ANNALES
POLONICI MATHEMATICI
LVI.1 (1991)

A univalence criterion for meromorphic functions

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Abstract. A sufficient univalence condition for meromorphic functions is given.

1. Let f denote a meromorphic and locally univalent function in $E = \{z : |z| > 1\}$, that is, $f'(z) \neq 0$ and any pole of f is simple.

In this note we give a univalence criterion for f in terms of the Schwarz derivative defined by

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

Epstein (see for example [4]) gives the following univalence criterion for meromorphic and locally univalent functions in the unit disk $D = \{z : |z| < 1\}$.

Theorem E. *Let f be meromorphic and g holomorphic in D. If both functions are locally univalent in D and if*

$$\left|\frac{1}{2}(1-|z|^2)^2(S_f(z) - S_g(z)) + (1-|z|^2)\overline{z}\,\frac{g''(z)}{g'(z)}\right| \le 1 , \qquad z \in D ,$$

then f is univalent in D.

In this section we transfer Theorem E to the exterior of the unit disk, which cannot be obtained immediately from Theorem E.

Theorem 1. *Let f and g be meromorphic and locally univalent functions in E and let $g(\zeta) = b\zeta + b_0 + b_1/\zeta + \ldots$ If there exists a holomorphic function h in E with $\operatorname{Re} h \ge 1/2$ in E and $h(\zeta) = 1 + h_2/\zeta^2 + \ldots$ such that*

$$\text{(1)} \qquad \left|\frac{1}{2}(|\zeta|^2-1)^2(S_f(\zeta) - S_g(\zeta))\frac{\zeta}{\overline{\zeta}}h(\zeta) - (|\zeta|^2-1)\left(\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right) - \frac{h(\zeta)-1}{h(\zeta)}|\zeta|^2\right| \le 1, \qquad \zeta \in E ,$$

1991 *Mathematics Subject Classification*: 30A36.
Key words and phrases: Löwner chain, Schwarz derivative.

then f is univalent in E.

Proof. Without loss of generality we can consider the functions of the form

$$f(\zeta)=\zeta+\frac{a_1}{\zeta}+\ldots,\qquad g(\zeta)=\zeta+\frac{b_1}{\zeta}+\ldots$$

since the Schwarzian derivative is invariant under Möbius transformations. The assumption $h(\infty)=1$ can be dropped (see [5]). Let

$$\begin{aligned}v(\zeta)&=\sqrt{\frac{g'(\zeta)}{f'(\zeta)}}=1+\frac{\beta_1}{\zeta^2}+\ldots,\\ u(\zeta)&=f(\zeta)v(\zeta)=\zeta+\frac{c_1}{\zeta}+\ldots\end{aligned}\tag{2}$$

The functions u and v are meromorphic in E since f and g do not have multiple poles and f' and g' are different from zero.

For $t\in I=[0,\infty), 1/\zeta=z$, we consider

$$f(z,t)=\left[\frac{u(\frac{e^t}{z})+(e^{-t}-e^t)\frac{1}{z}h(\frac{e^t}{z})u'(\frac{e^t}{z})}{v(\frac{e^t}{z})+(e^{-t}-e^t)\frac{1}{z}h(\frac{e^t}{z})v'(\frac{e^t}{z})}\right]^{-1},\qquad z\in D.\tag{3}$$

The function $f(z,t)$ is meromorphic in D. By (2) the denominator in (3) in square brackets is $1+O(z^2)$ as $z\to0$, uniformly in t. Hence there exist constants $r_0>0$ and K_0 such that

$$|f(z,t)|\le K_0e^t\qquad\text{for } |z|<r_0,\ t\in I.\tag{4}$$

By (2) the numerator in (3) is $e^{-t}/z+O(z^2)$ as $z\to0$. Hence

$$f(z,t)=e^tz+O(z^2)\qquad\text{as } z\to0.\tag{5}$$

We set

$$f'(z,t)=\frac{\partial f(z,t)}{\partial z},\qquad \dot f(z,t)=\frac{\partial f(z,t)}{\partial t}.$$

After simple calculations from (3) we obtain

$$\begin{aligned}w(z,t)&=\frac{\dot f(z,t)-zf'(z,t)}{\dot f(z,t)+zf'(z,t)}\\ &=-\left\{\left(\frac{1}{h}-1\right)e^{2t}+(e^{-t}-e^t)\frac{e^{2t}}{z}\left(\frac{h'}{h}+\frac{u''v-uv''}{u'v-uv'}\right)\right.\\ &\qquad\left.+(e^t-e^{-t})\frac{e^{2t}}{z^2}h\frac{u''v'-u'v''}{u'v-uv'}\right\},\end{aligned}\tag{6}$$

where

$$u'v-uv'=g',\qquad u''v-uv''=g'',$$
$$u''v'-u'v''=\tfrac12 g'(S_f-S_g),$$

and u, v, u', v', u'', v'' are calculated at e^t/z. Hence

$$(7)\qquad -w(z,t)=\frac{1}{2}(e^{-t}-e^t)^2\left(\frac{e^t}{z}\right)^2 h\left(\frac{e^t}{z}\right)\left(S_f\left(\frac{e^t}{z}\right)-S_g\left(\frac{e^t}{z}\right)\right)$$
$$+(1-e^{2t})\frac{e^t}{z}\left(\frac{h'(\frac{e^t}{z})}{h(\frac{e^t}{z})}+\frac{g''(\frac{e^t}{z})}{g'(\frac{e^t}{z})}\right)+\left(\frac{1}{h(\frac{e^t}{z})}-1\right)e^{2t}.$$

The right hand side is zero for $t=0$, and is holomorphic in $\overline{D}=\{z:|z|\le 1\}$ for $t>0$.

Putting $e^t/z=\widetilde{\zeta}\in E, \widetilde{\zeta}=\zeta e^t, e^t=|\widetilde{\zeta}|$ for $|z|=1$, from (7) by assumption (1) replacing $\widetilde{\zeta}$ by ζ we have

$$|w(z,t)|=\left|\frac{\dot f(z,t)-zf'(z,t)}{\dot f(z,t)+zf'(z,t)}\right|\le 1,$$

so $\dot f(z,t)=zf'(z,t)p(z,t)$, $\operatorname{Re}p(z,t)>0$, $z\in D$, $t\in I$.

Hence from (4) and (5) it follows that $f(z,t)$, $z\in D$, $t\in T$, is a Löwner chain (see [5], Th. 6.2) and so $f(z,t)$ is univalent in D. From (2) and (3) it follows in particular that

$$f(z,0)=\frac{1}{f(\zeta)}=\frac{v(\zeta)}{u(\zeta)},\qquad \frac{1}{\zeta}=z\in D. \ ■$$

For $h\equiv 1$ in E the inequality (1) reads

$$(8)\qquad \left|\frac{1}{2}(|\zeta|^2-1)^2\frac{\zeta}{\overline{\zeta}}(S_f(\zeta)-S_g(\zeta))-(|\zeta|^2-1)\frac{\zeta g''(\zeta)}{g'(\zeta)}\right|\le 1,\qquad \zeta\in E.$$

This inequality is a sufficient univalence condition of Epstein type on the exterior of the unit disk obtained earlier by the second author [6].

If in Theorem 1 we take $g(z)=z$, $h=1/c$, $|c-1|<1$, $c\ne 0$, then the resulting inequality

$$(9)\qquad \left|\frac{1}{2}(|\zeta|^2-1)\frac{\zeta}{\overline{\zeta}}S_f(\zeta)-c(1-c)|\zeta|^2\right|\le |c|,\qquad \zeta\in E,$$

is a sufficient univalence condition on the exterior of the unit disk of Ahlfors type [1] and for $c=1$ of Nehari type [3].

On putting $f=g$, $h=1/c$, $|c-1|\le 1$, $c\ne 0$ in Theorem 1, the inequality (1) reads

$$(10)\qquad \left|(|\zeta|^2-1)\frac{\zeta g''(\zeta)}{g'(\zeta)}-(1-c)|\zeta|^2\right|\le 1,\qquad \zeta\in E.$$

For $c=1$, this is a known univalence condition for functions in E obtained by Becker [2].

To show that Theorem 1 is an essential generalization of known univalence conditions for functions defined in the exterior of the unit disk we consider the following example.

EXAMPLE. Define

$$f(\zeta)=\frac{\zeta^2}{(\zeta-1)^2}\,,\qquad g(\zeta)=\frac{2\zeta^2}{2\zeta-1}\,,$$

and let

$$h(\zeta)=\frac{(\zeta-1/2)^2}{\zeta-1}\,.$$

Then $\operatorname{Re} h(\zeta)\ge 1/2, \zeta\in E$. It is easy to show that the left hand side of (1) is

$$\left|\frac{\zeta^2}{4\zeta(\zeta-1)+1}\right|=\left|\frac{\zeta}{2\zeta-1}\right|^2\le 1\,,\qquad \zeta\in E\,.$$

On the other hand, the left hand side of (8) is

$$\frac{|\zeta|^2-1}{|(\zeta-1)(2\zeta-1)|}\le\frac{|\zeta|+1}{2|\zeta|-1}\,.$$

So for $\zeta\in E$, f and g do not satisfy the inequality (8).

Neither (9) nor (10) are satisfied by the function f or g in E.

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Reçu par la Rédaction le 23.5.1990
Révisé le 25.1.1991