y \le -(1+x^2)/2$ and $z \in U$, we have $\operatorname{Re} m(ix, y; z) = (y-x^2)\operatorname{Re} Q(z) \le -(1+3x^2)\operatorname{Re}(Q(z)/2) \le -\delta$ and so applying Lemma A with $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > -\delta\}$, we get $\operatorname{Re} p(z) > 0$ in U. Hence the theorem.

For the proof of Theorem 2 we prove the following lemmas. Theorem 2 further improves Theorem 1.

LEMMA 1. *If* $g \in R(\beta)$ *then* G *defined by* (2) *belongs to* $R(\beta+(1-\beta)(3-4\ln 2))$ $(\beta < 1)$.

P r o o f. From (2) we deduce

$$zG'(z) + G(z) = 2g(z)\,, \tag{7}$$

$$zG''(z) + 2G'(z) = 2g'(z)\,. \tag{8}$$

Since $g \in R(\delta)$, by using (5), we obtain

$$G'(z) \prec \beta + (1-\beta)L(z), \qquad z \in U\,,$$

and so $\operatorname{Re} G'(z) > \beta + (1-\beta)(3-4\ln 2)$, $z \in U$. Here $L(z)$ is as defined earlier. This proves Lemma 1.

LEMMA 2. *Let* $M = 2(2\ln 2 - 1)(1 - \ln 2)$, $\theta = 0.911621907$, $N = \tan\theta$, $a = 4(1+M)^2 - \frac{4}{3}N^2(2\ln 2-1)^4 - 4$, $b = -4(1-2M)(1+M) - \frac{8}{3}(2\ln 2-1)^4N^2$, $c = (1-2M)^2 - \frac{2}{3}(2\ln 2-1)^2N$ *and* $\varrho = (-b-(b^2-4ac)^{1/2})/(2a)$. *Suppose that* Q *is a complex function with* $Q(0) = 1$ *satisfying*

$$Q(U) \subset E_1 \cap E_2 \cap E_3 \quad \textit{where} \tag{9}$$

$$E_1 = \{w \in \mathbb{C} : \operatorname{Re} w > 1 - 2M(1+\varrho)\}\,,$$

$$E_2 = \{w \in \mathbb{C} : |\arg(w - (1 - 2(2\ln 2 - 1)(\varrho + 1)))| < \theta\}\,,$$

$$E_3 = \{w \in \mathbb{C} : |\operatorname{Im} w| < 2(2\ln 2 - 1)(\varrho + 1)\pi\}\,.$$

If p *is analytic in* U *with* $p(0) = 1$ *and if*

$$\operatorname{Re} Q(z)[zp'(z) + p^2(z) + p(z)] > -2\varrho, \qquad z \in U\,,$$

then $\operatorname{Re} p(z) > 0$ *in* U.

Throughout the paper M, θ, N, a, b, c, ϱ, and E's are all as defined above.

P r o o f o f L e m m a 2. If we let $m(u, v; z) = Q(z)(v + u^2 + u)/2$ then for all x, $y \le -(1 + x^2)/2$ and $z \in U$, we have

$$\begin{aligned}\operatorname{Re} m(ix, y; z) &= [(y - x^2)\operatorname{Re} Q(z) - x \operatorname{Im} Q(z)]/2\\ &\le -[3x^2 \operatorname{Re} Q(z) + 2x \operatorname{Im} Q(z) + \operatorname{Re} Q(z)]/4\,.\end{aligned}$$

Thus $\operatorname{Re} m(ix, y; z) \le -\varrho$ if Q satisfies

$$[(X - 2\varrho)^2/(2\varrho)^2] - [Y^2/(3(2\varrho)^2)] \ge 1 \tag{10}$$

where $X = \operatorname{Re} Q(z)$ and $Y = \operatorname{Im} Q(z)$.

Since Q satisfies (9), to prove (10) it is enough to show that the point (X_0, Y_0) with $X_0 = 1-2M(\varrho+1)$ and $Y_0 = 2(2\ln 2-1)^2(\varrho+1)\tan\theta$ lies on the hyperbola $[(X-2\varrho)^2/(2\varrho)^2] - [Y^2/(3(2\varrho)^2)] = 1$. Thus by substituting this value in this hyperbola, we get, by a simple calculation,

$$a\varrho^2 + b\varrho + c = 0\,. \tag{12}$$

Hence by hypothesis, we deduce that $\operatorname{Re} m(ix, y; z) \le -\varrho$ for all $z \in U$. Now by Lemma A, with $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > -\varrho\}$, we obtain $\operatorname{Re} p(z) > 0$ in U. This completes the proof of Lemma 2.

Remark 1. If we let ϱ' and ϱ'' be the roots of the quadratic equation (12) then the approximate calculations show that

$$\varrho' = (-b - (b^2 - 4ac)^{1/2})/(2a) \approx 0.09032572\ldots,$$
$$\varrho'' = (-b + (b^2 - 4ac)^{1/2})/(2a) \approx 1.2113303378\ldots$$

(Here $a \approx 2.071919132\ldots$, $b \approx -2.701014071$, $c \approx 0.227066802\ldots$ and $b^2 - 4ac = \frac{80}{3}\tan^2\theta(2\ln 2-1)^4 + 16(1-2M)^2 \approx [2.326718893\ldots]^2$.)

THEOREM 2. *Let ϱ be as defined in Lemma 2, i.e., $\varrho \approx 0.09032572\ldots$ and $g \in R(-\varrho)$. Then the Libera transform G defined by (2) is in S^*.*

Proof. Since $g \in R(-\varrho)$, by using Lemma 1 we obtain $G \in R(\beta)$ with

$$\beta = -\varrho + (1+\varrho)(3-4\ln 2) = 1 - 2(2\ln 2 - 1)(\varrho+1)\,. \tag{13}$$

Now using (4) and the fact that $G \in R(\beta)$ we get

$$(G(z)/z) \prec \beta + (1-\beta)l(z), \quad z \in U\,, \tag{14}$$

where $l(z) = -1 - (2/z)\log(1-z)$. By (13), a simple calculation yields $\beta + (1-\beta)(2\ln 2 - 1) = 1 - 2M(1+\varrho)$. This, from (14) and the observation made earlier, shows that the complex function Q defined by $Q(z) = G(z)/z$ satisfies (9). If we set $p(z) = zG'(z)/G(z)$, by using (2) we obtain

$$zG''(z) + 2G'(z) = 2g'(z)\,. \tag{15}$$

Since $\operatorname{Re} g'(z) > -\varrho$ in U, by using (15) we easily get

$$\operatorname{Re}\{Q(z)[zp'(z) + p^2(z) + p(z)]\} > -2\varrho, \quad z \in U\,,$$

and by Lemma 2 we deduce $\operatorname{Re} p(z) > 0$ in U, which shows that $G \in S^*$. Hence the theorem.

The following theorem can be proved along similar lines and so we omit its proof.

THEOREM 3. *If $h \in A$ satisfies $\operatorname{Re}\{h'(z)h(z)/z\} > -\varrho$ in U then the function H defined by $H(z) = \int_0^z (h(t)/t)\,dt$ is starlike in U.*

Remark 2. In [6], the author showed that for $f \in A$ and $1/6 \le \beta < 1$, $\operatorname{Re}[h'(z)h(z)/z] > \beta((3\beta-1)/2)$ implies $\operatorname{Re}(f(z)/z) > \beta$ in U.

Remark 3. For $\alpha \geq 0$ and $\beta < 1$, let $R(\alpha, \beta)$ be the class of functions f in A satisfying $\mathrm{Re}[f'(z) + \alpha z f''(z)] > \beta$ for z in U. From a result of Ponnusamy and Karunakaran [5], we have $R(\alpha, \beta) \subset R(\alpha', \beta + (\alpha - \alpha')(1 - \beta)/(2+\alpha))$ for all $\alpha > \alpha' \geq 0$. This relation and Theorem 2 show that

$$R(\alpha, (-2\varrho(2+\alpha) + 1 - 2\alpha)/5) \subset S^* \quad \text{for all } \alpha \geq 1/2\,.$$

As an immediate consequence of the above observation, we have

THEOREM 4. *If* $f \in A$ *satisfies* $\mathrm{Re}[f'(z)] > (-2\varrho(2+\alpha) + 1 - 2\alpha)/5$, $z \in U$, *for* $\alpha \geq 1/2$, *then the function* F *defined by*

$$F(z) = \alpha z^{1-1/\alpha} \int_o^z f(t) t^{1/\alpha - 2}\, dt$$

is in S^*.

COROLLARY. *If* $f \in A$ *satisfies* $\mathrm{Re}\, f'(z) > -(6\varrho + 1)/5 \approx 0.3083908\ldots$ *for* z *in* U, *then the function* F *defined by* (1) *is starlike in* U.

The above corollary improves Theorem 1.

Remark 4. For g defined by $g(z) = z(2+z)/2(1-z)$ (and hence g satisfies $\mathrm{Re}[zg'(z)/g(z)] > -1/2$ in U) it is well known that the corresponding Libera transform G is starlike in U. On the other hand, a simple calculation shows that $g \in R(-1/8)$. Hence the natural problem which arises is to find the best possible ϱ' $(> \varrho)$ such that $g \in R(-\varrho')$ implies G is starlike in U.

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