A univalence criterion for meromorphic functions

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Abstract. A sufficient univalence condition for meromorphic functions is given.

1. Let \( f \) denote a meromorphic and locally univalent function in \( E = \{ z : |z| > 1 \} \), that is, \( f'(z) \neq 0 \) and any pole of \( f \) is simple.

In this note we give a univalence criterion for \( f \) in terms of the Schwarz derivative defined by

\[
S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

Epstein (see for example [4]) gives the following univalence criterion for meromorphic and locally univalent functions in the unit disk \( D = \{ z : |z| < 1 \} \).

Theorem E. Let \( f \) be meromorphic and \( g \) holomorphic in \( D \). If both functions are locally univalent in \( D \) and if

\[
\left| \frac{1}{2} (1 - |z|^2)^2 (S_f(z) - S_g(z)) + (1 - |z|^2) \frac{g''(z)}{g'(z)} \right| \leq 1, \quad z \in D,
\]

then \( f \) is univalent in \( D \).

In this section we transfer Theorem E to the exterior of the unit disk, which cannot be obtained immediately from Theorem E.

Theorem 1. Let \( f \) and \( g \) be meromorphic and locally univalent functions in \( E \) and let \( g(\zeta) = b_0 + b_1/\zeta + \ldots \). If there exists a holomorphic function \( h \) in \( E \) with \( \text{Re} h \geq 1/2 \) in \( E \) and \( h(\zeta) = 1 + h_2/\zeta^2 + \ldots \) such that

\[
(1) \quad \left| \frac{1}{2} ((|\zeta|^2 - 1)^2 (S_f(\zeta) - S_g(\zeta)) - \frac{\zeta^2 h(\zeta)}{\zeta} \right| \leq 1, \quad \zeta \in E,
\]

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then $f$ is univalent in $E$.

**Proof.** Without loss of generality we can consider the functions of the form

$$f(\zeta) = \zeta + \frac{a_1}{\zeta} + \ldots, \quad g(\zeta) = \zeta + \frac{b_1}{\zeta} + \ldots$$

since the Schwarzian derivative is invariant under Möbius transformations. The assumption $h(\infty) = 1$ can be dropped (see [5]). Let

$$v(\zeta) = \sqrt{\frac{g'(\zeta)}{f'(\zeta)}} = 1 + \frac{\beta_1}{\zeta^2} + \ldots, \quad u(\zeta) = f(\zeta)v(\zeta) = \zeta + \frac{c_1}{\zeta} + \ldots$$

The functions $u$ and $v$ are meromorphic in $E$ since $f$ and $g$ do not have multiple poles and $f'$ and $g'$ are different from zero.

For $t \in I = [0, \infty)$, $1/\zeta = z$, we consider

$$f(z, t) = \left[ \frac{u(z) + (e^{-t} - e^{t}) \frac{1}{2} h(\frac{z}{t}) u'(\frac{z}{t})}{v(z) + (e^{-t} - e^{t}) \frac{1}{2} h(\frac{z}{t}) v'(\frac{z}{t})} \right]^{-1}, \quad z \in D.$$  

The function $f(z, t)$ is meromorphic in $D$. By (2) the denominator in (3) in square brackets is $1 + O(z^2)$ as $z \to 0$, uniformly in $t$. Hence there exist constants $r_0 > 0$ and $K_0$ such that

$$|f(z, t)| \leq K_0 e^t \quad \text{for} \ |z| < r_0, \ t \in I.$$  

By (2) the numerator in (3) is $e^{-t}/z + O(z^2)$ as $z \to 0$. Hence

$$f(z, t) = e^t z + O(z^2) \quad \text{as} \ z \to 0.$$  

We set

$$f'(z, t) = \frac{\partial f(z, t)}{\partial z}, \quad \dot{f}(z, t) = \frac{\partial f(z, t)}{\partial t}.$$  

After simple calculations from (3) we obtain

$$w(z, t) = \frac{\dot{f}(z, t) - zf'(z, t)}{f(z, t) + zf'(z, t)}$$

$$= - \left\{ \left( \frac{1}{h} - 1 \right) e^{2t} + (e^{-t} - e^{t}) \frac{e^{2t}}{z} \left( \frac{h'}{h} + \frac{u''v - uv''}{u'v - uv'} \right) + (e^{-t} - e^{t}) \frac{e^{2t}}{z^2} \frac{u''v' - u'v''}{u'v - uv'} \right\},$$

where

$$u'v - uv' = g', \quad u''v - uv'' = g'',$$

$$u''v' - u'v'' = \frac{1}{2} g' (S_f - S_g).$$


and \( u, v, u', v', u'', v'' \) are calculated at \( e^t/z \). Hence

\[
-w(z,t) = \frac{1}{2} (e^{-t} - e^t)^2 \left( \frac{e^t}{z} \right)^2 h \left( \frac{e^t}{z} \right) \left( S_f \left( \frac{e^t}{z} \right) - S_g \left( \frac{e^t}{z} \right) \right) \\
+ (1 - e^{2t}) \frac{e^t}{z} \left( h' \left( \frac{e^t}{z} \right) g'' \left( \frac{e^t}{z} \right) + g' \left( \frac{e^t}{z} \right) h'' \left( \frac{e^t}{z} \right) \right) + \left( \frac{1}{h' \left( \frac{e^t}{z} \right)} - 1 \right) e^{2t}.
\]

The right hand side is zero for \( t = 0 \), and is holomorphic in \( D = \{ z : |z| \leq 1 \} \) for \( t > 0 \).

Putting \( e^t/z = \tilde{\zeta} \in E, \tilde{\zeta} = \zeta e^t, e^t = |\tilde{\zeta}| \) for \( |z| = 1 \), from (7) by assumption (1) replacing \( \tilde{\zeta} \) by \( \zeta \) we have

\[
|w(z,t)| = \left| \frac{\tilde{f}(z,t) - zf'(z,t)}{\tilde{f}(z,t) + zf'(z,t)} \right| \leq 1,
\]

so \( \tilde{f}(z,t) = zf'(z,t)p(z,t) \), \( \Re p(z,t) > 0, z \in D, t \in I \).

Hence from (4) and (5) it follows that \( f(z,t), z \in D, t \in T \), is a L"owner chain (see [5], Th. 6.2) and so \( f(z,t) \) is univalent in \( D \). From (2) and (3) it follows in particular that

\[
f(z,0) = \frac{1}{f(\zeta)} = \frac{v(\zeta)}{u(\zeta)}, \quad 1/\zeta = z \in D.
\]

For \( h \equiv 1 \) in \( E \) the inequality (1) reads

\[
\left| \frac{1}{2} (|\zeta|^2 - 1)^2 \zeta (S_f(\zeta) - S_g(\zeta)) - (|\zeta|^2 - 1) \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| \leq 1, \quad \zeta \in E.
\]

This inequality is a sufficient univalence condition of Epstein type on the exterior of the unit disk obtained earlier by the second author [6].

If in Theorem 1 we take \( g(z) = z, h = 1/c, |c - 1| < 1, c \neq 0 \), then the resulting inequality

\[
\left| \frac{1}{2} (|\zeta|^2 - 1) \zeta S_f(\zeta) - c(1-c)|\zeta|^2 \right| \leq |c|, \quad \zeta \in E.
\]

is a sufficient univalence condition on the exterior of the unit disk of Ahlfors type [1] and for \( c = 1 \) of Nehari type [3].

On putting \( f = g, h = 1/c, |c - 1| \leq 1, c \neq 0 \) in Theorem 1, the inequality (1) reads

\[
\left| (|\zeta|^2 - 1) \frac{\zeta g''(\zeta)}{g'(\zeta)} - (1 - c)|\zeta|^2 \right| \leq 1, \quad \zeta \in E.
\]

For \( c = 1 \), this is a known univalence condition for functions in \( E \) obtained by Becker [2].

To show that Theorem 1 is an essential generalization of known univalence conditions for functions defined in the exterior of the unit disk we
consider the following example.

**Example.** Define

\[ f(\zeta) = \frac{\zeta^2}{(\zeta - 1)^2}, \quad g(\zeta) = \frac{2\zeta^2}{2\zeta - 1}, \]

and let

\[ h(\zeta) = \frac{(\zeta - 1/2)^2}{\zeta - 1}. \]

Then \( \text{Re} h(\zeta) \geq 1/2, \zeta \in E \). It is easy to show that the left hand side of (1) is

\[ \frac{\zeta^2}{4\zeta(\zeta - 1) + 1} = \left| \frac{\zeta}{2\zeta - 1} \right|^2 \leq 1, \quad \zeta \in E. \]

On the other hand, the left hand side of (8) is

\[ \frac{|\zeta|^2 - 1}{|(\zeta - 1)(2\zeta - 1)|} \leq \frac{|\zeta| + 1}{2|\zeta| - 1}. \]

So for \( \zeta \in E \), \( f \) and \( g \) do not satisfy the inequality (8).

Neither (9) nor (10) are satisfied by the function \( f \) or \( g \) in \( E \).

**References**


