Classical solutions of hyperbolic partial differential equations with implicit mixed derivative

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Abstract. Let f be a continuous function from $[0,a] \times [0,\beta] \times (\mathbb{R}^n)^4$ into $\mathbb{R}^n$. Given $u_0, v_0 \in C^0([0,\beta], \mathbb{R}^n)$, with

$$f\left(0, x, \int_0^x u_0(s)\,ds, \int_0^x v_0(s)\,ds, u_0(x), v_0(x)\right) = v_0(x)$$

for every $x \in [0,\beta]$, consider the problem

(P)
$$\begin{cases} \dfrac{\partial^2 z}{\partial t \partial x} = f\left(t, x, z, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial t \partial x}\right), \\[2mm] z(t,0) = \vartheta_{\mathbb{R}^n}, \\[2mm] z(0,x) = \displaystyle\int_0^x u_0(s)\,ds, \\[2mm] \dfrac{\partial^2 z(0,x)}{\partial t \partial x} = v_0(x). \end{cases}$$

In this paper we prove that, under suitable assumptions, problem (P) has at least one classical solution that is local in the first variable and global in the other. As a consequence, we obtain a generalization of a result by P. Hartman and A. Wintner ([4], Theorem 1).

Introduction. Let a, β be two positive real numbers; n a positive integer; $\mathbb{R}^n$ the real Euclidean n-space, whose null element is denoted by $\vartheta_{\mathbb{R}^n}$; $f(t, x, z, z_1, z_2, z_3)$ a continuous function from $[0,a] \times [0,\beta] \times (\mathbb{R}^n)^4$ into $\mathbb{R}^n$.

Given $u_0, v_0 \in C^0([0,\beta], \mathbb{R}^n)$, with

$$f\left(0, x, \int_0^x u_0(s)\,ds, \int_0^x v_0(s)\,ds, u_0(x), v_0(x)\right) = v_0(x)$$

for every $x \in [0, \beta]$, consider the problem

(P)
$$\begin{cases} \dfrac{\partial^2 z}{\partial t \partial x} = f\left(t, x, z, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial t \partial x}\right), \\[2mm] z(t, 0) = \vartheta_{\mathbb{R}^n}, \\[2mm] z(0, x) = \displaystyle\int\limits_0^x u_0(s) \, ds, \\[2mm] \dfrac{\partial^2 z(0, x)}{\partial t \partial x} = v_0(x). \end{cases}$$

A function $z : [0, a] \times [0, \beta] \to \mathbb{R}^n$ is said to be a *classical solution* of (P) if $z, \partial z / \partial t, \partial z / \partial x, \partial^2 z / \partial t \partial x \in C^0([0, a] \times [0, \beta], \mathbb{R}^n)$ and, for every $(t, x) \in [0, a] \times [0, \beta]$, one has $\partial^2 z(t, x) / \partial t \partial x = f(t, x, z(t, x), \partial z(t, x) / \partial t, \partial z(t, x) / \partial x, \partial^2 z(t, x) / \partial t \partial x)$, $z(t, 0) = \vartheta_{\mathbb{R}^n}$, $z(0, x) = \int_0^x u_0(s) \, ds$, $\partial^2 z(0, x) / \partial t \partial x = v_0(x)$.

In this paper we prove that, under suitable assumptions, problem (P) has at least one classical solution that is local in the first variable and global in the other (see Theorems 2.1 and 2.2). Further, as a simple consequence of Theorem 2.2, we obtain a result (Theorem 2.3) which improves, in some directions, the well-known result by P. Hartman and A. Wintner (see [4], Theorem 1). For instance, it is worth noticing that the hypotheses of Theorem 2.3 on f do not imply that the function $(z_1, z_2, z_3) \to f(t, x, z, z_1, z_2, z_3)$ is uniformly Lipschitzian, with Lipschitz constant strictly less than one with respect to z_3.

As far as we know, this seems to be the first contribution to the study of hyperbolic partial differential equations, with implicit mixed derivative, in this direction.

The main tool we use in order to get our results is a recent existence theorem for implicit ordinary differential equations in a Banach space, namely Theorem 2.1 of [3].

1. Preliminaries. Let (X, d) be a metric space. For every $x \in X$ and every $r > 0$, we put $B(x, r) = \{z \in X : d(x, z) \leq r\}$ and $B(x, +\infty) = X$. Let V be a nonempty subset of X and let Ω be a bounded subset of V. The *Hausdorff measure of noncompactness* of Ω with respect to V is the following number:

$$\gamma_V(\Omega) = \inf\left\{r > 0 : \exists x_1, \ldots, x_k \in V, k \in \mathbb{N} : \Omega \subseteq \bigcup_{i=1}^k B(x_i, r)\right\}.$$

If $V = X$, we put $\gamma_V(\Omega) = \gamma(\Omega)$. It is easy to verify that one has

(1) $$\gamma(\Omega) \leq \gamma_V(\Omega) \leq 2\gamma(\Omega).$$

Let n be a positive integer and let $\mathbb{R}^n$ be the real Euclidean n-space, endowed with the norm $\|z\| = \max_{1 \le i \le n} |z_i|$, where $z = (z_1, ..., z_n) \in \mathbb{R}^n$. If I is a compact real interval, we denote by $C^0(I, \mathbb{R}^n)$ the space of all continuous functions from I into $\mathbb{R}^n$, endowed with the norm $\|u\|_{C^0(I,\mathbb{R}^n)} = \max_{x \in I} \|u(x)\|$, and by $C^1(I, \mathbb{R}^n)$ the space of all functions $u : I \to \mathbb{R}^n$ such that $u, du/dx \in C^0(I, \mathbb{R}^n)$.

On $C^1(I, \mathbb{R}^n)$ we consider the norm $\|u\|_{C^1(I,\mathbb{R}^n)} = \|u\|_{C^0(I,\mathbb{R}^n)} + \|du/dx\|_{C^0(I,\mathbb{R}^n)}$. Of course, the space $(C^1(I, \mathbb{R}^n), \| \cdot \|_{C^1(I,\mathbb{R}^n)})$ is complete.

For every $u \in C^0(I, \mathbb{R}^n)$, every nonempty subset U of $C^0(I, \mathbb{R}^n)$ and every $\sigma > 0$, we put

$$\omega(u, \sigma) = \sup\{\|u(x') - u(x'')\| : x', x'' \in I, \ |x' - x''| \le \sigma\} \,;$$
$$\omega(U, \sigma) = \sup_{u \in U} \omega(u, \sigma); \quad \omega_0(U) = \lim_{\sigma \to 0^+} \omega(U, \sigma) \,.$$

If U is bounded then, thanks to Theorem 7.1.2 of [1], one has

$$\omega_0(U) = 2\gamma(U) \,. \tag{2}$$

Let $Q \subseteq \mathbb{R}^2$ be a rectangle. We denote by $C^0(Q, \mathbb{R}^n)$ the space of all continuous functions from Q into $\mathbb{R}^n$ and by $E(Q, \mathbb{R}^n)$ the space of all functions $z(t, x) : Q \to \mathbb{R}^n$ such that $z, \partial z/\partial t, \partial z/\partial x, \partial^2 z/\partial t \partial x \in C^0(Q, \mathbb{R}^n)$.

In the sequel, we will apply the following lemma, whose simple proof is left to the reader.

LEMMA 1.1. *Let I be a compact real interval, J a real interval, and $C^1(J, C^1(I, \mathbb{R}^n))$ the space of all continuously differentiable functions from J into $C^1(I, \mathbb{R}^n)$. Then a function $w : J \to C^1(I, \mathbb{R}^n)$ belongs to $C^1(J, C^1(I, \mathbb{R}^n))$ if and only if the function $\widetilde{w} : J \times I \to \mathbb{R}^n$ defined by putting, for every $(t, x) \in J \times I, \widetilde{w}(t, x) = w(t)(x)$, belongs to $E(J \times I, \mathbb{R}^n)$.*

For the reader's convenience, we report now the statement of Theorem 2.1 of [3], which will be used in the sequel.

THEOREM 1.1. *Let $(B, \| \cdot \|)$ be a real or complex Banach space, whose null element is denoted by ϑ_B; $u_0, v_0 \in B$; $t_0 \in \mathbb{R}$; $a, b, c \in \mathbb{R}^+ \cup \{+\infty\}$; R the set $\{(t, u, v) \in \mathbb{R} \times B \times B : t_0 \le t \le t_0 + a, \|u - u_0\| \le b, \|v - v_0\| \le c\}$; $F(t, u, v)$ a function from R into B such that $F(t_0, u_0, v_0) = \vartheta_B$; $T(w)$ a function from B into itself such that $T(w) = \vartheta_B$ if and only if $w = \vartheta_B$; $G(t, u, v) = v + T(F(t, u, v))$ for every $(t, u, v) \in R$. Assume that:*

1) $A \ne \emptyset$ where A is the set of all $\bar{t} \in \mathbb{R} \cap \,]t_0, t_0 + a]$ for which there exists a function $\bar{d} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\liminf_{\varepsilon \to 0^+} \bar{d}(\varepsilon) = 0$, such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t', t'' \in [t_0, \bar{t}]$, $u', u'' \in B(u_0, b)$, $v', v'' \in B(v_0, c)$, and $|t' - t''| < \delta$, $\|u' - u''\| < \delta$, $\|v' - v''\| \le \bar{d}(\varepsilon)$ then

$$\|G(t', u', v') - G(t'', u'', v'')\| \le \bar{d}(\varepsilon) \,;$$

2) *if $c < +\infty$ one has $\|G(t, u, v) - v_0\| \le c$ for every $(t, u, v) \in R$ with $t \in A \cup \{t_0\}$, whereas if $c = +\infty$ there exists a continuous function $M(t) : A \cup \{t_0\} \to \mathbb{R}_0^+$ such that $\|G(t, u, v)\| \le M(t)(1 + \lambda\|u - u_0\|)$ for every $(t, u, v) \in R$ with $t \in A \cup \{t_0\}$, where*

$$\lambda = \begin{cases} 0 & \text{if } b < +\infty, \\ 1 & \text{if } b = +\infty; \end{cases}$$

3) *if $t^* = \sup A$ or t^* is a point of $[t_0, \sup A]$ such that $t^* - t_0 \le b/(\|v_0\| + c)$ or such that $\int_{t_0}^{t^*} M(t)\,dt \le b$, according to whether $b = +\infty$, or $b, c < +\infty$ or $b < +\infty$ and $c = +\infty$, and if*

$$A^* = \begin{cases} [t_0, t^*] & \text{if } t^* \in A, \\ [t_0, t^*[& \text{if } t^* \notin A, \end{cases}$$

then there exists a function $w(t, u, v) : A^* \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, nondecreasing with respect to u, and a number $\varrho > 0$ such that

(h_1) *for every $\bar{t} \in A^* \setminus \{t_0, t^*\}$ the conditions $v : [t_0, \bar{t}] \to \mathbb{R}$, v continuous, $0 \le v(t) < \varrho, v(t) \le w(t, \int_{t_0}^t v(\tau)\,d\tau, v(t))$ for each $t \in [t_0, \bar{t}]$, $v(t_0) = 0$ imply $v(t) = 0$ for every $t \in [t_0, \bar{t}]$;*

(h_2) *for every $t \in A^* \setminus \{t_0, t^*\}$, $U \subseteq B(u_0, b)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$, one has*

$$\gamma_{B(v_0,c)}(G(t, U, V)) \le w(t, \gamma(U), \gamma_{B(v_0,c)}(V)).$$

Then there exists a continuously differentiable function $\xi : A^* \to B$ such that $F(t, \xi(t), d\xi(t)/dt) = \vartheta_B$ for every $t \in A^*$ and $\xi(t_0) = u_0$, $d\xi(t_0)/dt = v_0$.

2. Results. Let a, β, b be three positive real numbers; $c \in \mathbb{R}^+ \cup \{+\infty\}$; $u_0, v_0 \in C^0([0, \beta], \mathbb{R}^n)$; $\Delta(a, \beta, b, c)$ the set $\{(t, x, z, z_1, z_2, z_3) \in [0, a] \times [0, \beta] \times (\mathbb{R}^n)^4 : \|z\| \le \beta(\|u_0\|_{C^0([0,\beta],\mathbb{R}^n)} + b), \|z_1\| \le \beta(\|v_0\|_{C^0([0,\beta],\mathbb{R}^n)} + c), \|z_2\| \le \beta\|u_0\|_{C^0([0,\beta],\mathbb{R}^n)} + b, \|z_3\| \le \beta\|v_0\|_{C^0([0,\beta],\mathbb{R}^n)} + c\}$; f a continuous function from $[0, a] \times [0, \beta] \times (\mathbb{R}^n)^4$ into $\mathbb{R}^n$ such that

$$(3) \qquad f\left(0, x, \int_0^x u_0(s)\,ds, \int_0^x v_0(s)\,ds, u_0(x), v_0(x)\right) = v_0(x)$$

for every $x \in [0, \beta]$. Our first result is the following:

THEOREM 2.1. *Assume that*:

(i) *there exists a function $d : \mathbb{R}^+ \to \mathbb{R}^+$, with $\liminf_{\varepsilon \to 0^+} d(\varepsilon) = 0$, such that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $t', t'' \in [0, a]$, $z', z'', z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2, 3$, and $|t' - t''| < \delta$, $\|z' - z''\| < \beta\delta$, $\|z_1' - z_1''\| \le \beta d(\varepsilon)$, $\|z_2' - z_2''\| < \delta$, $\|z_3' - z_3''\| \le d(\varepsilon)$ then*

$$\|f(t', x, z', z_1', z_2', z_3') - f(t'', x, z'', z_1'', z_2'', z_3'')\| \le d(\varepsilon)$$

for every $x \in [0, \beta]$;

(ii) *if* $c < +\infty$ *one has* $\|f(t, x, z, z_1, z_2, z_3) - v_0(x)\| \leq c$ *for every* $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, b, c)$, *whereas if* $c = +\infty$ *the function* $f(t, x, z, z_1, z_2, z_3)$ *is bounded on* $\Delta(a, \beta, b, c)$;

(iii) *if* M *is a positive constant such that* $\|f(t, x, z, z_1, z_2, z_3)\| \leq M$ *for every* $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, b, c)$ *and* $\delta^* = \min(a, b/M)$, *there exist a function* $w(t, u, v) : [0, \delta^* [\times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, *nondecreasing with respect to* u, *and a number* $\varrho > 0$ *such that*

1) *for every* $\bar{t} \in]0, \delta^*[$ *the conditions* $v : [0, \bar{t}] \to \mathbb{R}$, v *continuous*, $0 \leq v(t) < \varrho$, $v(t) \leq w(t, \int_0^t v(\tau) \, d\tau, v(t))$ *for each* $t \in [0, \bar{t}], v(0) = 0$ *imply* $v(t) = 0$ *for every* $t \in [0, \bar{t}]$;

2) *for every* $t \in]0, \delta^*[$, $U \subseteq B(u_0, b)$, $V \subseteq B(v_0, c)$, *with* $\gamma(U) < \varrho$ *and* $0 < \gamma(V) < \varrho$, *one has*

$$(4) \quad \mu \lim_{\sigma \to 0^+} \sup_{(u,v) \in U \times V} \sup \left\{ \left\| f\left(t, x', \int_0^{x'} u(s) \, ds, \int_0^{x'} v(s) \, ds, u(x'), v(x')\right) \right. \right.$$

$$\left. - f\left(t, x'', \int_0^{x''} u(s) \, ds, \int_0^{x''} v(s) \, ds, u(x''), v(x'')\right) \right\| :$$

$$x', x'' \in [0, \beta], \ |x' - x''| \leq \sigma \Big\} \leq w(t, \gamma(U), \gamma_{B(v_0,c)}(V)),$$

where

$$\mu = \begin{cases} 1 & \text{if } c < +\infty, \\ 1/2 & \text{if } c = +\infty. \end{cases}$$

Then there exists at least one solution of the problem (P) *in the class* $E([0, \delta^*] \times [0, \beta], \mathbb{R}^n)$.

P r o o f. Let B be the Banach space $(C^0([0, \beta], \mathbb{R}^n), \| \cdot \|_{C^0([0,\beta], \mathbb{R}^n)})$.

To simplify notation, we write $\| \cdot \|_B$ for $\| \cdot \|_{C^0([0,\beta], \mathbb{R}^n)}$. For every $(t, u, v) \in [0, a] \times B(u_0, b) \times B(v_0, c)$ and every $x \in [0, \beta]$, put

$$F(t, u, v)(x) = f\left(t, x, \int_0^x u(s) \, ds, \int_0^x v(s) \, ds, u(x), v(x)\right) - v(x).$$

The function F so defined takes values in B and, thanks to (3), one has $F(0, u_0, v_0) = \vartheta_B$, where ϑ_B is the null element of B. Now, let T be the identity operator on B. We prove that (i)$\Rightarrow$1) of Theorem 1.1, with $A =]0, a]$ and $\bar{d} = d$ for every $\bar{t} \in]0, a]$. Fix $\varepsilon > 0$ and observe that from our assumptions it follows that

$$(5) \qquad G(t, u, v)(x) = f\left(t, x, \int_0^x u(s) \, ds, \int_0^x v(s) \, ds, u(x), v(x)\right)$$

for every $(t, u, v) \in [0, a] \times B(u_0, b) \times B(v_0, c)$ and every $x \in [0, \beta]$. If $(t', u', v'), (t'', u'', v'') \in [0, a] \times B(u_0, b) \times B(v_0, c)$ and $|t' - t''| < \delta$, $\|u' - u''\|_B < \delta$, $\|v' - v''\|_B \leq d(\varepsilon)$ then, for every fixed $x \in [0, \beta]$, one has

$$\Big\| \int_0^x u'(s)\, ds - \int_0^x u''(s)\, ds \Big\| \leq \beta \|u' - u''\|_B < \beta\delta \,,$$

$$\Big\| \int_0^x v'(s)\, ds - \int_0^x v''(s)\, ds \Big\| \leq \beta \|v' - v''\|_B \leq \beta d(\varepsilon) \,,$$

$$\|u'(x) - u''(x)\| \leq \|u' - u''\|_B < \delta \,,$$
$$\|v'(x) - v''(x)\| \leq \|v' - v''\|_B \leq d(\varepsilon) \,.$$

Hence, by (i) and (5),

$$\|G(t', u', v')(x) - G(t'', u'', v'')(x)\| \leq d(\varepsilon)$$

for every $x \in [0, \beta]$. This implies that

$$\|G(t', u', v') - G(t'', u'', v'')\|_B \leq d(\varepsilon) \,.$$

It is trivial to check that (ii)$\Rightarrow$2) of Theorem 1.1. Let us prove that (iii)$\Rightarrow$3) of Theorem 1.1. To this end, it is enough to verify that (h_2) of Theorem 1.1 holds. Fix $t \in]0, \delta^*[$, $U \subseteq B(u_0, b)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$. If $c < +\infty$, then, by (1), (2) and 2), one has

$$\gamma_{B(v_0,c)}(G(t, U, V)) \leq 2\gamma(G(t, U, V))$$

$$= \lim_{\sigma \to 0^+} \sup_{(u,v) \in U \times V} \sup \Big\{ \Big\| f\Big(t, x', \int_0^{x'} u(s)\, ds, \int_0^{x'} v(s)\, ds, u(x'), v(x')\Big)$$

$$- f\Big(t, x'', \int_0^{x''} u(s)\, ds, \int_0^{x''} v(s)\, ds, u(x''), v(x'')\Big)\Big\| :$$

$$x', x'' \in [0, \beta], |x' - x''| \leq \sigma \Big\} \leq w(t, \gamma(U), \gamma_{B(v_0,c)}(V)) \,.$$

If $c = +\infty$ then, by (2) and 2), one has

$$\gamma(G(t, U, V)) =$$

$$\frac{1}{2} \lim_{\sigma \to 0^+} \sup_{(u,v) \in U \times V} \sup \Big\{ \Big\| f\Big(t, x', \int_0^{x'} u(s)\, ds, \int_0^{x'} v(s)\, ds, u(x'), v(x')\Big)$$

$$- f\Big(t, x'', \int_0^{x''} u(s)\, ds, \int_0^{x''} v(s)\, ds, u(x''), v(x'')\Big)\Big\| :$$

$$x', x'' \in [0, \beta], |x' - x''| \leq \sigma \Big\} \leq w(t, \gamma(U), \gamma(V)) \,.$$

At this point, we are able to apply Theorem 1.1. By that result, there exists $\xi \in C^1([0, \delta^*], B)$ such that $F(t, \xi(t), d\xi(t)/dt) = \vartheta_B$ for every $t \in [0, \delta^*], \xi(0) = u_0, d\xi(0)/dt = v_0$. Put, for every $(t, x) \in [0, \delta^*] \times [0, \beta]$,

$$z(t, x) = \int_0^x \xi(t)(s)\, ds\,.$$

Thanks to Lemma 1.1, the function $z : [0, \delta^*] \times [0, \beta] \to \mathbb{R}^n$ so defined belongs to $E([0, \delta^*] \times [0, \beta], \mathbb{R}^n)$ and, for each $(t, x) \in [0, \delta^*] \times [0, \beta]$, one has

$$\frac{\partial^2 z(t, x)}{\partial t \partial x} = \frac{d\xi(t)}{dt}(x)$$

$$= f\left(t, x,\ \int_0^x \xi(t)(s)\, ds,\ \int_0^x \frac{d\xi(t)}{dt}(s)\, ds, \xi(t)(x), \frac{d\xi(t)}{dt}(x)\right)$$

$$= f\left(t, x, z(t, x), \frac{\partial z(t, x)}{\partial t}, \frac{\partial z(t, x)}{\partial x}, \frac{\partial^2 z(t, x)}{\partial t \partial x}\right),$$

$$z(t, 0) = \vartheta_{\mathbb{R}^n}\,, \quad z(0, x) = \int_0^x \xi(0)(s)\, ds = \int_0^x u_0(s)\, ds\,,$$

$$\frac{\partial^2 z(0, x)}{\partial t \partial x} = \xi(0)(x) = v_0(x)\,.$$

This completes the proof. ∎

R e m a r k 2.1. Let f satisfy the following assumption:

(j) *for every $t \in [0, a]$ the function $(x, z, z_1, z_2, z_3) \to f(t, x, z, z_1, z_2, z_3)$ is uniformly continuous.*

For every $t \in [0, a]$ and every $\sigma > 0$, put

$$\omega_t(f, \sigma) = \sup \Big\{ \|f(t, x', z', z_1', z_2', z_3') - f(t, x'', z'', z_1'', z_2'', z_3'')\| :$$

$$x', x'' \in [0, \beta], z', z'', z_i', z_i'' \in \mathbb{R}^n\,, \ i = 1, 2, 3,$$

$$\Big((x' - x'')^2 + \|z' - z''\|^2 + \sum_{i=1}^3 \|z_i' - z_i''\|^2\Big)^{1/2} < \sigma \Big\}\,.$$

It is easy to check that, for each $t \in [0, a]$, the function $\omega_t(f, \cdot)$ is nondecreasing, continuous on $\mathbb{R}^+$ and such that $\lim_{\sigma \to 0+} \omega_t(f, \sigma) = 0$. A simple sufficient condition in order that 2) of (iii) of Theorem 2.1 holds is the following:

(jj) *for every $t \in\,]0, \delta^*[,\ U \subseteq B(u_0, b),\ V \subseteq B(v_0, c),\ \text{with } \gamma(U) < \varrho \text{ and } 0 < \gamma(V) < \varrho, \text{ one has}$*

$$\mu \omega_t(f, 2\sqrt{\gamma(U)^2 + \gamma_{B(v_0, c)}(V)^2}) \le w(t, \gamma(U), \gamma_{B(v_0, c)}(V))\,,$$

where

$$\mu = \begin{cases} 1 & \text{if } c < +\infty, \\ 1/2 & \text{if } c = +\infty. \end{cases}$$

P r o o f. Fix $t \in \,]0, \delta^*[$, $U \subseteq B(u_0, b)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$. If $\sigma > 0$, $(u, v) \in U \times V$, $x', x'' \in [0, \beta]$ and $|x' - x''| \le \sigma$, then

$$\left\| f\left(t, x', \int\limits_0^{x'} u(s)\, ds, \int\limits_0^{x'} v(s)\, ds, u(x'), v(x')\right) \right.$$

$$\left. - f\left(t, x'', \int\limits_0^{x''} u(s)\, ds, \int\limits_0^{x''} v(s)\, ds, u(x''), v(x'')\right) \right\|$$

$$\le \omega_t(f, \sqrt{(1 + \|u\|_B^2 + \|v\|_B^2)\sigma^2 + \omega(u, \sigma)^2 + \omega(v, \sigma)^2}\,).$$

(we still write B for $C^0([0, \beta], \mathbb{R}^n)$).

Taking into account that $\omega_t(f, \cdot)$ is nondecreasing, this implies that

$$\sup_{(u,v)\in U\times V} \sup \left\{ \left\| f\left(t, x', \int\limits_0^{x'} u(s)\, ds, \int\limits_0^{x'} v(s)\, ds, u(x'), v(x')\right) \right. \right.$$

$$\left. -f\left(t, x'', \int\limits_0^{x''} u(s)\, ds, \int\limits_0^{x''} v(s)\, ds, u(x''), v(x'')\right) \right\| :$$

$$\left. x', x'' \in [0, \beta], |x' - x''| \le \sigma \right\}$$

$$\le \omega_t(f, \sqrt{(1 + \sup_{u\in U}\|u\|_B^2 + \sup_{v\in V}\|v\|_B^2)\sigma^2 + \omega(U, \sigma)^2 + \omega(V, \sigma)^2})$$

for every $\sigma > 0$. Hence, $\omega_t(f, \cdot)$ being continuous,

$$\lim_{\sigma\to 0^+} \sup_{(u,v)\in U\times V} \sup \left\{ \left\| f\left(t, x', \int\limits_0^{x'} u(s)\, ds, \int\limits_0^{x'} v(s)\, ds, u(x'), v(x')\right) \right. \right.$$

$$\left. -f\left(t, x'', \int\limits_0^{x''} u(s)\, ds, \int\limits_0^{x''} v(s)\, ds, u(x''), v(x'')\right) \right\| :$$

$$x', x'' \in [0, \beta], |x' - x''| \le \sigma \left. \right\} \le \omega_t(f, \sqrt{\omega_0(U)^2 + \omega_0(V)^2}\,).$$

Now, the conclusion follows at once from (1), (2) and (jj). ∎

Now, assume that $a, b, c \in \mathbb{R}^+ \cup \{+\infty\}$ and put

$$A_1 = \begin{cases} [0, a] & \text{if } a < +\infty, \\ [0, a[& \text{if } a = +\infty. \end{cases}$$

Let f be a continuous function from $A_1 \times [0, \beta] \times (\mathbb{R}^n)^4$ into $\mathbb{R}^n$ and let $u_0, v_0 \in C^0([0, \beta], \mathbb{R}^n)$ such that (3) holds. Arguing as in the proof of the previous theorem, it is possible to verify the following

THEOREM 2.2. *Suppose that*:

(i_1) *$A \neq \emptyset$ where A is the set of all $\bar{t} \in A_1 \setminus \{0\}$ for which there exists a function $\bar{d} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\liminf_{\varepsilon \to 0^+} \bar{d}(\varepsilon) = 0$, such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $t', t'' \in [0, \bar{t}]$, $z', z'', z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2, 3$, and $|t' - t''| < \delta$, $\|z' - z''\| < \beta\delta$, $\|z_1' - z_1''\| \leq \beta\bar{d}(\varepsilon)$, $\|z_2' - z_2''\| < \delta$, $\|z_3' - z_3''\| \leq \bar{d}(\varepsilon)$ then*

$$\|f(t', x, z', z_1', z_2', z_3') - f(t'', x, z'', z_1'', z_2'', z_3'')\| \leq \bar{d}(\varepsilon)$$

for every $x \in [0, \beta]$;

(i_2) *if $c < +\infty$ one has $\|f(t, x, z, z_1, z_2, z_3) - v_0(x)\| \leq c$ for every $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, b, c)$, with $t \in A \cup \{0\}$, whereas if $c = +\infty$ there exists a continuous function $M(t) : A \cup \{0\} \to \mathbb{R}_0^+$ such that*

$$\|f(t, x, z, z_1, z_2, z_3)\| \leq M(t)(1 + \lambda\|z_2 - u_0(x)\|)$$

for every $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, b, c)$ with $t \in A \cup \{0\}$, where

$$\lambda = \begin{cases} 0 & \text{if } b < +\infty, \\ 1 & \text{if } b = +\infty; \end{cases}$$

(i_3) *if $t^* = \sup A$ or t^* is a point of $[0, \sup A]$ such that $t^* \leq b/(\|v_0\|_{C^0([0,\beta], \mathbb{R}^n)} + c)$ or such that $\int_0^{t^*} M(t)\, dt \leq b$, according to whether $b = +\infty$, or $b, c < +\infty$ or $b < +\infty$ and $c = +\infty$, and if*

$$A^* = \begin{cases} [0, t^*] & \text{if } t^* \in A, \\ [0, t^*[& \text{if } t^* \notin A, \end{cases}$$

then there exist a function $w(t, u, v) : A^* \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, nondecreasing with respect to u, and a number $\varrho > 0$ such that (h_1) of Theorem 1.1 holds (with $t_0 = 0$) and for every $t \in]0, t^*[$, $U \subseteq B(u_0, b)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$, (4) holds.

Then there exists at least one solution of the problem (P) *in the class $E(A^* \times [0, \beta], \mathbb{R}^n)$.*

A remarkable particular case of Theorem 2.2 is the following.

THEOREM 2.3. *Assume that $a < +\infty$, $b = +\infty$ and* (i) *of Theorem 2.1 holds. Moreover, suppose that*:

(j_1) *if $c < +\infty$ one has $\|f(t, x, z, z_1, z_2, z_3) - v_0(x)\| \leq c$ for every $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, +\infty, c)$, if $c = +\infty$ there exists a continuous*

function $M(t) : [0, a] \to \mathbb{R}_0^+$ such that $\|f(t, x, z, z_1, z_2, z_3)\| \leq M(t)(1 + \|z_2 - u_0(x)\|)$ for every $(t, x, z, z_1, z_2, z_3) \in \Delta(a, \beta, +\infty, +\infty)$;

(j_2) there exist a function $w(t, u, v) : [0, a] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, nondecreasing with respect to u, and a number $\varrho > 0$ such that

1) for every $\bar{t} \in]0, a[$ the conditions $v : [0, \bar{t}] \to \mathbb{R}$, v continuous, $0 \leq v(t) < \varrho$, $v(t) \leq w(t, \int_0^t v(\tau) \, d\tau, v(t))$ for each $t \in [0, \bar{t}]$, $v(0) = 0$ imply $v(t) = 0$ for every $t \in [0, \bar{t}]$;

2) for every $t \in]0, a[$, $U \subseteq C^0([0, \beta], \mathbb{R}^n)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$, (4) holds.

Then there exists at least one solution of the problem (P) in the class $E([0, a] \times [0, \beta], \mathbb{R}^n)$.

Theorem 2.3 contains, as a particular case, the following well-known result by P. Hartman and A. Wintner (see [4], Theorem 1):

THEOREM A. Let $f(t, x, z, z_1, z_2)$ be a continuous function from $[0, a] \times [0, \beta] \times (\mathbb{R}^n)^3$ into $\mathbb{R}^n$ and let $\varphi \in C^1([0, a], \mathbb{R}^n)$, $\psi \in C^1([0, \beta], \mathbb{R}^n)$ such that $\varphi(0) = \psi(0)$. Assume that:

(a_1) there exists a positive constant r such that $\|f(t, x, z, z_1, z_2)\| \leq r$ for every $(t, x) \in [0, a] \times [0, \beta]$ and every $z, z_1, z_2 \in \mathbb{R}^n$;

(a_2) there exist $L_1, L_2 \geq 0$ such that, for every $(t, x) \in [0, a] \times [0, \beta]$ and every $z, z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2$, one has

$$\|f(t, x, z, z_1', z_2') - f(t, x, z, z_1'', z_2'')\| \leq L_1 \|z_1' - z_1''\| + L_2 \|z_2' - z_2''\|.$$

Then the Darboux problem

(DP)
$$\begin{cases} \dfrac{\partial^2 z}{\partial t \partial x} = f\left(t, x, z, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial x}\right), \\ z(t, 0) = \varphi(t), \\ z(0, x) = \psi(x), \end{cases}$$

has at least one solution in the class $E([0, a] \times [0, \beta], \mathbb{R}^n)$.

To verify our assertion, observe first that a function $z \in E([0, a] \times [0, \beta], \mathbb{R}^n)$ is a solution of (DP) if and only if there exists a solution $w \in E([0, a] \times [0, \beta], \mathbb{R}^n)$ of the problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t \partial x} = f_1\left(t, x, w, \dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}\right), \\ w(t, 0) = \vartheta_{\mathbb{R}^n}, \\ w(0, x) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

where $f_1(t, x, w, w_1, w_2) = f(t, x, w + \varphi(t) + \psi(x) - \varphi(0), w_1 + d\varphi(t)/dt,$

$w_2 + d\psi(x)/dx), (t, x) \in [0, a] \times [0, \beta], w, w_1, w_2 \in \mathbb{R}^n$, such that $z(t, x) = \varphi(t) + \psi(x) - \varphi(0) + w(t, x)$ for every $(t, x) \in [0, a] \times [0, \beta]$. Since f_1 satisfies (a$_1$) and (a$_2$), we may assume that $\varphi(t) = \vartheta_{\mathbb{R}^n}$, $\psi(x) = \vartheta_{\mathbb{R}^n}$ for every $(t, x) \in [0, a] \times [0, \beta]$. Moreover, it is easy to check that (3) holds.

Now, observe that, thanks to our assumptions, the problem (DP) is equivalent to the following integral equation:

$$y(t, x) = f\left(t, x, \int\limits_0^t \int\limits_0^x y(\tau, s)\, d\tau\, ds, \int\limits_0^x y(t, s)\, ds, \int\limits_0^t y(\tau, x) d\tau\right),$$
$$(t, x) \in [0, a] \times [0, \beta].$$

Therefore, by Lemma 3.4 of [2], we may suppose that f is uniformly continuous.

Finally, there is no loss of generality in supposing $L_1\beta < 1$, for otherwise the rectangle $[0, a] \times [0, \beta]$ can be divided into a finite number of sufficiently small rectangles, and Theorem 2.3 applied successively to each of these sub-rectangles (in a suitable order) (see [4], p. 839, lines 26–29). Let us prove that (i) of Theorem 2.1 holds. Put, for every $\varepsilon > 0$, $d(\varepsilon) = \varepsilon$ and fix $\varepsilon > 0$. There exists $\delta^* > 0$ such that if $t', t'' \in [0, a]$, $z', z'', z_2', z_2'' \in \mathbb{R}^n$, and $|t' - t''| < \delta^*$, $\|z' - z''\| < \delta^*$, $\|z_2' - z_2''\| < \delta^*$, then

(6) $\qquad \|f(t', x, z', z_1, z_2') - f(t'', x, z'', z_1, z_2'')\| < (1 - L_1\beta)\, d(\varepsilon)$

for every $x \in [0, \beta]$, $z_1 \in \mathbb{R}^n$. Let $\delta = \min(\delta^*, \delta^*/\beta)$ and let $t', t'' \in [0, a]$, $z', z'', z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2$, such that $|t' - t''| < \delta$, $\|z' - z''\| < \beta\delta$, $\|z_1' - z_1''\| \leq \beta d(\varepsilon)$, $\|z_2' - z_2''\| < \delta$.

Thanks to (a$_2$) and (6), one has

$$\|f(t', x, z', z_1', z_2') - f(t'', x, z'', z_1'', z_2'')\|$$
$$\leq \|f(t', x, z', z_1', z_2') - f(t', x, z', z_1'', z_2')\|$$
$$+ \|f(t', x, z', z_1'', z_2') - f(t'', x, z'', z_1'', z_2'')\|$$
$$< L_1\beta d(\varepsilon) + (1 - L_1\beta)d(\varepsilon) = d(\varepsilon)$$

for every $x \in [0, \beta]$. Now, let us take $c = +\infty$. Since (j$_1$) of Theorem 2.3 follows at once from (a$_1$), to complete the proof we must only verify that (j$_2$) of Theorem 2.3 holds. Fix $\varrho > 0$ and, for every $(t, u, v) \in [0, a] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, put $w(t, u, v) = 2L_2 u$. Of course, the function w is nondecreasing with respect to u and, thanks to Gronwall's Lemma, 1) of (j$_2$) of Theorem 2.3 holds. Let $t \in\,]0, a[$, $U \subseteq C^0([0, \beta], \mathbb{R}^n)$, $V \subseteq B(v_0, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$. If $\sigma > 0$, $(u, v) \in U \times V$, $x', x'' \in [0, \beta]$ and $|x' - x''| \leq \sigma$,

then

$$\Big\| f\Big(t, x', \int\limits_0^{x'} u(s)\, ds, \int\limits_0^{x'} v(s)\, ds, u(x')\Big)$$

$$- f\Big(t, x'', \int\limits_0^{x''} u(s)\, ds, \int\limits_0^{x''} v(s)\, ds, u(x'')\Big)\Big\|$$

$$\leq L_1 \|v\|_{C^0([0,\beta],\mathbb{R}^n)} |x' - x''| + L_2 \|u(x') - u(x'')\|$$

$$+ \omega(f, \sqrt{(1 + \|u\|_{C^0([0,\beta],\mathbb{R}^n)}^2)(x' - x'')^2}\,).$$

Taking into account that f is uniformly continuous, this implies that

$$\lim_{\sigma \to 0^+} \sup_{(u,v) \in U \times V} \sup \Big\{ \Big\| f\Big(t, x', \int\limits_0^{x'} u(s)\, ds, \int\limits_0^{x'} v(s)\, ds, u(x')\Big)$$

$$- f\Big(t, x'', \int\limits_0^{x''} u(s)\, ds, \int\limits_0^{x''} v(s)\, ds, u(x'')\Big)\Big\| :$$

$$x', x'' \in [0, \beta], |x' - x''| \leq \sigma \Big\}$$

$$\leq \lim_{\sigma \to 0^+} [\sigma L_1 \sup_{v \in V} \|v\|_{C^0([0,\beta],\mathbb{R}^n)} + L_2 \omega(U, \sigma)$$

$$+ \omega(f, \sqrt{(1 + \sup_{u \in U} \|u\|_{C^0([0,\beta],\mathbb{R}^n)}^2)\sigma^2}\,)]$$

$$= L_2 \omega_0(U) = 2 L_2 \gamma(U) = w(t, \gamma(U), \gamma(V)).$$

This shows 2) of (j_2) of Theorem 2.3 and completes the proof.

To give an idea of the nature of the previous existence theorems of classical solutions for a Darboux problem for an hyperbolic partial differential equation with implicit mixed derivative, we recall here the following result (see [1], p. 85, and [2], p. 114), which is a simple consequence of Theorem A.

THEOREM B. *Let f be a continuous function from $[0, a] \times [0, \beta] \times (\mathbb{R}^n)^4$ into $\mathbb{R}^n$ and let $\varphi \in C^1([0, a]\mathbb{R}^n)$, $\psi \in C^1([0, \beta], \mathbb{R}^n)$ such that $\varphi(0) = \psi(0)$. Assume that:*

(b_1) *there exists a positive constant r such that $\|f(t, x, z, z_1, z_2, z_3)\| \leq r$ for every $(t, x) \in [0, a] \times [0, \beta]$ and every $z, z_i \in \mathbb{R}^n$, $i = 1, 2, 3$;*

(b_2) *there exist $L_1, L_2 \geq 0$, $N \in [0, 1[$ such that, for every $(t, x) \in [0, a] \times [0, \beta]$ and every $z, z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2, 3$, one has*

$$\|f(t, x, z, z_1', z_2', z_3') - f(t, x, z, z_1'', z_2'', z_3'')\|$$

$$\leq L_1 \|z_1' - z_1''\| + L_2 \|z_2' - z_2''\| + N \|z_3' - z_3''\|.$$

Then the Darboux problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial t \partial x} = f\Big(t, x, z, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial t \partial x}\Big), \\ z(t, 0) = \varphi(t), \\ z(0, x) = \psi(x), \end{cases}$$

has at least one solution in the class $E([0, a] \times [0, \beta], \mathbb{R}^n)$.

P r o o f. Thanks to (b_2), for every fixed $(t, x, z, z_1, z_2) \in [0, a] \times [0, \beta] \times (\mathbb{R}^n)^3$ there exists a unique point $f_0(t, x, z, z_1, z_2) \in \mathbb{R}^n$ such that

(7) $$f(t, x, z, z_1, z_2, f_0(t, x, z, z_1, z_2)) = f_0(t, x, z, z_1, z_2).$$

Hence, to prove our assertion, it suffices to verify that the function $f_0 : [0, a] \times [0, \beta] \times (\mathbb{R}^n)^3 \to \mathbb{R}^n$ so defined satisfies the assumptions of Theorem A.

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence in $[0, a] \times [0, \beta] \times (\mathbb{R}^n)^3$ such that $\lim_{k \to \infty} \xi_k = \xi$. The functions $f(\xi, \cdot), f(\xi_k, \cdot)$ $(k \in \mathbb{N})$ are contractions on $\mathbb{R}^n$ with the same constant N and, for every $\eta \in \mathbb{R}^n$, one has $\lim_{k \to \infty} f(\xi_k, \eta) = f(\xi, \eta)$. This implies that $\lim_{k \to \infty} f_0(\xi_k) = f_0(\xi)$. Hence, the function f_0 is continuous.

Assumption (a_1) of Theorem A follows at once from (7) and (b_1). Let us prove (a_2). Fix $(t, x) \in [0, a] \times [0, \beta]$, $z, z_i', z_i'' \in \mathbb{R}^n$, $i = 1, 2$. Thanks to (7) and (b_2), one has

$$\|f_0(t, x, z, z_1', z_2') - f_0(t, x, z, z_1'', z_2'')\| \le L_1 \|z_1' - z_1''\| + L_2 \|z_2' - z_2''\| \\ + N \|f_0(t, x, z, z_1', z_2') - f_0(t, x, z, z_1'', z_2'')\|.$$

Taking into account that $N < 1$, we get

$$\|f_0(t, x, z, z_1', z_2') - f_0(t, x, z, z_1'', z_2'')\| \le \frac{L_1}{1 - N} \|z_1' - z_1''\| + \frac{L_2}{1 - N} \|z_2' - z_2''\|.$$

This completes the proof. ∎

Finally, we give an example of application of Theorem 2.3, where it is impossible to apply Theorem B, since (b_2) does not hold.

EXAMPLE 2.1. *Let* $g(t, x, z, z_2) : [0, a] \times [0, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ *be a uniformly continuous function such that*:

(g_1) *there exists a positive constant* r *such that* $|g(t, x, z, z_2)| \le r$ *for every* $(t, x) \in [0, a] \times [0, \beta]$ *and every* $z, z_2 \in \mathbb{R}$;

(g_2) *there exists* $L_2 \ge 0$ *such that, for every* $(t, x) \in [0, a] \times [0, \beta]$ *and every* $z, z_2', z_2'' \in \mathbb{R}$, *one has*

$$|g(t, x, z, z_2') - g(t, x, z, z_2'')| \le L_2 |z_2' - z_2''|.$$

Let $h : [0, a] \times [0, \beta] \to \mathbb{R}$ *be a continuous function such that*

(h) $$|h(t, x)| \le 1$$

for every $(t, x) \in [0, a] \times [0, \beta]$ and let $k : \mathbb{R} \to \mathbb{R}$ be a Lipschitzian function, with Lipschitz constant 1, such that $k(0) = 0$ and

(k) *there exists $\vartheta \in [0, 1/2[$ such that $|k(z_3') - k(z_3'')| \leq \vartheta|z_3' - z_3''|$ for every $z_3', z_3'' \in [-(1 + r), (1 + r)]$.*

Finally, let $u_0 \in C^0([0, \beta]\mathbb{R})$ such that

$$(8) \qquad g\left(0, x, \int_0^x u_0(s)\, ds, u_0(x)\right) = 0$$

for all $x \in [0, \beta]$.

Then the Darboux problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial t \partial x} = g\left(t, x, z, \dfrac{\partial z}{\partial x}\right) + h(t, x) \sin k\left(\dfrac{\partial^2 z}{\partial t \partial x}\right), \\[2mm] z(t, 0) = 0, \\[2mm] z(0, x) = \displaystyle\int_0^x u_0(s)\, ds, \end{cases}$$

has at least one solution in the class $E([0, a] \times [0, \beta], \mathbb{R})$.

P r o o f. For every $(t, x) \in [0, a] \times [0, \beta]$, $z, z_2, z_3 \in \mathbb{R}$, put

$$f(t, x, z, z_2, z_3) = g(t, x, z, z_2) + h(t, x) \sin k(z_3)\,.$$

The function $f : [0, a] \times [0, \beta] \times \mathbb{R}^3 \to \mathbb{R}$ so defined is continuous and, if we take $v_0 = \vartheta_B$, thanks to (8), (3) holds. Now, fix $\varepsilon \in]0, \pi]$ and take $\varepsilon_1, \varepsilon_2 \in]0, \pi]$ such that

$$\varepsilon_1 + \varepsilon_2 + 2\sin(\varepsilon/2) \leq \varepsilon\,.$$

There exists $\delta^* > 0$ such that if $t', t'' \in [0, a]$, $z', z'', z_2', z_2'' \in \mathbb{R}$ and $|t' - t''| < \delta^*$, $|z' - z''| < \delta^*$, $|z_2' - z_2''| < \delta^*$ then

$$(9) \qquad |g(t', x, z', z_2') - g(t'', x, z'', z_2'')| < \varepsilon_1, \qquad |h(t', x) - h(t'', x)| < \varepsilon_2$$

for every $x \in [0, \beta]$. Let $\delta = \min(\delta^*, \delta^*/\beta)$ and let $t', t'' \in [0, a]$, z', z'', z_i', z_i'' $\in \mathbb{R}$, $i = 2, 3$, such that $|t' - t''| < \delta$, $|z' - z''| < \beta\delta$, $|z_2' - z_2''| < \delta$, $|z_3' - z_3''| \leq \varepsilon$. Taking into account (9) and (k), we obtain

$$\begin{aligned} |f(t', x, z', z_2', z_3') &- f(t'', x, z'', z_2'', z_3'')| \\ &\leq |g(t', x, z', z_2') - g(t'', x, z'', z_2'')| + |h(t', x) - h(t'', x)| \\ &\quad + |\sin k(z_3') - \sin k(z_3'')| < \varepsilon_1 + \varepsilon_2 + 2\sin(|k(z_3') - k(z_3'')|/2) \\ &\leq \varepsilon_1 + \varepsilon_2 + 2\sin(\varepsilon/2) \leq \varepsilon \end{aligned}$$

for every $x \in [0, \beta]$. Hence,

$$|f(t', x, z', z_2', z_3') - f(t'', x, z'', z_2'', z_3'')| \leq d(\varepsilon)$$

for all $x \in [0, \beta]$, where

$$d(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon \in]0, \pi], \\ 2 + \varepsilon & \text{if } \varepsilon \in]\pi, +\infty[. \end{cases}$$

This shows (i) of Theorem 2.1.

Next, take $c = 1 + r$ and observe that (j_1) of Theorem 2.3 follows at once from (g_1) and (h). Let us prove that (j_2) of Theorem 2.3 holds. To this end, assume $\varrho = 1 + r$ and, for every $(t, u, v) \in [0, a] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, put $w(t, u, v) = 2L_2 u + 2\vartheta v$. Of course, the function w is nondecreasing with respect to u and since $\vartheta < 1/2$, thanks to Gronwall's Lemma, 1) holds. Now, fix $t \in]0, a[$, $U \subseteq C^0([0, \beta], \mathbb{R})$, $V \subseteq B(\vartheta_B, c)$, with $\gamma(U) < \varrho$ and $0 < \gamma(V) < \varrho$. Let $\sigma > 0$, $(u, v) \in U \times V$, $x', x'' \in [0, \beta]$ such that $|x' - x''| \leq \sigma$. Taking account of (g_2) and (k), we get

$$\left| f\left(t, x', \int_0^{x'} u(s)\, ds, u(x'), v(x')\right) - f\left(t, x'', \int_0^{x''} u(s)\, ds, u(x''), v(x'')\right) \right|$$
$$\leq L_2 |u(x') - u(x'')| + \omega(g, \sqrt{(1 + \|u\|_{C^0([0,\beta],\mathbb{R})}^2) \sigma^2})$$
$$+ \omega(h, \sigma) + \vartheta |v(x') - v(x'')|.$$

From this, by means of the usual reasoning, it follows that

$$\lim_{\sigma \to 0^+} \sup_{(u,v) \in U \times V} \sup \left\{ \left| f\left(t, x', \int_0^{x'} u(s)\, ds, u(x'), v(x')\right) \right.\right.$$
$$\left.\left. - f\left(t, x'', \int_0^{x''} u(s)\, ds, u(x''), v(x'')\right) \right| : x', x'' \in [0, \beta], |x' - x''| \leq \sigma \right\}$$
$$\leq 2L_2 \gamma(U) + 2\vartheta \gamma(V) \leq w(t, \gamma(U), \gamma_{B(\vartheta_B, c)}(V)).$$

This shows 2) and we can apply Theorem 2.3. It yields that there exists at least one solution of $(DP)_1$ in the class $E([0, a] \times [0, \beta], \mathbb{R})$, as desired. $\blacksquare$

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