ANNALES
POLONICI MATHEMATICI
LVI.2 (1992)

On the density of extremal solutions of differential inclusions

by F. S. De Blasi (Roma) and G. Pianigiani (Siena)

Abstract. An existence theorem for the Cauchy problem (∗) $\dot{x} \in \operatorname{ext} F(t,x)$, $x(t_0) = x_0$, in Banach spaces is proved, under assumptions which exclude compactness. Moreover, a type of density of the solution set of (∗) in the solution set of $\dot{x} \in F(t,x)$, $x(t_0) = x_0$, is established. The results are obtained by using an improved version of the Baire category method developed in [8]–[10].

1. Introduction. Let $\mathbb{E}$ be a separable reflexive real Banach space. Let F be a continuous multifunction defined on a nonempty open subset of $\mathbb{R} \times \mathbb{E}$ with values in the space of closed convex bounded subsets of $\mathbb{E}$ with nonempty interior. We shall consider the Cauchy problems

$$\dot{x} \in F(t,x)\,, \quad x(t_0) = x_0\,, \tag{1.1}$$

$$\dot{x} \in \operatorname{ext} F(t,x)\,, \quad x(t_0) = x_0\,, \tag{1.2}$$

where $\operatorname{ext} F(t,x)$ denotes the set of extreme points of $F(t,x)$.

By a result of Pliś ([2], p. 127) the solution set $\mathcal{M}_{\operatorname{ext} F}$ of (1.2) is not, in general, dense in the solution set $\mathcal{M}_F$ of (1.1). Nevertheless, elements of $\mathcal{M}_{\operatorname{ext} F}$ do approximate some significant subsets of $\mathcal{M}_F$. More specifically, we shall prove that, for any selection f of F in an admissible class which includes locally α-Lipschitz selections, if we denote by K_f the solution set of the Cauchy problem

$$\dot{x} = f(t,x)\,, \quad x(t_0) = x_0\,, \tag{1.3}$$

then $\mathcal{M}_{\operatorname{ext} F}$ has nonempty intersection with every neighborhood of K_f. In particular, the Cauchy problem (1.2) has solutions.

In finite dimensions this type of approximation result has been established by Pianigiani [16], by using the technique of Antosiewicz and Cellina [1]. Additional difficulties occur in infinite dimensions because, in this setting, the existence theory for differential equations is more delicate [12].

1991 *Mathematics Subject Classification*: 34A60, 47H05.

For recent contributions, see Tolstonogov [17], Bahi [3], Tolstonogov and Finogenko [18], Papageorgiou [14], [15].

The approach used in the present paper is a variant of the Baire category method introduced in [8]–[10] in order to prove the existence of solutions for nonconvex-valued differential inclusions in Banach spaces. We mention that recently this method has been improved by Bressan and Colombo [4], who have obtained an existence theorem containing both the existence theorem of [10] and Filippov's theorem [11] (see also Kaczyński and Olech [13], Antosiewicz and Cellina [1]). The property that $\mathcal{M}_{\mathrm{ext}\,F} \neq \emptyset$ has been proved in [10], under stronger hypotheses; subsequently the same result has been established in [7], by following the method and the techniques of [10].

2. Preliminaries and auxiliary results. Let $\mathbb{E}$ be a reflexive separable real Banach space with norm $\|\cdot\|$. We denote by $\mathcal{B}$ the metric space of all closed convex bounded subsets of $\mathbb{E}$, with nonempty interior, endowed with the Hausdorff distance h.

Let Z be a metric space. A multifunction $G : Z \to \mathcal{B}$ is said to be continuous, bounded, if it so as a function from Z to the metric space $\mathcal{B}$. Let X be a nonempty subset of Z. A single-valued function $f : X \to \mathbb{E}$ satisfying $f(x) \in G(x)$ for every $x \in X$ is called a *selection* of G on X (a selection of G if $X = Z$). For any subset X of Z, the interior of X and the closure of X are denoted by $\mathrm{int}\, X$ and $\overline{X}$, respectively. Moreover, if $X \subset Z$ is bounded, $\alpha[X]$ stands for the Kuratowski measure of noncompactness of X. In Z an open (resp. closed) ball with center $x \in Z$ and radius $r > 0$ is denoted by $B(x,r)$ (resp. $\widetilde{B}(x,r)$). The unit open ball in a normed space Z is denoted by B; moreover, for any subset X of Z, $\mathrm{ext}\, X$ stands for the set of extreme points of X.

Let J be a nonempty bounded interval of $\mathbb{R}$. As usual, $C(J,\mathbb{E})$ denotes the Banach space of all continuous bounded functions $x : J \to \mathbb{E}$ endowed with the norm of uniform convergence. Furthermore, by $|J|$ we mean the length of J. The space $\mathbb{R} \times \mathbb{E}$ will be equipped with the norm $\|(t,x)\| = \max\{|t|, \|x\|\}$, $(t,x) \in \mathbb{R} \times \mathbb{E}$. In the sequel, when a set $X \subset Z$ is considered as a metric space, it is understood that X retains the metric of Z.

Let U be a nonempty subset of $\mathbb{R}\times\mathbb{E}$. A function $f : U \to \mathbb{E}$ is said to be *α-Lipschitzean* (with constant k) if f is continuous and bounded on U, and there exists a constant $k \geq 0$ such that $\alpha[f(X)] \leq k\alpha[X]$ for every bounded set $X \subset U$. A function $f : U \to \mathbb{E}$ is said to be *locally Lipschitzean* (resp. *locally α-Lipschitzean*) if f is bounded (resp. continuous and bounded), and for each $(s,u) \in U$ there exist $\delta_{s,u} > 0$ and $k_{s,u} \geq 0$ such that f restricted to $B((s,u), \delta_{s,u})$ is Lipschitzean (resp. α-Lipschitzean) with constant $k_{s,u}$.

Let J be a nonempty bounded interval of the form $[a,b[$. We denote by $\mathcal{I}(J)$ the class of all countable families $\{J_i\}$ of nonempty pairwise disjoint intervals $J_i = [a_i, b_i[$ such that $\bigcup_i J_i = J$. A member of $\mathcal{I}(J)$ is called, for short, a *partition* of J. Let $\{J_i\}$ be a partition of J; the set of end points of the intervals J_i is called the *mesh* of the partition, and the number $\sup |J_i|$ the *norm* of the partition. Let $J = [a, b[$ be nonempty and bounded, and let $B(x_0, r) \subset \mathbb{E}$, $r > 0$. A function $f : J \times B(x_0, r) \to \mathbb{E}$ is said to be *piecewise locally Lipschitzean* (resp. *piecewise locally α-Lipschitzean*) if f is bounded and there exists a partition $\{J_i\} \in \mathcal{I}(J)$ of J such that the restriction of f to each set $J_i \times B(x_0, r)$ is locally Lipschitzean (resp. locally α-Lipschitzean).

We shall denote by $\mathcal{L}(J \times B(x_0, r))$ and $\mathcal{L}^\alpha(J \times B(x_0, r))$ the class of all functions $f : J \times B(x_0, r) \to \mathbb{E}$ which are, respectively, piecewise locally Lipschitzean and piecewise locally α-Lipschitzean.

Let $F : I \times B(x_0, r) \to \mathcal{B}$ be a multifunction, where $I = [t_0, T[$ and $B(x_0, r) \subset \mathbb{E}$ $(r > 0)$. We suppose:

(H_1) F is continuous on $I \times B(x_0, r)$,

(H_2) F is bounded on $I \times B(x_0, r)$ by a constant $M \geq 1$,

(H_3) $0 < T - t_0 < r/(2M)$.

By a *solution* of (1.1) (resp. (1.2), (1.3)) we mean a Lipschitzean function $x : J \to \mathbb{E}$ defined on a nondegenerate interval J containing t_0, satisfying (1.1) (resp. (1.2), (1.3)) a.e. in J. Set

$$\mathcal{M}_F = \{x : I \to \mathbb{E} \mid x \text{ is a solution of } (1.1)\},$$
$$\mathcal{M}_{\mathrm{ext}\,F} = \{x : I \to \mathbb{E} \mid x \text{ is a solution of } (1.2)\}.$$

The space $\mathcal{M}_F$, endowed with the metric of uniform convergence, is complete [8].

For F satisfying (H_1)–(H_3), set $\mathcal{S}_F = \{f \in \mathcal{L}(I \times B(x_0, r)) \mid f$ is a selection of $F\}$, $\mathcal{S}_F^\alpha = \{f \in \mathcal{L}^\alpha(I \times B(x_0, r)) \mid f$ is a selection of $F\}$. Clearly $\mathcal{S}_F$, $\mathcal{S}_F^\alpha$ are nonempty. For $f \in \mathcal{S}_F^\alpha$, we set $K_f = \{x : I \to \mathbb{E} \mid x$ is a solution of (1.3)$\}$. K_f is a nonempty compact subset of $\mathcal{M}_F$ and, if $f \in \mathcal{S}_F$, then K_f is a singleton.

PROPOSITION 2.1. *Let F satisfy (H_1)–(H_3). Let $f \in \mathcal{S}_F^\alpha$ and $\eta > 0$. Then there exists $\varrho = \varrho_f(\eta)$, $0 < \varrho < r/2$, such that if $x \in C(I, \mathbb{E})$ satisfies $\|x(t) - x_0\| < r$ and*

$$\Big\| \int_{t_0}^{t} [\dot{x}(s) - f(s, x(s))]\, ds \Big\| < \varrho \quad \textit{for every } t \in I,$$

then $x \in K_f + \eta B$.

Proof. Suppose the statement is not true. Then there exist $f \in \mathcal{S}_F^\alpha$, $\eta > 0$, and a sequence $\{x_n\} \subset C(I, \mathbb{E})$, with $\|x_n(t) - x_0\| < r$, $t \in I$, satisfying for each $n \in \mathbb{N}$

$$\Big\| \int\limits_{t_0}^{t} [\dot{x}_n(s) - f(s, x_n(s))]\, ds \Big\| < \frac{r}{2n} \qquad \text{for every } t \in I\,,$$

and $x_n \notin K_f + \eta B$. By a standard argument one can prove that $\alpha[\{x_n(t)\}] = 0$ for every $t \in I$. Hence the sequence $\{x_n\} \subset C(I, \mathbb{E})$ is compact. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to x, say. As $x \in K_f$, for k large enough we have $x_{n_k} \in K_f + \eta B$, a contradiction. This completes the proof.

PROPOSITION 2.2. *Let T, X be metric spaces. Let $G : T \times X \to \mathcal{B}$ be a continuous multifunction. Let $u_0 \in \mathbb{E}$ be such that $u_0 \in \operatorname{int} G(t,x)$ for every $(t,x) \in T \times \widetilde{B}(x_0, \delta)$, where $x_0 \in X$ and $\delta > 0$. Then there exists a locally Lipschitzean selection g of G satisfying $g(t,x) = u_0$ for every $(t,x) \in T \times \widetilde{B}(x_0, \delta)$.*

Proof. Let $(s,z) \in T \times X$. Suppose that $d(z, x_0) = \delta$, where d is the metric of X. Since $u_0 \in \operatorname{int} G(s,z)$, and G is continuous, there exists a ball $B((s,z), \delta_{s,z}) \subset T \times X$ such that $u_0 \in G(t,x)$ for every $(t,x) \in B((s,z), \delta_{s,z})$.

Suppose $d(z, x_0) > \delta$. In this case choose any $u_{s,z} \in \operatorname{int} G(s,z)$. Since G is continuous there exists a ball $B((s,z), \delta_{s,z}) \subset T \times X$ not intersecting $T \times \widetilde{B}(x_0, \delta)$ such that $u_{s,z} \in G(t,x)$ for every $(t,x) \in B((s,z), \delta_{s,z})$. Denote by $\mathcal{U} = \{U\}$ the family whose members are $T \times B(x_0, \delta)$ and each of the sets $B((s,z), \delta_{s,z})$ constructed above. $\mathcal{U}$ is an open covering of $T \times X$. For $U \in \mathcal{U}$, set

$$y_U = \begin{cases} u_0 & \text{if } U = T \times B(x_0, \delta)\,, \\ u_{s,z} & \text{if } U = B((s,z), \delta_{s,z})\,. \end{cases}$$

Let $\{p_U\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to $\mathcal{U}$ [6]. Without loss of generality we suppose that the functions $p_U : T \times X \to [0,1]$ are locally Lipschitzean. Now, define $g : T \times X \to \mathbb{E}$ by

$$g(t,x) = \sum_{U \in \mathcal{U}} p_U(t,x) y_U\ .$$

It is straightforward to verify that g is a locally Lipschitzean selection of G such that $g(t,x) = u_0$ for every $(t,x) \in T \times \widetilde{B}(x_0, \delta)$. This completes the proof.

Let $\mathbb{E}^*$ be the topological dual of $\mathbb{E}$. Let $\{e_n\} \subset \mathbb{E}^*$, $\|e_n\| = 1$, be a sequence dense in the unit sphere of $\mathbb{E}^*$ (recall that $\mathbb{E}$ is separable and reflexive). Let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathbb{E}^*$ and $\mathbb{E}$. Let $F : I \times B(x_0, r) \to \mathcal{B}$ satisfy (H_1)–(H_3). Following Choquet [6] and Castaing and Valadier [5], define $\varphi_F : I \times B(x_0, r) \times \mathbb{E} \to [0, +\infty]$ by

$$\varphi_F(t,x,v) = \begin{cases} \sum_{n=1}^{\infty} \langle e_n, v \rangle^2 / 2^n & \text{if } v \in F(t,x)\,, \\ +\infty & \text{if } v \notin F(t,x)\,. \end{cases}$$

Let $\mathcal{A}$ denote the class of all continuous affine functions $a : \mathbb{E} \to \mathbb{R}$. We associate with φ_F the function $\widehat{\varphi}_F : I \times B(x_0, r) \times \mathbb{E} \to [-\infty, +\infty[$ given by $\widehat{\varphi}_F(t, x, v) = \inf\{a(v) \mid a \in \mathcal{A} \text{ and } a(z) \geq \varphi_F(t, x, z) \text{ for every } z \in F(t, x)\}$. Now, define the Choquet function $d_F : I \times B(x_0, r) \times \mathbb{E} \to [-\infty, +\infty[$ by

$$d_F(t, x, v) = \widehat{\varphi}_F(t, x, v) - \varphi_F(t, x, v).$$

Some known properties of the Choquet function d_F are collected in the following proposition (see [5], [3]).

PROPOSITION 2.3. *Let F satisfy (H_1)–(H_3). Then we have*:

(i) *For each $(t, x) \in I \times B(x_0, r)$ and $v \in F(t, x)$ we have $0 \leq d_F(t, x, v) \leq M^2$. Moreover, $d_F(t, x, v) = 0$ if and only if $v \in \operatorname{ext} F(t, x)$.*

(ii) *For each $(t, x) \in I \times B(x_0, r)$ the function $v \to d_F(t, x, v)$ is concave on $\mathbb{E}$ and strictly concave on $F(t, x)$.*

(iii) *d_F is upper semicontinuous on $I \times B(x_0, r) \times \mathbb{E}$.*

(iv) *For each solution $x : I \to \mathbb{E}$ of (1.1), the function $t \to d_F(t, x(t), \dot{x}(t))$ is nonnegative, bounded and Lebesgue measurable.*

(v) *If $\{x_n\} \subset \mathcal{M}_F$ converges uniformly to $x \in \mathcal{M}_F$, then*

$$\limsup_{n\to+\infty} \int_I d_F(t, x_n(t), \dot{x}_n(t))\, dt \leq \int_I d_F(t, x(t), \dot{x}(t))\, dt.$$

3. Main result.

Let F satisfy (H_1)–(H_3). For $\theta > 0$, define

$$\mathcal{M}_\theta = \Big\{x \in \mathcal{M}_F \;\Big|\; \int_I d_F(t, x(t), \dot{x}(t))\, dt < \theta\Big\}.$$

LEMMA 3.1. *Let F satisfy (H_1)–(H_3). Then for every $\theta > 0$ the set $\mathcal{M}_\theta$ is open in $\mathcal{M}_F$.*

Proof. Let $\{x_n\} \subset \mathcal{M}_F \setminus \mathcal{M}_\theta$ be any sequence converging to $x \in \mathcal{M}_F$. By virtue of Proposition 2.3(v), we have

$$\int_I d_F(t, x(t), \dot{x}(t))\, dt \geq \limsup_{n\to+\infty} \int_I d_F(t, x_n(t), \dot{x}_n(t))\, dt \geq \theta,$$

and so $x \in \mathcal{M}_F \setminus \mathcal{M}_\theta$. Hence $\mathcal{M}_F \setminus \mathcal{M}_\theta$ is closed, completing the proof.

LEMMA 3.2. *Let F satisfy (H_1)–(H_3). Let $f \in \mathcal{S}_F^\alpha$. Let $\eta > 0$ and $\theta > 0$. Then there exists $g \in \mathcal{S}_F$ such that*

$$K_g \in \mathcal{M}_\theta \cap (K_f + \eta B). \tag{3.1}$$

Proof. The construction of g is realized in three steps. In Step 1, g is constructed locally on a set of the form $I_\delta \times B(x_0, r)$ for some interval $I_\delta \subset I$. In Step 2, g is extended to the whole set $I \times B(x_0, r)$ and it is shown that $g \in \mathcal{S}_F$. In Step 3, it is proved that for such g, (3.1) is satisfied.

Let $f \in \mathcal{S}_F^\alpha$, $\eta > 0$ and $\theta > 0$. Let $\varrho = \varrho_f(\eta)$ correspond to f and η according to Proposition 2.1. Fix σ with

$$0 < \sigma < \min\{\varrho, \theta\}\,. \tag{3.2}$$

Denote by $\{L_j\} \in \mathcal{I}(I)$ a partition of I associated with f (according to the definition of a piecewise α-Lipschitzean function) and let L_j be the interval of such partition containing t_0.

Step 1 (*Local construction of* g). Since $f(t_0, x_0) \in F(t_0, x_0)$, by the Krein–Milman theorem there exist $v_k \in \operatorname{ext} F(t_0, x_0)$ and $0 < \lambda_k \le 1$ ($k = 1, \ldots, p$), with $\sum_{k=1}^p \lambda_k = 1$, such that

$$\Big\| f(t_0, x_0) - \sum_{k=1}^p \lambda_k v_k \Big\| < \frac{\sigma}{4|I|}\,.$$

By Proposition 2.3(i), (iii), there exist $u_k \in \operatorname{int} F(t_0, x_0)$ ($k = 1, \ldots, p$) such that $d_F(t_0, x_0, u_k) < \sigma/|I|$, and

$$\Big\| f(t_0, x_0) - \sum_{k=1}^p \lambda_k u_k \Big\| < \frac{\sigma}{4|I|}\,. \tag{3.3}$$

Since f and F are continuous at (t_0, x_0), and d_F is upper semicontinuous at (t_0, x_0, u_k), there exists a δ_0, with $[t_0, t_0 + \delta_0[\subset L_j$, such that for every $(t, x) \in [t_0, t_0 + \delta_0[\times \widetilde{B}(x_0, \delta_0)$ we have

$$\|f(t, x) - f(t_0, x_0)\| \le \sigma/(4|I|)\,, \tag{3.4}$$

$$u_k \in \operatorname{int} F(t, x)\,, \quad k = 1, \ldots, p\,, \tag{3.5}$$

$$d_F(t, x, u_k) \le \sigma/|I|\,, \quad k = 1, \ldots, p\,. \tag{3.6}$$

Consider the interval $I_\delta = [t_0, t_0 + \delta[$, where

$$0 < \delta < \min\{\delta_0/M, \sigma/(4M)\} \tag{3.7}$$

($M \ge 1$ is the constant in (H_2)). Let $\{J_k\}_{k=1}^p$ be the partition of I_δ given by

$$J_k = [t_{k-1}, t_k[\,, \quad t_k = t_0 + \sum_{h=1}^k \lambda_h \delta\,, \quad k = 1, \ldots, p\,.$$

By Proposition 2.2, there exists a function $g : I_\delta \times B(x_0, r) \to \mathbb{E}$ which is a selection of F on $I_\delta \times B(x_0, r)$ and, moreover, for each k, $1 \le k \le p$, the restriction of g to $J_k \times B(x_0, r)$ is locally Lipschitzean and satisfies

$$g(t, x) = u_k \quad \text{for every } (t, x) \in J_k \times B(x_0, \delta_0)\,. \tag{3.8}$$

Let $x : I_\delta \to \mathbb{E}$ be the solution of the Cauchy problem

$$\dot{x} = g(t, x)\,, \quad x(t_0) = x_0\,. \tag{3.9}$$

We claim that

$$d_F(t, x(t), \dot{x}(t)) \le \sigma/|I|\,, \qquad t \in I_\delta \text{ a.e.,} \tag{3.10}$$

$$\Big\| \int_{t_0}^{t_0+\delta} [\dot{x}(s) - f(s, x(s))]\, ds \Big\| \le \frac{\sigma\delta}{2|I|}\,. \tag{3.11}$$

In order to prove (3.10), observe that for each $t \in I_\delta$ we have $\|x(t)-x_0\| < M\delta \le \delta_0$, thus

$$(t, x(t)) \in I_\delta \times B(x_0, \delta_0) \qquad \text{for every } t \in I_\delta\,. \tag{3.12}$$

Then, by (3.12), (3.8) and (3.6), for almost all $t \in \operatorname{int} I_\delta$ we have

$$d_F(t, x(t), \dot{x}(t)) = d_F(t, x(t), g(t, x(t))) = d_F(t, x(t), u_k) \le \sigma/|I|\,,$$

and (3.10) is satisfied.

Let us prove (3.11). We have

$$\begin{aligned}\Big\| \int_{t_0}^{t_0+\delta} [\dot{x}(s) - f(s, x(s))]\, ds \Big\| &= \Big\| \delta \sum_{k=1}^{p} \lambda_k u_k - \int_{t_0}^{t_0+\delta} f(s, x(s))\, ds \Big\| \\ &\le \Big\| \delta \sum_{k=1}^{p} \lambda_k u_k - \delta f(t_0, x_0) \Big\| + \Big\| \int_{t_0}^{t_0+\delta} [f(s, x(s)) - f(t_0, x_0)]\, ds \Big\| \\ &\le \delta \Big\| \sum_{k=1}^{p} \lambda_k u_k - f(t_0, x_0) \Big\| + \int_{t_0}^{t_0+\delta} \|f(s, x(s)) - f(t_0, x_0)\|\, ds\,.\end{aligned}$$

From this, by virtue of (3.3), (3.12), and (3.4), we have

$$\Big\| \int_{t_0}^{t_0+\delta} [\dot{x}(s) - f(s, x(s))]\, ds \Big\| < \delta \frac{\sigma}{4|I|} + \delta \frac{\sigma}{4|I|} = \frac{\sigma\delta}{2|I|}\,,$$

and also (3.11) is satisfied.

Step 2 (*Global construction of* g). Denote by $\mathcal{G}$ the class of all functions $g : D_g \times B(x_0, r) \to \mathbb{E}$, $D_g = [t_0, t_g[$, $t_0 < t_g \le T$, such that:

(i) g is a selection of F on $D_g \times B(x_0, r)$,

(ii) g is a piecewise locally Lipschitzean function,

(iii) the solution $x : D_g \to \mathbb{E}$ of the Cauchy problem (3.9) satisfies

$$d_F(t, x(t), \dot{x}(t)) \le \sigma/|I|\,, \quad t \in D_g \text{ a.e.,} \tag{3.13}$$

(iv) D_g admits a partition $\{I_i\} \in \mathcal{I}(D_g)$ of norm strictly less than $\sigma/(4M)$ such that, at each mesh point t_i, we have

$$\Big\| \int_{t_0}^{t_i} [\dot{x}(s) - f(s, x(s))]\, ds \Big\| \le \frac{\sigma(t_i - t_0)}{2|I|}\,. \tag{3.14}$$

$\mathcal{G}$ is nonempty, for the function $g : I_\delta \times B(x_0, r) \to \mathbb{E}$ constructed in Step 1 satisfies (i)–(iv). Now, let us introduce in $\mathcal{G}$ a partial order. For $g_k : D_{g_k} \times B(x_0, r) \to \mathbb{E}$ $(k = 1, 2)$, define $g_1 \prec g_2$ if and only if $t_{g_1} \le t_{g_2}$ and the restriction of g_2 to the set $D_{g_1} \times B(x_0, r)$ is equal to g_1. Let $\{g_j\}_{j \in \Gamma}$ be an arbitrary chain in $\mathcal{G}$. Let $\tau = \sup\{t_{g_j} \mid j \in \Gamma\}$. Define $g : D_g \times B(x_0, r) \to \mathbb{E}$, where $D_g = [t_0, \tau[$, by $g(t, x) = g_j(t, x)$ if $(t, x) \in D_{g_j} \times B(x_0, r)$. Clearly $g \in \mathcal{G}$ is an upper bound of the chain $\{g_j\}_{j \in \Gamma}$. By Zorn's Lemma there exists in $\mathcal{G}$ a maximal element, say g, where $g : D_g \times B(x_0, r) \to \mathbb{E}$ and $D_g = [t_0, t_g[$. We claim that $t_g = T$. Suppose $t_g < T$. Let $x : D_g \to \mathbb{E}$ be the solution of the Cauchy problem (3.9). Let u be the limit of $x(t)$ as t tends to t_g. As in Step 1 we construct a piecewise locally Lipschitzean selection of F on $\Delta \times B(x_0, r)$, say $h : \Delta \times B(x_0, r) \to \mathbb{E}$ (where $\Delta = [t_g, t_g + \delta[$ and $0 < \delta < \sigma/(4M)$), such that the solution $y : \Delta \to \mathbb{E}$ of the Cauchy problem $\dot y = h(t, y)$, $y(t_g) = u$, satisfies (3.10) and (3.11) (with y, Δ, t_g in place of x, I_δ, t_0). Now, defining $\gamma : [t_0, t_g + \delta[\times B(x_0, r) \to \mathbb{E}$ by

$$\gamma(t, x) = \begin{cases} g(t, x) & \text{if } (t, x) \in D_g \times B(x_0, r)\,, \\ h(t, x) & \text{if } (t, x) \in \Delta \times B(x_0, r)\,, \end{cases}$$

one can easily see that $\gamma \in \mathcal{G}$ and $g \prec \gamma$, $g \ne \gamma$, a contradiction. Thus $t_g = T$ and the existence of a map $g : I \times B(x_0, r) \to \mathbb{E}$ satisfying (i)–(iv) is proved, completing Step 2.

Step 3 (*The solution x of* (3.9) *satisfies* $x \in \mathcal{M}_\theta \cap (K_f + \eta B)$). Let $g : I \times B(x_0, r) \to \mathbb{E}$ satisfy (i)–(iv) (with I in place of D_g). By construction $g \in \mathcal{S}_F$. Let $x : I \to \mathbb{E}$ be the solution of (3.9). From (3.13) and (3.2), we have

$$\int_I d_F(t, x(t), \dot x(t))\, dt < \theta\,,$$

thus $x \in \mathcal{M}_\theta$. Now, let $t \in I$. With the notations of (iv) for some mesh point t_i of the partition $\{I_i\} \in \mathcal{I}(I)$, we have $|t - t_i| < \theta/(4M)$. From this inequality and (3.14) it follows that

$$\begin{aligned} \Big\| \int_{t_0}^{t} [\dot x(s) - f(s, x(s))]\, ds \Big\| & \\ &\le \Big\| \int_{t_0}^{t_i} [\dot x(s) - f(s, x(s))]\, ds \Big\| + \Big\| \int_{t_i}^{t} [\dot x(s) - f(s, x(s))]\, ds \Big\| \\ &\le \frac{\sigma(t_i - t_0)}{2|I|} + |t - t_i| 2M < \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma\,. \end{aligned}$$

As the last inequality is satisfied for arbitrary $t \in I$ and $\sigma < \varrho$ (by (3.2)), Proposition 2.1 implies that $x \in K_f + \eta B$. Hence $x \in \mathcal{M}_\theta \cap (K_f + \eta B)$ and thus $K_g \in \mathcal{M}_\theta \cap (K_f + \eta B)$, for $K_g = x$. This completes the proof.

THEOREM 3.3. *Let F satisfy (H_1)–(H_3). Let $f \in \mathcal{S}_F^\alpha$. Then for every $\eta > 0$ we have*

$$\mathcal{M}_{\operatorname{ext} F} \cap (K_f + \eta B) \neq \emptyset \,. \tag{3.15}$$

In particular, $\mathcal{M}_{\operatorname{ext} F}$ *is nonempty.*

Proof. Fix $f \in \mathcal{S}_F^\alpha$, $\eta > 0$ and set $\theta_n = 1/n$ ($n \in \mathbb{N}$). We denote by $B(u,r)$ and $\widetilde{B}(u,r)$ an open and a closed ball in the space $\mathcal{M}_F$. By Lemma 3.2 there exists $g_1 \in \mathcal{S}_F$ such that $K_{g_1} \in \mathcal{M}_F \cap (K_f + \eta B)$ and thus, for some $0 < \eta_1 < \theta_1$ we have

$$\widetilde{B}(K_{g_1}, \eta_1) \subset \mathcal{M}_F \cap (K_f + \eta B) \,.$$

By Lemma 3.2 there exists $g_2 \in \mathcal{S}_F$ such that $K_{g_2} \in \mathcal{M}_{\theta_1} \cap B(K_{g_1}, \eta_1)$. Since, by Lemma 3.1, this set is open in $\mathcal{M}_F$, there exists $0 < \eta_2 < \theta_2$ such that

$$\widetilde{B}(K_{g_2}, \eta_2) \subset \mathcal{M}_{\theta_1} \cap B(K_{g_1}, \eta_1) \,.$$

Continuing in this way gives a decreasing sequence of closed balls $\widetilde{B}(K_{g_n}, \eta_n) \subset \mathcal{M}_F$, where $g_n \in \mathcal{S}_F$ and $0 < \eta_n < \theta_n$, with diameters tending to zero, satisfying

$$\widetilde{B}(K_{g_{n+1}}, \eta_{n+1}) \subset \mathcal{M}_{\theta_n} \cap B(K_{g_n}, \eta_n) \,, \qquad n \in \mathbb{N} \,.$$

As $\mathcal{M}_F$ is complete, by Cantor's intersection theorem there is one (and only one) point, say x, lying in all the balls $\widetilde{B}(K_{g_n}, \eta_n)$. Since $x \in \mathcal{M}_{\theta_n}$, $n \in \mathbb{N}$, we have

$$\int_I d_F(t, x(t), \dot{x}(t)) \, dt = 0 \,.$$

Thus, by Proposition 2.3(i), $\dot{x}(t) \in \operatorname{ext} F(t, x(t))$ a.e., showing that $x \in \mathcal{M}_{\operatorname{ext} F}$. On the other hand, $x \in \widetilde{B}(K_{g_1}, \eta_1) \subset K_f + \eta B$. Hence (3.15) is proved. This completes the proof.

References

[1] A. Antosiewicz and A. Cellina, *Continuous selections and differential relations*, J. Differential Equations 19 (1975), 386–398.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin 1984.

[3] S. Bahi, *Quelques propriétés topologiques de l'ensemble des solutions d'une classe d'équations différentielles multivoques (II)*, Séminaire d'Analyse Convexe, Montpellier, 1983, exposé No. 4.

[4] A. Bressan and G. Colombo, *Generalized Baire category and differential inclusions in Banach spaces*, J. Differential Equations 76 (1988), 135–158.

[5] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin 1977.

[6] G. Choquet, *Lectures on Analysis*, Benjamin, Reading 1969.

[7] P. V. Chuong, *Un résultat d'existence de solutions pour des équations différentielles multivoques*, C. R. Acad. Sci. Paris 301 (1985), 399–402.

[8] F. S. De Blasi and G. Pianigiani, *A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces*, Funkcial. Ekvac. (2) 25 (1982), 153–162.

[9] —, —, *The Baire category method in existence problems for a class of multivalued differential equations with nonconvex right hand side*, ibid. 28 (1985), 139–156.

[10] —, —, *Differential inclusions in Banach spaces*, J. Differential Equations 66 (1987), 208–229.

[11] A. F. Filippov, *The existence of solutions of generalized differential equations*, Math. Notes 10 (1971), 608–611.

[12] A. N. Godunov, *Peano's theorem in Banach spaces*, Funktsional. Anal. i Prilozhen. 9 (1) (1974), 59–60 (in Russian).

[13] H. Kaczyński and C. Olech, *Existence of solutions of orientor fields with nonconvex right hand side*, Ann. Polon. Math. 29 (1974), 61–66.

[14] N. S. Papageorgiou, *On the solution set of differential inclusions in Banach spaces*, Appl. Anal. 25 (1987), 319–329.

[15] —, *On measurable multifunctions with applications to random multivalued equations*, Math. Japon. 32 (1987), 437–464.

[16] G. Pianigiani, *On the fundamental theory of multivalued differential equations*, J. Differential Equations 25 (1977), 30–38.

[17] A. A. Tolstonogov, *On differential inclusions in Banach spaces*, Soviet Math. Dokl. 20 (1979), 186–190.

[18] A. A. Tolstonogov and I. A. Finogenko, *On solutions of a differential inclusion with lower semicontinuous nonconvex right-hand side in a Banach space*, Math. USSR-Sb. 53 (1986), 203–231.

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI ROMA II
VIA FONTANILE DI CARCARICOLA
00133 ROMA, ITALY

ISTITUTO DI MATEMATICA
UNIVERSITÀ DI SIENA
VIA DEL CAPITANO 15
53100 SIENA, ITALY

Reçu par la Rédaction le 3.1.1990
Révisé le 1.8.1990