ANNALES
POLONICI MATHEMATICI
LVI.2 (1992)

On the solvability of nonlinear elliptic equations in Sobolev spaces

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Abstract. We consider the existence of solutions of the system

$$(*) \qquad P(D)u^l = F(x, (\partial^\alpha u)), \quad l = 1, \ldots, k, \ x \in \mathbb{R}^n$$

($u = (u^1, \ldots, u^k)$) in Sobolev spaces, where P is a positive elliptic polynomial and F is nonlinear.

1. Introduction. We study the existence of solutions of the system $(*)$ in Sobolev spaces. We make such assumptions that the right sides of the considered equations are locally integrable for u belonging to a space of solutions. In this way, we can understand these equations in the sense of distributions.

The other assumptions concerning the right sides of equations $(*)$ give a priori bounds of solutions. We consider, for example, assumptions of the Bernstein type. Assumptions of this kind can be found in the papers [1], [4] concerning equations on a bounded interval, in [8] concerning equations on the half-line and in [3] concerning equations on the line.

We shall denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the euclidean norm in $\mathbb{R}^l$ for any positive integer l.

The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}f)(\xi) := \int e^{-i\langle x, \xi \rangle} f(x)\, dx,$$

where $\int = \int_{\mathbb{R}^n}$. We define the Fourier transformation in the space of tempered distributions in the standard way.

By $\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^n)$, for real $s \geq 0$, we denote the real Sobolev space of real tempered distributions u such that

$$\|u\|_s^2 := (2\pi)^{-n} \int |(\mathcal{F}u)(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty.$$

1991 *Mathematics Subject Classification*: Primary 35J65.

(We have $\mathcal{H}^0 = L^2 = L^2(\mathbb{R}^n)$.) We denote the local Sobolev space by $\mathcal{H}^s_{\mathrm{loc}} = \mathcal{H}^s_{\mathrm{loc}}(\mathbb{R}^n)$ and treat it as a Fréchet space in the standard way (see for example [5]).

We denote the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ with compact support by $\mathcal{C}^\infty_0$ and the space of Schwartz distributions on $\mathbb{R}^n$ by $\mathcal{D}'$.

By $\alpha = (\alpha_1, \ldots, \alpha_n)$ we denote a multi-index, $|\alpha| := \sum_{i=1}^n \alpha_i$. We set $\partial_j := \partial/\partial x_j$ and $D_j := -i\partial_j$.

2. Existence theorem for a single equation. We prove the following

THEOREM 1. *Let P be a polynomial of n variables and degree T such that the polynomial $P(-i\partial)$ of the variable ∂ has real coefficients. Assume that*

$$1 + |\xi|^T \le C|P(\xi)|, \qquad \xi \in \mathbb{R}^n, \tag{1}$$

for some constant C.

Let $t \in [0, T[$ and

$$m := \sum_{0 \le l \le t} n^l, \qquad m' := \sum_{0 \le l < t - n/2} n^l,$$

where l is an integer variable. (*We set $\sum_{l\in\emptyset} n^l := 0$.*)

Suppose that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory condition: $F(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^n$ and $F(\cdot, (v_\alpha)_{|\alpha|\le t})$ is measurable for all $(v_\alpha)_{|\alpha|\le t} \in \mathbb{R}^m$.

For any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^{m'}$, let there exist a function $h_K \in L^2(\mathbb{R}^n)$ and a constant C_K such that

$$|F(x, (v_\alpha)_{|\alpha|\le t})| \le h_K(x) + C_K |(v_\alpha)_{t-n/2 \le |\alpha| \le t}| \tag{2}$$

for all $(x, (v_\alpha)_{|\alpha|<t-n/2}) \in K$ almost everywhere with respect to x (a.e. x). (*We omit the last term in* (2) *if $m = m'$.*)

Suppose that there exist a sequence of open bounded sets $U_1 \subset U_2 \subset \ldots$, $\bigcup U_j = \mathbb{R}^n$, and a constant M such that any equation

$$P(D)u = \lambda F_j(x, (\partial^\alpha u)_{|\alpha|\le t}), \qquad j = 1, 2, \ldots, \ \lambda \in [0, 1], \tag{3}$$

$$F_j(x, (v_\alpha)_{|\alpha|\le t}) := \begin{cases} F(x, (v_\alpha)_{|\alpha|\le t}) & \text{for } x \in U_j, \\ 0 & \text{for } u \notin U_j, \end{cases}$$

has no solution in the set $\{u \in \mathcal{H}^t : \|u\|_t > M\}$.

Under these assumptions, the equation

$$P(D)u = F(x, (\partial^\alpha u)_{|\alpha|\le t}) \tag{4}$$

has a solution u in $\mathcal{H}^t$ for which $\|u\|_t \le M$.

The proof of Theorem 1 is based on several lemmas.

LEMMA 1. *If $u \in \mathcal{H}^s$, then any $\partial^\alpha u$, for $|\alpha| < s - n/2$, is a continuous bounded function and there exists a constant C such that*

$$\sup_{x\in\mathbb{R}^n} \sup_{|\alpha|<s-n/2} |\partial^\alpha u(x)| \le C\|u\|_s. \tag{5}$$

Proof. See [5], Corollary 7.9.4. One can obtain inequality (5) by standard calculus.

LEMMA 2. *The Nemytskiĭ operator $u \mapsto F_j(\cdot, (\partial^\alpha u(\cdot))_{|\alpha|\le t})$ transforms $\mathcal{H}^t_{\mathrm{loc}}$ into L^2 continuously.*

Proof. If $u \in \mathcal{H}^t_{\mathrm{loc}}$, then $F_j(\cdot, (\partial^\alpha u(\cdot))_{|\alpha|\le t})$ is measurable by the Carathéodory condition (see [2], appendix, or [6], §17). The set

$$K := \overline{U}_j \times \bigtimes_{|\alpha|<t-n/2} \partial^\alpha u(\overline{U}_j)$$

is compact because of the continuity of $\partial^\alpha u$ for $|\alpha| < s - n/2$ due to Lemma 1. We have $F_j(\cdot, (\partial^\alpha u(\cdot))_{|\alpha|\le t}) \in L^2$ by (2) for K defined above. The required continuity is now proved as in [2], appendix, where the case $s = 0$ is considered.

Observe that, in $\mathcal{H}^t$, equation (3) is equivalent to

$$u = \lambda A_j u, \tag{6}$$

where

$$A_j u := \mathcal{F}^{-1}\left(\frac{1}{P}\mathcal{F}F_j(\cdot, (\partial^\alpha u(\cdot))_{|\alpha|\le t})\right). \tag{7}$$

DEFINITION 1. A continuous operator between locally convex spaces is called *completely continuous* if it sends bounded sets into precompact ones.

The following lemma is very important for the proof of Theorem 1.

LEMMA 3. *The embedding $\mathcal{H}^s_{\mathrm{loc}} \to \mathcal{H}^{s'}_{\mathrm{loc}}$ for $s > s' \ge 0$ is completely continuous.*

The proof is in [5], Theorem 10.1.27.

LEMMA 4. *The operator A_j defined by* (7) *is completely continuous from $\mathcal{H}^T$ into $\mathcal{H}^T$.*

Proof. Observe that, for $u \in \mathcal{H}^t_{\mathrm{loc}}$,

$$(1 + |\xi|^T)\mathcal{F}(A_j u)(\xi) = b(\xi)\mathcal{F}F_j(\cdot, (\partial^\alpha u(\cdot))_{|\alpha|\le t})(\xi) \tag{8}$$

where $b(\xi) := (P(\xi))^{-1}(1 + |\xi|^T)$ is bounded by (1).

Consequently, $A_j u \in \mathcal{H}^T + i\mathcal{H}^T$. We have $A_j u \in \mathcal{H}^T$ because the polynomial $P(-i\partial)$ of the variable ∂ has real coefficients.

It follows easily from Lemma 2 and (8) that A_j transforms $\mathcal{H}^t_{\mathrm{loc}}$ into $\mathcal{H}^T$ continuously. But the embedding $\mathcal{H}^T \to \mathcal{H}^t_{\mathrm{loc}}$ is completely continuous by Lemma 3, which proves the lemma.

Lemma 5. *There exists a constant* M_j *such that* $\|u_{j\lambda}\|_T \le M_j$ *for any solution* $u_{j\lambda}$ *of equation* (6) *for* $\lambda \in [0,1]$.

Proof. From the assumption, we have $\|u_{j\lambda}\|_t \le M$, hence from (5) and (2)

$$\|F_j(\cdot,(\partial^\alpha u_{j\lambda}(\cdot))_{|\alpha|\le t})\|_0 \le M'_j$$

for some constant M'_j. Using (8), we obtain the result.

Lemma 6. *Equation* (6) *has a solution in* $\mathcal{H}^T$ *for* $\lambda = 1$ *and any* j.

Proof. Write (6) in the form

$$(I - \lambda A_j)u = 0$$

where I stands for the identity mapping. We treat $I - \lambda A_j$ as a mapping from the ball $B(0, M_j+1) \subset \mathcal{H}^T$ into $\mathcal{H}^T$ and use the Leray–Schauder degree theory (see for instance [7]), since A_j is completely continuous by Lemma 4. From Lemma 5, we know that $(I-\lambda A_j)u \ne 0$ for $\|u\|_T = M_j+1$, so for the Leray–Schauder degree we obtain

$$\deg(I - A_j, B(0, M_j+1), 0) = \deg(I, B(0, M_j+1), 0) = 1 \ne 0.$$

Therefore, equation (6) has a solution in $\mathcal{H}^T$ for $\lambda = 1$.

Lemma 7. *The set* $\{u_j\}$ *of solutions of equations* (3), *for* $\lambda = 1$, *in the space* $\mathcal{H}^T$ *is bounded in the space* $\mathcal{H}^s_{\mathrm{loc}}$, *where* $s := \min\{t+1, T\}$.

Proof. Let $\phi \in \mathcal{C}^\infty_0$. We have to estimate $\|\phi u_j\|_s$ by a constant depending on ϕ only. From (1) and the Leibniz–Hörmander formula ([5], (1.1.10)), we obtain

$$\begin{aligned}
\|\phi u_j\|_s &= \Big((2\pi)^{-n} \int (1+|\xi|^2)^s |\mathcal{F}(\phi u_j)(\xi)|^2\, d\xi\Big)^{1/2}\\
&= \Big((2\pi)^{-n} \int (1+|\xi|^2)^{s-T}(1+|\xi|^2)^T |\mathcal{F}(\phi u_j)(\xi)|^2\, d\xi\Big)^{1/2}\\
&\le C_1\Big(\int (1+|\xi|^2)^{s-T}|P(\xi)|^2 |\mathcal{F}(\phi u_j)(\xi)|^2\, d\xi\Big)^{1/2}\\
&\le C_1\Big(\int (1+|\xi|^2)^{s-T}|\mathcal{F}P(D)(\phi u_j)(\xi)|^2\, d\xi\Big)^{1/2}\\
&\le C_1\Big(\int (1+|\xi|^2)^{s-T}\Big|\mathcal{F}\Big(\sum_{\alpha\in\mathbb{N}^n} \partial^\alpha\phi(\partial^\alpha P)(D)u_j/\alpha!\Big)(\xi)\Big|^2\, d\xi\Big)^{1/2}
\end{aligned}$$

$$\le C_1\Big(\int (1+|\xi|^2)^{s-T}|\mathcal{F}(\phi P(D)u_j)(\xi)|^2\,d\xi\Big)^{1/2} + C_1\sum_{\alpha\ne(0,\dots,0)}\Big(\int (1+|\xi|^2)^{s-T}|\mathcal{F}(\partial^\alpha\phi(\partial^\alpha P)(D)u_j/\alpha!)(\xi)|^2\,d\xi\Big)^{1/2}$$

for some constant C_1. We will estimate the integrals in the last expression:

$$\begin{aligned}\int (1+|\xi|^2)^{s-T}|\mathcal{F}(\phi P(D)u_j)(\xi)|^2\,d\xi &\le \int |\mathcal{F}(\phi P(D)u_j)(\xi)|^2\,d\xi\\ &\le (2\pi)^n\int |\phi(x)P(D)u_j(x)|^2\,dx\\ &= (2\pi)^n\int |\phi(x)F_j(x,(\partial^\alpha u_j(x))_{|\alpha|\le t})|^2\,dx \le C_2\end{aligned}$$

for some constant C_2 depending on ϕ. In the last step we have used the estimate $\|u_j\|_t\le M$ and (2) for the set

$$K := (\operatorname{supp}\phi)\times \bigtimes_{|\alpha|<t-n/2}[-M,M].$$

Note that the map $\mathcal{H}^{s-T}\ni w\mapsto \psi w\in\mathcal{H}^{s-T}$ is continuous for fixed $\psi\in\mathcal{C}_0^\infty$ (see [5], Theorem 10.1.15). This implies that, for $\alpha\ne(0,\dots,0)$,

$$\begin{aligned}&\int (1+|\xi|^2)^{s-T}|\mathcal{F}(\partial^\alpha\phi(\partial^\alpha P)(D)u_j/\alpha!)(\xi)|^2\,d\xi\\ &\qquad\le C_\alpha\int (1+|\xi|^2)^{s-T}|\mathcal{F}(\partial^\alpha P)(D)u_j(\xi)|^2\,d\xi\\ &\qquad= C_\alpha\int (1+|\xi|^2)^{s-T}|(\partial^\alpha P)(\xi)\mathcal{F}u_j(\xi)|^2\,d\xi\\ &\qquad= C'_\alpha\int (1+|\xi|^2)^{s-T}|\mathcal{F}u_j(\xi)|^2\,d\xi\le C'_\alpha\|u_j\|_t\le C'_\alpha M\end{aligned}$$

for some constants C_α, C'_α depending on ϕ. Hence

$$\|\phi u_j\|_s\le C$$

for some C depending on ϕ.

Let (u_j) be a sequence of $\mathcal{H}^T$-solutions of equations (3) for $\lambda=1$. The set $\{u_j\}$ is bounded in $\mathcal{H}^{\min\{t+1,T\}}_{\mathrm{loc}}$ by Lemma 7. Using Lemma 3, take a subsequence of (u_j) (denoted once more by (u_j) for simplicity of notation) which is convergent to some u in the topology of $\mathcal{H}^t_{\mathrm{loc}}$. We have $u\in\mathcal{H}^t$ and $\|u\|_t\le M$, since $\|u_j\|_t\le M$.

We shall demonstrate that u is a solution of equation (4). Notice that

$$F_j(\cdot,(\partial^\alpha u_j(\cdot))_{|\alpha|\le t})\to F(\cdot,(\partial^\alpha u(\cdot))_{|\alpha|\le t})$$

in $\mathcal{D}'$. Indeed, let $\phi\in\mathcal{C}_0^\infty$. For large j, we have

$$\begin{aligned}\int \phi(x)F_j(x,(\partial^\alpha u_j(x))_{|\alpha|\le t})\,dx &= \int_{\operatorname{supp}\phi}\phi(x)F(x,(\partial^\alpha u_j(x))_{|\alpha|\le t})\,dx\\ &\to \int \phi(x)F(x,(\partial^\alpha u(x))_{|\alpha|\le t})\,dx.\end{aligned}$$

In the last step, we have used Lemma 2.

The convergence $u_j \to u$ in $\mathcal{H}^t_{\mathrm{loc}}$ implies the convergence $u_j \to u$ in $\mathcal{D}'$, so also the convergence $P(D)u_j \to P(D)u$ in $\mathcal{D}'$. Hence u is a solution of (4).

Example 1. We now define a class of equations for which Theorem 1 is valid.

Assume that P is a real polynomial of degree $T = 2t$, positive for $\xi \in \mathbb{R}^n$ and such that the polynomial $P(-i\partial)$ of the variable ∂ has real coefficients and (1) is valid.

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfy the Carathéodory condition as above. Let F satisfy (2).

Assume that there exist constants $0 < a < 2$, $L > 0$ and nonnegative functions $f \in L^{2/a}$, $g \in L^{2/(2-a)}$ such that

$$v_{(0,\ldots,0)} F(x, (v_\alpha)_{|\alpha|\le t}) \le 0 \quad \text{if } |v_{(0,\ldots,0)}| \ge g(x) \text{ a.e. } x, \tag{10}$$

$$|F(x, (v_\alpha)_{|\alpha|\le t})| \le f(x) + L|(v_\alpha)_{|\alpha|\le t}|^a \quad \text{if } |v_{(0,\ldots,0)}| \le g(x) \text{ a.e. } x. \tag{11}$$

We show that our assumptions give an a priori bound for solutions of equations (3) where, for example,

$$F_j(x, (v_\alpha)_{|\alpha|\le t}) = \begin{cases} F(x, (v_\alpha)_{|\alpha|\le t}) & \text{for } |x| < j, \\ 0 & \text{for } |x| \ge j. \end{cases}$$

Let $u = u_{j\lambda} \in \mathcal{H}^t$ be a solution of (3). Compute

$$\begin{aligned}
\|u\|_t^2 &= \int (1 + |\xi|^2)^t |\mathcal{F}u(\xi)|^2 \, d\xi \le C \int P(\xi)\mathcal{F}u(\xi)\overline{\mathcal{F}u(\xi)} \, d\xi \\
&= C \int \mathcal{F}u(\xi)\overline{\mathcal{F}(P(D)u)(\xi)} \, d\xi = (2\pi)^n C \int u(x)\overline{P(D)u(x)} \, dx \\
&= (2\pi)^n C \int u(x)P(D)u(x) \, dx \\
&= (2\pi)^n C\lambda \int u(x)F_j(x, (\partial^\alpha u(x))_{|\alpha|\le t}) \, dx \\
&\le (2\pi)^n C \int u(x)F_j(x, (\partial^\alpha u(x))_{|\alpha|\le t}) \, dx \\
&\le (2\pi)^n C \int\limits_{\{x : |u(x)|\le g(x)\}} u(x)F_j(x, (\partial^\alpha u(x))_{|\alpha|\le t}) \, dx \\
&\le (2\pi)^n C \int\limits_{\{x : |u(x)|\le g(x)\}} |u(x)| \, |F_j(x, (\partial^\alpha u(x))_{|\alpha|\le t})| \, dx \\
&\le (2\pi)^n C \int g(x)(f(x) + L|(\partial^\alpha u(x))_{|\alpha|\le t}|^a) \, dx \\
&\le (2\pi)^n C\Big(\int |g(x)|^{2/(2-a)} \, dx\Big)^{(2-a)/2}\Big(\int |f(x)|^{2/a} \, dx\Big)^{a/2} \\
&\quad + (2\pi)^n CL\Big(\int |g(x)|^{2/(2-a)} \, dx\Big)^{(2-a)/2} \|u\|_t^a \le C_1(1 + \|u\|_t^a)
\end{aligned}$$

for some constant C_1. In the last steps we have used assumptions (10), (11) and the Hölder inequality.

Now, it is easy to see that $\|u\|_t \le M$ for some M.

3. Existence theorem for a system of equations. We formulate a similar theorem for a system of equations.

THEOREM 2. *Let P_r, $r = 1, \ldots, k$, be polynomials of n variables and degrees T_r such that the polynomials $P_r(-i\partial)$ of the variable ∂ have real coefficients. Assume that*

$$1 + |\xi|^{T_r} \le C_r P_r(\xi), \quad \xi \in \mathbb{R}^n, \tag{13}$$

for some constants C_r, $r = 1, \ldots, k$. Let $t_r \in [0, T_r[$ and

$$m := \sum_{r=1}^{k} \sum_{0 \le l \le t_r} n^l, \quad m' = \sum_{r=1}^{k} \sum_{0 \le l < t_r - n/2} n^l.$$

Suppose that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ satisfies the Carathéodory condition and, for any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exist a function $h_K \in L^2(\mathbb{R}^n)$ and a constant C_K that

$$|F(x, (v_\alpha^r)_{|\alpha| \le t_r, r=1,\ldots,k})| \le h_K(x) + C_K |(v_\alpha^r)_{t_r - n/2 \le |\alpha| \le t_r, r=1,\ldots,k}| \tag{14}$$

for $(x, (v_\alpha^r)_{|\alpha| < t_r - n/2, r=1,\ldots,k}) \in K$ a.e. x. (We omit the last term in (14) if $m = m'$.) Suppose that there exist a sequence of open bounded sets $U_1 \subset U_2 \subset \ldots$, $\bigcup U_j = \mathbb{R}^n$, and a constant $M > 0$ such that the system of equations

$$P_l(D)u^l = \lambda F_j^l(x, (\partial^\alpha u^r)_{|\alpha| \le t_r, r=1,\ldots,k}), \quad l = 1, \ldots, k \tag{15}$$

$(F = F^1, \ldots, F^k))$, has no solution in the set

$$\Big\{ u = (u^1, \ldots, u^k) \in \bigtimes_{r=1}^{k} \mathcal{H}^{t_r} : \sum_{r=1}^{k} \|u^r\|_{t_r}^2 > M^2 \Big\}$$

for $j = 1, 2, \ldots$, $\lambda = [0, 1]$. The functions F_j^l used above are defined by

$$F_j^l(x, (v_\alpha^r)_{|\alpha| \le t_r, r=1,\ldots,k}) := \begin{cases} F^l(x, (v_\alpha^r)_{|\alpha| \le t_r, r=1,\ldots,k}) & \text{for } x \in U_j, \\ 0 & \text{for } x \notin U_j. \end{cases}$$

Under these assumptions, the system of equations

$$P_l(D)u^l = F^l(x, (\partial^\alpha u^r)_{|\alpha| \le t_r, r=1,\ldots,k}), \quad l = 1, \ldots, k,$$

has a solution u in $\bigtimes_{r=1}^{k} \mathcal{H}^{t_r}$ for which

$$\sum_{r=1}^{k} \|u^r\|_{t_r}^2 \le M^2.$$

We omit the proof, similar to the proof of Theorem 1.

Example 2 (cf. Example 1). We define a class of systems for which Theorem 2 is valid.

Assume that P_r, $r = 1, \ldots, k$, are real polynomials of degrees $T_r = 2t_r$, positive for $\xi \in \mathbb{R}^n$ and such that the polynomials $P_r(-i\partial)$ of the variable ∂ have real coefficients and (13) is valid. Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfy the Carathéodory condition and (14). Assume that there exist constants $0 < a < 2$, $L > 0$ and nonnegative functions $f \in L^{2/a}$, $g \in L^{2/(2-a)}$ such that

$$\langle v_{(0,\ldots,0)}, F(x, (v^r_\alpha)_{|\alpha| \le t_r,\, r=1,\ldots,k})\rangle \le 0 \quad \text{if } |v_{(0,\ldots,0)}| \ge g(x) \text{ a.e. } x$$

$(F = (F^1, \ldots, F^k))$, and

$$|F(x, (v^r_\alpha)_{|\alpha| \le t_r, r=1,\ldots,k})| \le f(x) + L|(v^r_\alpha)_{|\alpha| \le t_r,\, r=1,\ldots,k}|^a \quad \text{if } |v_{(0,\ldots,0)}| \le g(x) \text{ a.e. } x.$$

These assumptions give (13) and an a priori bound for solutions of the system (15) in the space $\bigtimes_{r=1}^k \mathcal{H}^{t_r}$. The proof is similar to the one in Example 1 so it can be omitted.

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Reçu par la Rédaction le 1.8.1990
Révisé le 15.1.1991