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A constant in pluripotential theory

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Abstract. We compute the constant

$$\sup\left\{\frac{1}{\deg P}\left(\max_S \log|P| - \int_S \log|P|\,d\sigma\right) : P \text{ a polynomial in } \mathbb{C}^n\right\},$$

where S denotes the euclidean unit sphere in $\mathbb{C}^n$ and σ its unitary surface measure.

Let S be the euclidean unit sphere in $\mathbb{C}^n$ and σ its unitary surface measure. Setting

$$L := \{u \in \mathrm{PSH}(\mathbb{C}^n) : \sup\{u(z) - \log_+|z| : z \in \mathbb{C}^n\} < \infty\}$$

we define

$$c_n := \sup\left\{\max_S u - \int_S u\,d\sigma : u \in L\right\}. \tag{1}$$

The aim of this note is to find the exact value of the constant c_n (for its applications see [2]). We start with the following

LEMMA 1. *For $a \in \mathbb{C}$ we have*

$$\int_S \log|z_n + a|\,d\sigma(z) = \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{j=1}^{n-1}\frac{1}{j}(1-|a|^2)^j & \text{if } |a| \le 1,\\ \log|a| & \text{if } |a| \ge 1.\end{cases}$$

P r o o f. The function $z \mapsto \log|z_n + a|$ is harmonic in $\{|z| < |a|\}$, thus for $|a| \ge 1$ the statement is obvious. Now let $|a| < 1$. Without loss of generality we may assume $a \ge 0$. We have

$$\int_S \log|z_n + a|\,d\sigma(z) = \frac{n-1}{\pi}\int_0^1 r(1-r^2)^{n-2}\int_0^{2\pi} \log|re^{it} + a|\,dt\,dr$$

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(see [3], p. 15). Using the equality

$$\frac{1}{2\pi}\int_0^{2\pi}\log|re^{it}+a|\,dt=\max\{\log r,\log a\}$$

we see that

$$\begin{aligned}\int_S \log|z_n+a|\,d\sigma(z) &= 2(n-1)\log a\int_0^a r(1-r^2)^{n-2}\,dr\\ &\quad + 2(n-1)\int_a^1 r(1-r^2)^{n-2}\log r\,dr\\ &= -\log a[(1-r^2)^{n-1}]_0^a\\ &\quad + \Big[-(1-r^2)^{n-1}\log r+\log r+\frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{j}(1-r^2)^j\Big]_a^1\\ &= -\frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{j}(1-a^2)^j\,. \ \blacksquare\end{aligned}$$

LEMMA 2. *For every* $n\ge 2$ *there exists a unique* $\varrho_n\in(0,1)$ *such that* $(1+\varrho_n)(1-\varrho_n^2)^{n-1}=1$. *Moreover*, $\{\varrho_n\}$ *decreases to* 0.

Proof. For $\varrho\in[0,1]$ let

$$f_n(\varrho):=(1+\varrho)(1-\varrho^2)^{n-1}-1\,.$$

We have

$$f_n'(\varrho)=-(2n-1)(1+\varrho)(1-\varrho^2)^{n-2}\left(\varrho-\frac{1}{2n-1}\right).$$

Hence $f_n(0)=0$, f_n increases in $(0,1/(2n-1))$, decreases in $(1/(2n-1),1)$ and $f_n(1)=-1$. Thus the first statement is clear. Now since $f_{n+1}(\varrho_n)<f_n(\varrho_n)=0$ we see that the sequence $\{\varrho_n\}$ is decreasing. Let then $\varepsilon:=\lim\varrho_n$. Since

$$0=\lim f_n(\varrho_n)\le\lim 2(1-\varepsilon^2)^{n-1}-1$$

we conclude that ε must be equal to 0. ■

THEOREM. *The supremum in* (1) *is attained for the function* $z\mapsto\log|z_n+\varrho_n|$ *where* $\varrho_1=1$ *and for* $n\ge 2$, ϱ_n *is given by Lemma* 2. *Therefore by Lemma* 1 *we have*

$$\begin{aligned}c_1&=\log 2\,,\\ c_n&=\log(1+\varrho_n)+\frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{j}(1-\varrho_n^2)^j\quad \textit{for } n\ge 2\,.\end{aligned}$$

Proof. Let $u \in L$. First we want to show that it is enough to take functions in the supremum which depend on one variable. For $n \geq 2$ we have

$$\int_S u\,d\sigma = \int_D \int_{\Gamma_{z_n}} u(z', z_n)\,d\sigma_{z_n}(z')\,A(|z_n|)\,d\lambda(z_n)\,,$$

where D is the unit disc in $\mathbb{C}$, λ its Lebesgue measure, $z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$,

$$\Gamma_{z_n} := \{(z', z_n) \in \mathbb{C}^n : |z'|^2 = 1 - |z_n|^2\} \subset S\,,$$

σ_{z_n} the unitary surface measure on Γ_{z_n} and

$$A(r) := \frac{n-1}{\pi}(1 - r^2)^{n-2}\,.$$

By subharmonicity of the function $z' \mapsto u(z', z_n)$ we have

$$\int_{\Gamma_{z_n}} u(z', z_n)\,d\sigma_{z_n}(z') \geq u(0, z_n)$$

and therefore

$$\int_S u\,d\sigma \geq \int_S u(0, z_n)\,d\sigma(z)\,.$$

Since σ is invariant under unitary transformations we may assume that $\max_S u = u(0, \ldots, 0, 1)$, hence

$$\max_S u - \int_S u\,d\sigma \leq \max_{|z_n|=1} u(0, z_n) - \int_S u(0, z_n)\,d\sigma(z)\,.$$

We may thus consider only functions from L depending on the one variable z_n. Define

$$\begin{aligned} c'_n &:= \sup\left\{\frac{1}{d}\Big(\max_S \log|P| - \int_S \log|P|\,d\sigma\Big) : P \in \mathcal{P}_d(\mathbb{C}^n)\right\} \\ &= \sup\left\{\frac{1}{d}\Big(\max_{|z_n|=1} \log|P(z_n)| - \int_S \log|P(z_n)|\,d\sigma(z)\Big) : P \in \mathcal{P}_d(\mathbb{C})\right\} \end{aligned}$$

where $\mathcal{P}_d(\mathbb{C}^n)$ denotes the set of all polynomials in $\mathbb{C}^n$ of degree $\leq d$. If $P(\zeta) = \alpha(\zeta + a_1)\ldots(\zeta + a_d)$ then

$$\begin{aligned} &\frac{1}{d}\Big(\max_{|z_n|=1} \log|P(z_n)| - \int_S \log|P(z_n)|\,d\sigma(z)\Big) \\ &\qquad \leq \frac{1}{d}\sum_{k=1}^{d}\Big(\max_{|z_n|=1} \log|z_n + a_k| - \int_S \log|z_n + a_k|\,d\sigma(z)\Big) \\ &\qquad \leq \max_{k=1,\ldots,d}\Big(\max_{|z_n|=1} \log|z_n + a_k| - \int_S \log|z_n + a_k|\,d\sigma(z)\Big)\,. \end{aligned}$$

Therefore

$$c'_n = \sup\Big\{\max_{|z_n|=1}\log|z_n+\varrho| - \int_S \log|z_n+\varrho|\,d\sigma(z) : \varrho\in[0,1]\Big\}$$

(since for $\varrho>1$ one has $\log(1+\varrho)-\log\varrho<\log 2$, and by Lemma 1, it is enough to take only $\varrho\in[0,1]$ in the supremum). For $\varrho\in[0,1]$ put

$$g_n(\varrho) := \max_{|z_n|=1}\log|z_n+\varrho| - \int_S \log|z_n+\varrho|\,d\sigma(z)\,. \tag{2}$$

By Lemma 1

$$g_n(\varrho) = \log(1+\varrho) + \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{j}(1-\varrho^2)^j\,.$$

Therefore $g'_1(\varrho)=1/(1+\varrho)$ and for $n\ge 2$

$$\begin{aligned} g'_n(0) &= 1\,,\\ g'_n(\varrho) &= \frac{(1+\varrho)(1-\varrho^2)^{n-1}-1}{(1+\varrho)\varrho} \qquad \text{for } \varrho>0\,,\\ g'_n(1) &= -1/2\,. \end{aligned}$$

Hence, by Lemma 2 it is enough to prove that $c'_n=c_n$. Of course $c'_n\le c_n$, let us then take any $u\in L$. By the approximation property of L (see [4]) there exists a sequence of polynomials $\{P_k\}$ such that $P_k\in\mathcal{P}_k$ and

$$u = \Big(\limsup_{k\to\infty}\frac{1}{k}\log|P_k|\Big)^*$$

(v^* denotes the upper regularization of v). Now Fatou's Lemma gives

$$\max_S u - \int_S u\,d\sigma \le \limsup_{k\to\infty}\frac{1}{k}\Big(\max_S\log|P_k| - \int_S\log|P_k|\,d\sigma\Big)\,.$$

The proof is complete. ∎

Put

$$H := \{v\in \mathrm{PSH}(\mathbb{C}^n) : v(az)=v(z)+\log|a| \text{ for all } a\in\mathbb{C}\,,\ z\in\mathbb{C}^n\}$$

and define

$$\kappa_n := \sup\Big\{\max_S v - \int_S v\,d\sigma : v\in H\Big\}\,.$$

Then $H\subset L$ and

$$\kappa_n = -\int_S \log|z_n|\,d\sigma(z) = \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{j}$$

(see [4] and [1]).

COROLLARY. *$c_1 = \log 2$, $\kappa_n < c_n < \kappa_n + \log(1+\varrho_n)$ for $n \ge 2$, $\lim_{n\to\infty}(c_n - \kappa_n) = 0$ and $\{c_n\}$ is increasing.*

Proof. Everything but the last statement is clear. Taking g_n defined by (2) we see that $g_n \le g_{n+1}$, hence

$$c_n = \max_{[0,1]} g_n < \max_{[0,1]} g_{n+1} = c_{n+1}\,. \ \blacksquare$$

References

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