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On two new functional equations for generalized Joukowski transformations

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Abstract. The purpose of this paper is to solve two functional equations for generalized Joukowski transformations and to give a geometric interpretation to one of them. Here the Joukowski transformation means the function $\frac{1}{2}(z+z^{-1})$ of a complex variable z.

1. Introduction and statement of the results. A function

$$Az^p + Bz^{-p},$$

where A, B are arbitrary complex constants and p is an arbitrary integer, is said to be a generalized Joukowski transformation (see [4], [5], [7], [8]). In [8] the following functional equation for the generalized Joukowski transformations was studied:

$$|f(re^{i\theta})|^2 + |f(1)|^2 = |f(r)|^2 + |f(e^{i\theta})|^2 \tag{1}$$

for all real r (> 0) and θ. For (1) the following theorem was proved:

Theorem A. *Suppose that a complex-valued function f of a complex variable z is analytic for $0 < |z| < +\infty$ and is either analytic or has a pole at $z = 0$. Then the only solution of* (1) *is given by*

$$f(z) = Az^p + Bz^{-p},$$

where A, B are arbitrary complex constants and p is an arbitrary integer.

In this paper we consider the following two new functional equations which the generalized Joukowski transformations satisfy:

$$|f(re^{i\theta}) + f(1)| = |f(r) + f(e^{i\theta})| \tag{2}$$

and

$$|f(re^{i\theta}) - f(1)| = |f(r) - f(e^{i\theta})|, \tag{3}$$

where r and θ are real variables with $r > 0$.

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Let $f(z) = Az^p + Bz^{-p}$. By Theorem A we obtain

$$|f(re^{i\theta})|^2 + |f(1)|^2 = |f(r)|^2 + |f(e^{i\theta})|^2 .$$

Furthermore, after some calculations, we have

$$\operatorname{Re}(f(re^{i\theta})\overline{f(1)}) = \operatorname{Re}(f(r)\overline{f(e^{i\theta})}) .$$

Since

$$|A+B|^2 = |A|^2 + |B|^2 + 2\operatorname{Re}(A\overline{B})$$

and

$$|A-B|^2 = |A|^2 + |B|^2 - 2\operatorname{Re}(A\overline{B})$$

hold for all complex numbers A, B, we see that the generalized Joukowski transformations satisfy (2) and (3).

The purpose of this paper is to solve (2) and (3), i.e., to prove the following two theorems and to present a geometric interpretation of (3) (see Section 4).

THEOREM 1. *Suppose that a complex-valued function f of a complex variable z is analytic for $0 < |z| < +\infty$. Then the only solution of* (2) *is given by*

$$f(z) = Az^p + Bz^{-p} ,$$

where A, B are arbitrary complex constants and p is an arbitrary integer.

THEOREM 2. *Suppose that a complex-valued function f of a complex variable z is analytic for $0 < |z| < +\infty$. Then the only solution of* (3) *is given by*

$$f(z) = Az^p + Bz^{-p} + C ,$$

where A, B, C are arbitrary complex constants and p is an arbitrary integer.

Thus, by Theorems 1 and 2 we find that the generalized Joukowski transformations are characterized up to an additive complex constant C by (2), (3), respectively.

2. Lemma. To prove Theorems 1 and 2 we shall apply the following lemma:

LEMMA. *Suppose that a complex-valued function f of a complex variable z is analytic for $0 < |z| < +\infty$ and is a solution of* (2). *Then $g(z) \overset{\text{def}}{=} f(1/z)$ is also a complex-valued function of z and is analytic for $0 < |z| < +\infty$. Furthermore, g is a solution of* (2).

Proof. We have only to prove that g is a solution of (2). Replacing r and θ by $1/r$ and $-\theta$, respectively, in (2) and observing that

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

yields the desired result. ∎

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We may assume that f is not a complex constant. Since, by hypothesis, f is analytic for $0 < |z| < +\infty$, we can expand f in a Laurent series for $0 < |z| < +\infty$. We show that it can be expressed by

$$f(z) = \sum_{|n| \le p} a_n z^n \tag{4}$$

for $0 < |z| < +\infty$, where p is a positive integer, i.e., neither $z = \infty$ nor $z = 0$ is an essential singularity. Let

$$h(r) = |f(r)| \,. \tag{5}$$

Then either $h(r) \to +\infty$ as $r \to +\infty$ or there exists a sequence $\{r_n\}$ such that

$$h(r_n) = O(1) \quad (n = 1, 2, 3, \ldots) \qquad \text{as } n \to +\infty \,. \tag{6}$$

In the first case we have for $0 < |z| < +\infty$, by (2), (5) and by the triangle inequality,

$$|f(z)| \ge h(r) - |f(e^{i\theta})| - |f(1)| \ge h(r) - |f(e^{i\theta})| \ge h(r) - M \,,$$

where $z = re^{i\theta}$, $M = \sup_{|\theta| \le \pi} |f(e^{i\theta})| + |f(1)|$. By the assumption of the first case there exists $R > 0$ such that $h(r) > M + 1$ for $r > R$. Hence $|f(z)| > 1$ for $|z| > R$. This contradicts the Picard Theorem.

In the second case we have, by (2), (6) and the triangle inequality,

$$\begin{aligned} |f(r_n e^{i\theta})| &\le |f(r_n)| + |f(e^{i\theta})| + |f(1)| = h(r_n) + |f(e^{i\theta})| + |f(1)| \\ &\le K + \sup_{|\theta| \le \pi} |f(e^{i\theta})| + |f(1)| \overset{\text{def}}{=} M \qquad (n = 1, 2, \ldots) \,, \end{aligned}$$

for all real θ. Here K is a positive real constant. Hence we obtain

$$|f(r_n e^{i\theta})| \le M \qquad (n = 1, 2, \ldots)$$

for all real θ. By this inequality and Riemann's Theorem (in a sharp form) concerning removable singularities f becomes analytic at $z = \infty$. This is a contradiction. Summarizing the above, $z = \infty$ is not an essential singularity for f. Similarly, we can prove that $z = 0$ is not an essential singularity for f. Thus (4) holds for some positive integer p in $0 < |z| < +\infty$. We may assume that at least one of a_p and a_{-p} is nonzero. We discuss three cases.

Case 1: $p \ge 2$ and

$$a_p \ne 0 \,. \tag{7}$$

By (2) and by the identity $|\gamma|^2 = \gamma\overline{\gamma}$ (for all complex γ) we obtain

$$(f(re^{i\theta}) + f(1))(\overline{f(re^{i\theta})} + \overline{f(1)}) = (f(r) + f(e^{i\theta}))(\overline{f(r)} + \overline{f(e^{i\theta})}) .$$

By (4) we have

$$\Big(\sum_{|n|\le p} a_n r^n e^{in\theta} + f(1)\Big)\Big(\sum_{|n|\le p} \overline{a}_n r^n e^{-in\theta} + \overline{f(1)}\Big) = \Big(\sum_{|n|\le p} a_n r^n + f(e^{i\theta})\Big)\Big(\sum_{|n|\le p} \overline{a}_n r^n + f(e^{i\theta})\Big).$$

Equating the coefficients of r^n on both sides for $p+1 \le n \le 2p-1$ ($p \ge 2$) yields

$$\sum_{k=n-p}^{p} a_k \overline{a}_{n-k} e^{i(2k-n)\theta} = \sum_{k=n-p}^{p} a_k \overline{a}_{n-k} \tag{8}$$

for all real θ, and for $n = p$

$$a_p e^{ip\theta}(\overline{a}_0 + \overline{f(1)}) + \overline{a}_p e^{-ip\theta}(a_0 + f(1)) = a_p(\overline{a}_0 + \overline{f(e^{i\theta})}) + \overline{a}_p(a_0 + f(e^{i\theta})) \tag{9}$$

for all real θ. Since $e^{ik\theta}$ ($k \in \mathbb{Z}$) are linearly independent, by (8) we obtain

$$a_k \overline{a}_{n-k} = 0 \tag{10}$$

for all k satisfying $n - p \le k \le p$ and

$$2k - n \ne 0 \tag{11}$$

with $p+1 \le n \le 2p-1$ ($p \ge 2$).

If we set $k = p$ in (10), then, by the fact that $k = p$ satisfies (11) and by (7), we have $\overline{a}_{n-p} = 0$ for $p+1 \le n \le 2p-1$ ($p \ge 2$), and so $a_1 = a_2 = \ldots = a_{p-1} = 0$. By (4) we obtain

$$f(z) = a_{-p} z^{-p} + \sum_{n=1}^{p-1} a_{-n} z^{-n} + a_0 + a_p z^p .$$

Substituting the above into (9) yields

$$a_p(\overline{f(1)} + \overline{a}_0 - \overline{a}_p - \overline{a}_{-p})e^{ip\theta} + \overline{a}_p(f(1) + a_0 - a_p - a_{-p})e^{-ip\theta} = a_p \sum_{n=1}^{p-1} \overline{a}_{-n} e^{in\theta} + \overline{a}_p \sum_{n=1}^{p-1} a_{-n} e^{-in\theta} + 2(\overline{a}_0 a_p + a_0 \overline{a}_p) . \tag{12}$$

Since $e^{ik\theta}$ ($k \in \mathbb{Z}$) are linearly independent, by (12) we obtain $a_p \overline{a}_{-n} = 0$ for $1 \le n \le p-1$, and so, by (7),

$$a_{-1} = a_{-2} = \ldots = a_{-p+1} = 0 . \tag{13}$$

By (13), we have

$$f(z) = a_{-p}z^{-p} + a_0 + a_p z^p \,. \tag{14}$$

By (12), (13), (14) we obtain

$$\overline{a}_0 a_p e^{ip\theta} + a_0 \overline{a}_p e^{-ip\theta} = \overline{a}_0 a_p + a_0 \overline{a}_p \,.$$

Since $e^{ip\theta}$, $e^{-ip\theta}$, 1 are linearly independent, we obtain $\overline{a}_0 a_p = 0$ and so, by (7), $a_0 = 0$. By (14) we thus have

$$f(z) = a_{-p}z^{-p} + a_p z^p \,.$$

Case 2: $a_p = 0$ and $a_{-p} \neq 0$ with $p \geq 2$. We consider

$$g(z) = f(1/z) \,. \tag{15}$$

By the lemma g is a solution of (2). Applying our result of Case 1 to g yields

$$g(z) = a_p z^{-p} + a_{-p} z^p = 0 \cdot z^{-p} + a_{-p} z^p = a_{-p} z^p \,.$$

Case 3: $p = 1$. In this case, by (4) we obtain $f(z) = a_{-1}z^{-1} + a_0 + a_1 z$. Using a similar method to that of Case 1 yields $a_0 = 0$. ∎

By a similar method to that of solving (2) we can prove Theorem 2.

4. A geometric interpretation of the functional equation (3). Before presenting a geometric interpretation of the functional equation (3) we state the following definition:

Definition (see [9], p. 308). (i) Points on two confocal ellipses which have the same eccentric angles are called *corresponding points*.

(ii) Points on two confocal hyperbolas which have the same eccentric angles are called *corresponding points*.

Now we give a preliminary consideration about mapping properties of the Joukowski transformation $w = f(z) = \frac{1}{2}(z + 1/z)$ (see [1], [2], [3], [6]).

(i) The family of concentric circles with common centre at $z = 0$ are transformed by $w = f(z) = \frac{1}{2}(z + 1/z)$ into the family of confocal ellipses with common foci at 1 and -1.

(ii) The family of rays emanating from $z = 0$ are transformed by $w = f(z) = \frac{1}{2}(z + 1/z)$ into the family of confocal hyperbolas with common foci at 1 and -1.

We may now present a geometric interpretation of the functional equation (3), i.e., we shall prove geometrically by using the following theorem (see [9], p. 308) that $w = f(z) = \frac{1}{2}(z + 1/z)$ is a solution of (3).

Theorem B. (i) *Consider two arbitrary ellipses E_1 and E_2 of a family of confocal ellipses. If P, Q are any corresponding points on E_1 and if R, S are any corresponding points on E_2, then $\overline{PR} = \overline{QS}$.*

(ii) *Consider two arbitrary hyperbolas H_1 and H_2 of a family of confocal hyperbolas. If P, Q are any corresponding points on H_1 and if R, S are any corresponding points on H_2, then $\overline{PR} = \overline{QS}$.*

We discuss two cases. In the first case, we set $f(z) = \frac{1}{2}(z + 1/z)$ and $z = re^{i\theta}$ where r (> 0), θ are real with $r \neq 1$ and consider another point $se^{i\theta}$ where s (> 0) is real with $s \neq 1$. It is easy to see that the two points $f(re^{i\theta})$, $f(se^{i\theta})$ are corresponding points on the two ellipses

$$\frac{x^2}{(\frac{1}{2}(r+1/r))^2} + \frac{y^2}{(\frac{1}{2}(r-1/r))^2} = 1$$

and

$$\frac{x^2}{(\frac{1}{2}(s+1/s))^2} + \frac{y^2}{(\frac{1}{2}(s-1/s))^2} = 1\,,$$

respectively. (Their same eccentric angle is θ.) Similarly, the two points $f(r)$, $f(s)$ are also corresponding points on the same two ellipses, respectively. (Their same eccentric angle is 0.) Hence, by Theorem B(i) we obtain

$$|f(re^{i\theta}) - f(s)| = |f(r) - f(se^{i\theta})|\,.$$

Letting $s \to 1$ yields

$$|f(re^{i\theta}) - f(1)| = |f(r) - f(e^{i\theta})|\,.$$

Consequently, $f(z) = \frac{1}{2}(z + 1/z)$ is a solution of (3).

The second case, when we use Theorem B(ii), can be handled similarly. ■

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