The fixed points of holomorphic maps on a convex domain

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Abstract. We give a simple proof of the result that if $D$ is a (not necessarily bounded) hyperbolic convex domain in $\mathbb{C}^n$ then the set $V$ of fixed points of a holomorphic map $f : D \to D$ is a connected complex submanifold of $D$; if $V$ is not empty, $V$ is a holomorphic retract of $D$. Moreover, we extend these results to the case of convex domains in a locally convex Hausdorff vector space.

1. Introduction. In [15] J.-P. Vigué investigated the structure of the fixed point set of a holomorphic map from a bounded convex domain in $\mathbb{C}^n$ into itself. He proved the following. Let $D$ be a bounded convex domain in $\mathbb{C}^n$. Then the set $V$ of fixed points of a holomorphic map $f : D \to D$ is a connected complex submanifold of $D$ and, if $V$ is not empty, $V$ is a holomorphic retract of $D$. His main tools were the results of Vesentini [13], [14] and Lempert [10], [11] about complex geodesics. However, his proof was rather long.

Our purpose in this article is to give a brief and simple proof of this theorem in the general case of (not necessarily bounded) hyperbolic convex domains in $\mathbb{C}^n$. Moreover, we shall investigate the fixed point sets of holomorphic maps from a convex domain in a locally convex Hausdorff vector space into itself.

We now recall some definitions and properties.

(i) We shall frequently make use of the Kobayashi pseudodistance $d_M$ and the Carathéodory pseudodistance $c_M$ on a complex manifold $M$ (see Kobayashi [9]).

(ii) A complex manifold $M$ is called taut [7] if whenever $N$ is a complex manifold and $f_k : N \to M$ is a sequence of holomorphic maps, then either there exists a subsequence which converges uniformly on compact subsets to a holomorphic map $f : N \to M$ or a subsequence which is compactly divergent. In order for $M$ to be taut, it suffices that this condition holds for

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$N = \Delta$, the unit disk in $\mathbb{C}^n$ [1]. Also, every complete hyperbolic complex space is taut, and a taut complex manifold is hyperbolic [7].

(iii) Let $D$ be a domain in a locally convex Hausdorff topological vector space $E$. A holomorphic map $\varphi : \Delta \to D$ is called a complex geodesic [13] if $c_\Delta(\zeta_1, \zeta_2) = c_D(\varphi(\zeta_1), \varphi(\zeta_2))$ for all $\zeta_1, \zeta_2 \in \Delta$. Vesentini [13] proved that $\varphi$ is a complex geodesic iff there exist two distinct points $\zeta_0, \zeta_1 \in \Delta$ such that $c_\Delta(\zeta_0, \zeta_1) = c_D(\varphi(\zeta_0), \varphi(\zeta_1))$.

The theorems of the present paper in the infinite-dimensional case were suggested by my friend Ngo Hoang Huy. I wish to thank him for his help.

2. The finite-dimensional case. In this section we always assume that $D$ is a (not necessarily bounded) hyperbolic convex domain in $\mathbb{C}^n$ and $f : D \to D$ is a holomorphic map. Denote the fixed point set of $f$ by $V = \text{Fix}(f)$.

2.1. Theorem. If $V$ is not empty then $V$ is a holomorphic retract of $D$, i.e. there exists a holomorphic map $\varphi : D \to D$ such that $\varphi(D) \subset V$ and $\varphi|_V = \text{Id}$.

Proof. The space $\text{Hol}(D, \mathbb{C}^n)$ of all holomorphic maps $g : D \to \mathbb{C}^n$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Consider its subset $K = \{g \in \text{Hol}(D, D) : g|_V = \text{Id}\}$ with the induced topology. Clearly, $K$ is a nonempty convex subset of $\text{Hol}(D, D)$. Since $D$ is a hyperbolic convex domain, $D$ is taut (see Barth [2]). Hence $K$ is compact in $\text{Hol}(D, \mathbb{C}^n)$.

Consider the continuous operator 

$$T : \text{Hol}(D, D) \to \text{Hol}(D, D), \quad g \mapsto f \circ g.$$ 

It is easy to see that $T(K) \subset K$. By the Schauder fixed point theorem (see Edwards [4]), there exists $\varphi \in K$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(D) \subset V$. Since $\varphi|_V = \text{Id}$, $V$ is a holomorphic retract of $D$. \hfill \blacksquare

By a result of Rossi (see Fischer [5, p. 102]), we deduce the following

2.2. Corollary. The fixed point set $V$ of $f$ is a complex submanifold of $D$.

2.3. Proposition. For any two distinct fixed points $x$ and $y$ of $f$, there exists a complex geodesic $\varphi$ which passes through $x$, $y$ and satisfies $\varphi(\Delta) \subset V = \text{Fix}(f)$.

Proof. $\text{Hol}(\Delta, \mathbb{C}^n)$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Assume that $x, y \in V$ and $x \neq y$. Choose $\eta \in \Delta$ such that $c_\Delta(0, \eta) = c_D(x, y)$. Consider the subset $\Gamma = \{g \in \text{Hol}(\Delta, D) : g(0) = x, \ g(\eta) = y\}$ of $\text{Hol}(\Delta, \mathbb{C}^n)$ with the induced topology.
By the results of Lempert [10], [11] and Royden–Wong [12], we have $c_D(x, y) = d_D(x, y) = \delta_D(x, y) = \inf\{c_\Delta(0, \zeta) : \exists \varphi : \Delta \to D \text{ holomorphic with } \varphi(0) = x, \varphi(\zeta) = y\}$. Thus there exists a sequence $\{\varphi_n\} \subset \text{Hol}(\Delta, D)$ and a sequence $\{\zeta_n\} \subset \Delta$ such that $\varphi_n(0) = x$, $\varphi_n(\zeta_n) = y$ and $\lim_{n \to \infty} c_\Delta(0, \zeta_n) = c_D(x, y) < \infty$. We can assume that $\{\zeta_n\}$ converges to a point $\zeta_0 \in \Delta$. Since $D$ is taut [2], we may assume that $\{\varphi_n\}$ converges in $\text{Hol}(\Delta, D)$ to a map $\varphi_0 \in \text{Hol}(\Delta, D)$. Clearly $\varphi_0(0) = x$, $\varphi_0(\zeta_0) = y$ and $c_\Delta(0, \zeta_0) = c_D(x, y)$.

Take an automorphism $T$ of $\Delta$ such that $T(0) = 0$, $T(\eta) = \zeta_0$. Then $\varphi_0 \circ T \in \Gamma$. Thus $\Gamma$ is a nonempty convex subset of $\text{Hol}(\Delta, D)$. On the other hand, since $D$ is taut, $\Gamma$ is compact in $\text{Hol}(\Delta, \mathbb{C}^n)$.

Consider the continuous operator

$$T : \text{Hol}(\Delta, D) \to \text{Hol}(\Delta, D), \quad g \mapsto f \circ g.$$ 

It is easy to see that $T(\Gamma) \subset \Gamma$. By the Schauder fixed point theorem, there is $\varphi \in \Gamma$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(\Delta) \subset V$. ■

Corollary 2.2 and Proposition 2.3 yield the following

2.4. THEOREM. The fixed point set $V$ of $f$ is a connected complex submanifold of $D$.

2.5. PROPOSITION. Assume that $V$ is a one-dimensional connected complex submanifold of $D$. Then the following are equivalent:

(i) $V$ is the fixed point set of some holomorphic map $f : D \to D$.

(ii) $V$ is the image of some complex geodesic $\varphi : \Delta \to D$.

(iii) $V$ is a holomorphic retract of $D$.

Proof. (i)$\Rightarrow$(ii). Assume that $V = \text{Fix}(f)$, where $f : D \to D$ is a holomorphic map. Take two distinct $x, y \in V$. By Proposition 2.3, there exists a complex geodesic which passes through $x, y$ and satisfies $\varphi(\Delta) \subset V$. Then $\varphi(\Delta) = V$, because $\varphi(\Delta)$ is open and closed in $V$.

(ii)$\Rightarrow$(iii). Assume that $\varphi : \Delta \to D$ is a complex geodesic and $V = \varphi(\Delta)$. Take two distinct points $z_1, z_2 \in \Delta$. We have $c_D(\varphi(z_1), \varphi(z_2)) = \sup\{c_\Delta(0, g(\varphi(z_2))) : g \in \text{Hol}(D, \Delta) \text{ with } g(\varphi(z_1)) = 0\}$. By the normality of $\text{Hol}(D, \Delta)$, there exists $g \in \text{Hol}(D, \Delta)$ such that

$$c_D(\varphi(z_1), \varphi(z_2)) = c_\Delta(g(\varphi(z_1)), g(\varphi(z_2))).$$

Hence $c_\Delta(z_1, z_2) = c_\Delta(g \circ \varphi(z_1), g \circ \varphi(z_2))$. Thus $g \circ \varphi$ is an automorphism of $\Delta$ having two distinct fixed points $z_1, z_2$. By the Schwarz lemma, $g \circ \varphi = \text{Id}$. Therefore $\varphi \circ g : D \to \varphi(\Delta)$ is a retraction on $\varphi(\Delta) = V$.

(iii)$\Rightarrow$(i). The proof follows immediately from the definition of a holomorphic retract of $D$. ■
From Proposition 2.5 we have the following

2.6. COROLLARY. Let $f$ be a holomorphic map of a hyperbolic convex domain $D$ in $\mathbb{C}^2$ into itself having a fixed point in $D$. Then one of the following cases necessarily occurs:

(i) $f$ has a unique fixed point.
(ii) The fixed point set of $f$ is the image of a complex geodesic $\varphi : \Delta \to D$.
(iii) $f$ is the identity map.

3. The infinite-dimensional case. Assume that $D$ is a domain in a locally convex Hausdorff vector space $E$.

The Kobayashi pseudodistance $d_D$ on $D$ is defined as in [6]. If $d_D$ is a distance and if the topology defined by $d_D$ is equivalent to the relative topology of $D$ in $E$, the domain $D$ is said to be hyperbolic (see [6]).

In this section we always assume that $D$ is a convex domain in a locally convex Hausdorff vector space $E$ such that $D'$ is contained in a hyperbolic domain $D'$ of $E$ and $f : D \to D$ is a holomorphic map such that the image $f(D)$ of $f$ is contained in some compact convex subset $K$ of $E$.

3.1. THEOREM. If the fixed point set $V$ of $f$ is not empty then $V$ is a holomorphic retract of $D$.

Proof. The space $\text{Hol}(D,E)$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Consider its subset $N = \{g \in \text{Hol}(D,D) : g|V = \text{Id} \text{ and } g(D) \subset K\}$ with the induced topology. Then $N$ is a nonempty convex subset of $\text{Hol}(D,E)$.

Now we prove that $N$ is compact in $\text{Hol}(D,E)$. Suppose that a sequence $\{g_n\} \subset N$ converges in $\text{Hol}(D,E)$ to a map $g \in \text{Hol}(D,E)$. Clearly $g|V = \text{Id}$ and $g(D) \subset K$. We must prove that $g(D) \subset D$. Indeed, we have $D = \bigcap_{\gamma \in \partial D} \{x_\gamma^* < a_\gamma\}$, where $x_\gamma^*$ are (real) linear functionals on $E$. Therefore $x_\gamma^* g$ is plurisubharmonic on $D$, $x_\gamma^* g(z) \leq a_\gamma$ for all $z \in D$ and $x_\gamma^* g(z) < a_\gamma$ for all $z \in V$. By the maximum principle, $x_\gamma^* g(z) < a_\gamma$ for all $z \in D$, i.e. $g(D) \subset D$. Thus $N$ is a closed subset in $\text{Hol}(D,E)$.

Now we prove that $\text{Hol}(D,D)$ is an even family [8]. Indeed, let $x \in D$, $y \in E$ be any points and let $U$ be a neighbourhood of $y$ in $E$. Without loss of generality we can assume that $y \in D' \subset D'$.

Take $r > 0$ such that $B_r = \{q \in D' : d_D(q,y) < r\} \subset U$. Since $D$ is hyperbolic, $V = \{p \in D : d_D(x,p) < r/2\}$ is an open neighbourhood of $x$ in $D$. Analogously, the ball $W = B_{r/2} = \{q \in D' : d_D(y,q) < r/2\}$ is an open neighbourhood of $y$ in $E$. It is easy to see that $\tilde{f}(V) \subset U$ whenever $\tilde{f}(x) \in W$ (for all $\tilde{f} \in \text{Hol}(D,D)$).
By Arzelà–Ascoli’s theorem (see [8, Theorems 7.6 and 7.21]), $N$ is compact in $\text{Hol}(D, E)$.

Consider the continuous operator

$$T : \text{Hol}(D, D) \to \text{Hol}(D, D), \quad g \mapsto f \circ g.$$ 

Obviously $T(N) \subset N$. By the Schauder fixed point theorem, there is $\varphi \in N$ such that $f \circ \varphi = \varphi$. As in Theorem 2.1, we have $\varphi(D) \subset V$ and $\varphi|V = \text{Id}$. Thus $V$ is a holomorphic retract of $D$. ■

3.2. Theorem. For any two distinct fixed points $x$ and $y$ of $f$, there exists a complex geodesic $\varphi : \Delta \to D$ which passes through $x, y$ and satisfies $\varphi(\Delta) \subset \text{Fix}(f)$.

Proof. Consider the space $\text{Hol}(\Delta, E)$ with the compact-open topology.

By our assumption, $D$ is a hyperbolic convex domain and hence $c_D(x, y) = d_D(x, y) = \delta_D(x, y) = \inf\{c_\Delta(0, \zeta) : \exists \varphi : \Delta \to D \text{ holomorphic with } \varphi(0) = x, \varphi(\zeta) = y\}$ (see [3]). Thus there exist a sequence $\{\varphi_n\} \subset \text{Hol}(\Delta, D)$ and a sequence $\{\zeta_n\} \subset \Delta$ such that $\varphi_n(0) = x, \varphi_n(\zeta_n) = y$ and $\lim_{n \to \infty} c_\Delta(0, \zeta_n) = c_D(x, y) < \infty$. We can assume that $\{\zeta_n\}$ converges to a point $\zeta_0 \in \Delta$ and $|\zeta| \leq r < 1$ for all $i \geq 0$. Put $\psi_n = f \circ \varphi_n$ for all $n \geq 1$.

Consider the subset $A = \{\theta \in \text{Hol}(\Delta, D) : \theta(0) = x, \theta(\zeta) = y \text{ for some } |\zeta| \leq r \text{ and } \theta(\Delta) \subset K\}$ of $\text{Hol}(\Delta, E)$ with the induced topology. Reasoning as in Theorem 3.1, we find that $A$ is closed in $\text{Hol}(\Delta, E)$ and $\text{Hol}(\Delta, D)$ is an even family. By Arzelà–Ascoli’s theorem, $A$ is compact.

Since $\{\psi_n\} \subset A$, we can assume that $\{\psi_n\}$ converges in $\text{Hol}(\Delta, D)$ to a map $\psi_0 \in \text{Hol}(\Delta, D)$. We have $\psi_0(0) = x, \psi_0(\zeta_0) = y$ and $c_\Delta(0, \zeta_0) = c_D(x, y)$, i.e. $\psi_0$ is a complex geodesic passing through $x$ and $y$.

Consider the subset $N = \{\varphi \in \text{Hol}(\Delta, D) : \varphi(0) = x, \varphi(\zeta_0) = y \text{ and } \varphi(\Delta) \subset K\}$ of $\text{Hol}(\Delta, E)$ with the induced topology. Just as in Theorem 3.1, $N$ is closed in $\text{Hol}(\Delta, E)$ and hence it is a nonempty compact convex subset of $\text{Hol}(\Delta, E)$.

Consider the continuous operator

$$T : \text{Hol}(\Delta, D) \to \text{Hol}(\Delta, D), \quad g \mapsto f \circ g.$$ 

Again as in Theorem 3.1, there is $\varphi \in N$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(\Delta) \subset \text{Fix}(f)$. ■

Theorems 3.1 and 3.2 yield the following

3.3. Corollary. Let $D$ be a bounded convex domain in a Banach complex space $E$. Assume that $f : D \to D$ is a holomorphic map whose image $f(D)$ is contained in some compact convex subset $K$ of $E$. Then

(i) $\text{Fix}(f)$ is a holomorphic retract of $D$ if $\text{Fix}(f) \neq \emptyset$. 

(ii) For any two distinct fixed points $x$, $y$ of $f$, there exists a complex geodesic $\varphi : \Delta \to D$ passing through $x$, $y$ and satisfying $\varphi(\Delta) \subset \text{Fix}(f)$.

References